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POSITIVE POINTS IN POLAR LATTICES

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Abstract

The authors investigate

$$\max \min \mu F(x)\, F(y)$$

for two standard distance functions F in R^2, where μ denotes the area of $\{x \in R^2; F(x) \leqslant 1\}$, the maximum is over all (geometric) lattices Λ in R^2 and the minimum is over all positive points $x \in \Lambda$ and $y \in \Lambda^*$ (the polar lattice of Λ). An application is given to a problem on fractional parts.

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Let Λ be a (geometric) lattice in R^n and let

$$\Lambda^* = \{y \in R^n : y.x \in Z \text{ for all } x \in \Lambda\}$$

denote the polar lattice. It is well known that if $\Lambda = T\Gamma$, where T is a nonsingular linear transformation and Γ is the lattice of all points with integer coordinates, then $\Lambda^* = T^*\Gamma$, where T^* is the inverse transpose of T. Let F be a distance function for which the set

$$C_F = \{x \in R^n : F(x) < 1\}$$

is a symmetric convex body. If μ denotes the volume of C_F, then Minkowski's convex body theorem states that there exists a point $x \neq 0$ of Λ such that

$$F(x) \leqslant 2\mu^{-1/n}\, d(\Lambda)^{1/n}.$$

Since $d(\Lambda^*)\, d(\Lambda) = 1$ we therefore have the existence of nonzero points $x \in \Lambda$, $y \in \Lambda^*$ such that

$$\mu^{2/n} F(x)\, F(y) \leqslant 4. \tag{1}$$

For $n = 2$ this result is best possible when $F(x_1, x_2) = |x_1| + |x_2|$ and Λ is the lattice generated by $(1,0)$ and $(1/2, 1/2)$. In this particular case we can in fact choose $x \in \Lambda$, $y \in \Lambda^*$ satisfying (1) and the further condition that they lie in the interior P^0 of the positive cone

$$P = \{(x_1, x_2) \in R^2 \colon x_1 \geqslant 0, x_2 \geqslant 0\}.$$

One is therefore led to ask what can be said about $\mu^{2/n} F(x) F(y)$ with x, y nonzero points of Λ, Λ^*, respectively, when x and y are restricted to lie in either P or P^0. A little trial and error suggests that $x \in P^0$, $y \in P^0$ is a little too strict a condition.† Considering $F(x_1, x_2) = |x_1| + |x_2|$ and Λ generated by $(1,0)$ and $(-t, t)$, where t is a positive integer, we have

$$\mu F(x) F(y) \geqslant 2(t + t^{-1} + 2)$$

for all $x \in P^0$, $y \in P^0$, and so any upper bound on $\mu F(x) F(y)$ under these conditions would depend on both F and Λ. The next condition to be considered would be $x \in P$, $y \in P^0$ (or, equivalently, $x \in P^0$, $y \in P$), and the aim of this paper is to give an upper bound depending only on F for a couple of standard distance functions F. We show

THEOREM 1. *Let F_t denote the distance function on R^2 defined by*

$$F_t(x_1, x_2) = |x_1| + t|x_2|,$$

where $t > 0$, and let Λ be a lattice in R^2. Then there exist nonzero $x \in P \cap \Lambda$, $y \in P^0 \cap \Lambda^*$ such that

$$\mu F_t(x) F_t(y) \leqslant 2(t + t^{-1}).$$

THEOREM 2. *Let G_t denote the distance function on R^2 defined by*

$$G_t(x_1, x_2) = (x_1^2 + t^2 x_2^2)^{\frac{1}{2}},$$

where $t > 0$, and let Λ denote a lattice in R^2. Then there exist nonzero $x \in P \cap \Lambda$, $y \in P^0 \cap \Lambda^*$ such that

$$\mu G_t(x) G_t(y) \leqslant \pi (t^2 + t^{-2})^{\frac{1}{2}}.$$

Theorems 1 and 2 are best possible in that, for example, equality holds for any multiple of the lattice generated by $(1,0)$ and $(0, t^{-1})$.

The applications of these theorems are to lattices Λ containing Γ, for then Λ^* consists entirely of integral points. For example, we have

† Henceforth x will always stand for a non zero point of Λ and y a non zero point of Λ^*.

COROLLARY 1. *Let Λ be a lattice in R^2 containing Γ. Then there exists $(x_1, x_2) \in \Lambda \cap P$ and positive integers l, m such that*

$$lx_1 + mx_2 \in Z \tag{2}$$

and

$$0 < x_1 + x_2 \leqslant 2/(l+m).$$

This is just the case $t = 1$ of Theorem 1, though (2) is weaker than the requirement that (x_1, x_2) and (l, m) lie in polar lattices.

COROLLARY 2. *Let θ_1, θ_2 be rationals. Then there exist positive integers l, m such that $l\theta_1 + m\theta_2 \in Z$, and a positive integer t such that*

$$0 < \{t\theta_1\} + \{t\theta_2\} \leqslant 2/(l+m), \tag{3}$$

where $\{x\}$ denotes the fractional part of x.

This is just Corollary 1 applied to the lattice of all points of the form $(t\theta_1 + a_1, t\theta_2 + a_2)$ with a_1, a_2 integral. Of course using Theorem 2 in place of Theorem 1 would replace (3) by

$$0 < \{t\theta_1\}^2 + \{t\theta_2\}^2 \leqslant 2/(l^2+m^2). \tag{3'}$$

In addition, if $\|\theta x\|$ denotes $\min(\{x\}, 1-\{x\})$, then, since $\|t\theta\| \leqslant \{t\theta\}$ and $\|t\theta\| = 0$ only when $\{t\theta\} = 0$, we can clearly replace $\{t\theta\}$ by $\|t\theta\|$ in (3) and (3′).

It will be observed that Corollary 2 is best possible in some cases. For example, if k is integral then when $\theta_1 = \theta_2 = k^{-1}$ and when $\theta_1 = k^{-1}$, $\theta_2 = 1 - k^{-1}$ we are dealing with lattices requiring equality in Theorem 1 (when $t = 1$) and Corollary 2.

The inequality (3) can be compared with the results for θ_1, θ_2 not necessarily rational, which are as follows.

Firstly, if θ_1, θ_2, 1 are independent over the rationals then the right-hand side of (3) can be replaced by ε for any $\varepsilon > 0$. A similar situation occurs when θ_1, θ_2 satisfy a relation of the type $l\theta_1 - m\theta_2 = n$ with l, m, n positive integers. However, if θ_1, θ_2 satisfy a relation of the type $l\theta_1 + m\theta_2 = n$ with l, m, n positive integers, then if $(l, m, n) = 1$ the best that can be said is that the right-hand side of (3) can be replaced by $1/\max(l, m)$.

PROOF OF THEOREM 1. The proof uses the fact that $(d, c)/d(\Lambda)$ lies in Λ^* when $(-c, d) \in \Lambda$. Let $\alpha_1 = (a, b) \neq (0, 0)$ be a point of $\Lambda \cap P$ such that $a + bt$ is minimal and let $(-c, d)$ be a point of Λ in the second quadrant

$$Q = \{(x_1, x_2) \in R^2 \colon x_1 < 0, x_2 > 0\}$$

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such that $ct+d$ is minimal. It is clear that

$$\{(x_1, x_2) \in R^2 : |x_1 + tx_2| < a + bt, |tx_1 - x_2| < ct + d\}$$

contains no nonzero point of Λ, and so by Minkowski's linear forms theorem (Cassels (1959), p. 73) we have

$$d(\Lambda) \geqslant (a+bt)(ct+d)/(1+t^2).$$

The theorem now follows immediately on setting $x = \alpha_1$, $y = (d, c)/d(\Lambda)$ and $\mu = 2t^{-1}$.

PROOF OF THEOREM 2. The basic idea behind the proof of this theorem is the same as for the previous theorem, but Minkowski's theorem is not strong enough in this case. Let $x = (a, b) \neq (0, 0)$ be a point of $\Lambda \cap P$ such that $A^2 = a^2 + t^2 b^2$ is minimal, and let $(-c, d)$ be a point of $\Lambda \cap Q$ such that $B^2 = t^2 c^2 + d^2$ is minimal. Since $y = (d, c)/d(\Lambda)$ lies in $\Lambda^* \cap P^0$, we only need to show

$$t^{-1} AB \leqslant (t^2 + t^{-2})^{\frac{1}{2}} d(\Lambda) \tag{4}$$

in order to prove the theorem. We assume without loss of generality that $a \neq 0$ and that $m \leqslant t$ where $b = am$, for interchanging the roles of the coordinates replaces m by m^{-1} and t by t^{-1}.

Since x is a primitive lattice point there is a basis $\{x, \gamma\}$ of Λ, and replacing γ by $-\gamma$ if necessary we may take γ to lie on the line

$$L = \{(x_1, x_2) \in R^2 : x_2 = mx_1 + e\},$$

where $e > 0$. Furthermore, the lattice points $\gamma + nx$, for integral n, lie on L at equally spaced intervals of length $l_1 = a(1+m^2)^{\frac{1}{2}}$ and so L cannot intersect

$$T = \{(x_1, x_2) \in Q : t^2 x_1^2 + x_2^2 < B^2\} \cup \{(x_1, x_2) \in P : x_1^2 + t^2 x_2^2 < A^2\}$$

in a continuous segment of length greater than l_1. We now establish the inequality

$$Be^{-1} \leqslant \{(t^4+1)/(1+m^2 t^2)\}^{\frac{1}{2}}, \tag{5}$$

which is equivalent to (4).

We note firstly that (5) is trivial if $B \leqslant e$ and so we assume $B > e$. We let

$$f = e\{m + [(Be^{-1})^2 (m^2 + t^2) - t^2]^{\frac{1}{2}}\}/(m^2 + t^2)$$

and

$$h = e\{[(Ae^{-1})^2 (1 + m^2 t^2) - t^2]^{\frac{1}{2}} - mt^2\}/(1 + m^2 t^2).$$

Furthermore, we set $\sigma_1 = (-f, e - mf)$, $\sigma_2 = (-em^{-1}, 0)$, $\rho_1 = (0, e)$, $\rho_2 = (h, mh + e)$ and $e_0 = At^{-1} = a(m^2 + t^{-2})^{\frac{1}{2}}$.

Then L meets T in a continuous line segment, of length l_2, say, from the point σ, which is either σ_1 or σ_2, to the point ρ, where $\rho = \rho_1$ if $e \geqslant e_0$ and $\rho = \rho_2$ if $e < e_0$.

We consider the various possibilities.

(i) If $e \geqslant e_0$ then $em^{-1} > a$ and so $l_2 > l_1$ if $\sigma = \sigma_2$. However, when $\sigma = \sigma_1$ we have

$$l_2 e^{-1} = fe^{-1}(1+m^2)^{\frac{1}{2}},$$

which is an increasing funtcion of Be^{-1}. When

$$Be^{-1} = [(t^4+1)(1+m^2 t^2)]^{\frac{1}{2}}, \tag{6}$$

we see that

$$\begin{aligned} l_2 e^{-1} &= \{m(1+m^2 t^2)^{\frac{1}{2}} + (t^6+m^2)^{\frac{1}{2}}\}(1+m^2)^{\frac{1}{2}}(m^2+t^2)^{-1}(1+m^2 t^2)^{-\frac{1}{2}} \\ &> (m^2 t + t^3)(1+m^2)^{\frac{1}{2}}(m^2+t^2)^{-1}(1+m^2 t^2)^{-\frac{1}{2}} \\ &= t(1+m^2)^{\frac{1}{2}}(1+m^2 t^2)^{-\frac{1}{2}}. \end{aligned}$$

Hence if (5) is false we have, since $e \geqslant e_0$, that

$$l_2 \geqslant (l_2 e^{-1}) e_0 > t(1+m^2)^{\frac{1}{2}}(1+m^2 t^2)^{-\frac{1}{2}} e_0 = l_1,$$

which is impossible. Hence (5) must hold if $e \geqslant e_0$.

(ii) If $e < e_0$ we note that $l_2 \leqslant l_1$ is equivalent to the inequality $f+h \leqslant a$. Since for fixed e, $f+h$ is an increasing function of B, (5) will follow on showing that $f+h \geqslant a$ when (6) holds. Suppose to the contrary that $f+h < a$ when (6) holds, that is,

$$\begin{aligned} &\{m + [(t^6+m^2)(1+m^2 t^2)]^{\frac{1}{2}}\}(m^2+t^2)^{-1} - mt^2(1+m^2 t^2)^{-1} \\ &\qquad < \omega - [\omega^2(1+m^2 t^2)^2 - t^2]^{\frac{1}{2}}(1+m^2 t^2)^{-1}, \end{aligned} \tag{7}$$

on writing ω for ae^{-1} and re-arranging. The right hand side of this inequality decreases as ω increases, and $\omega > (m^2+t^{-2})^{-\frac{1}{2}}$ and so (7) implies that

$$m + [(t^6+m^2)(1+m^2 t^2)]^{\frac{1}{2}}(m^2+t^2)^{-1} < t(1+m^2 t^2)^{-1}.$$

Since squaring this yields

$$[(t^6+m^2)(1+m^2 t^2)]^{\frac{1}{2}} < m(t^4-1),$$

which is obviously impossible, we have reached the desired contradiction, and our proof of Theorem 2 is complete.

Reference

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