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A Free Logarithmic Sobolev Inequality on the Circle

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Abstract. Free analogues of the logarithmic Sobolev inequality compare the relative free Fisher information with the relative free entropy. In the present paper such an inequality is obtained for measures on the circle. The method is based on a random matrix approximation procedure, and a large deviation result concerning the eigenvalue distribution of special unitary matrices is applied and discussed.

1 Introduction

Logarithmic Sobolev inequalities have played a role in the study of norm estimates for the diffusion semigroup since the first systematic study done by L. Gross [6] in 1975 who recognized the relation to hypercontractive estimates. Afterwards many authors have discussed the logarithmic Sobolev inequality (LSI) in various contexts, in particular, in close connection with the notions of hypercontractivity and spectral gap. An LSI can be understood to compare the relative Fisher information with the relative entropy. Its simplest form is

$$\int_{\mathbf{R}} f(t)^2 \log f(t)^2 \, d\gamma(t) \le \int_{\mathbf{R}} f'(t)^2 \, d\gamma(t) \tag{1.1}$$

for any smooth function f on $\mathbf{R}$ and $d\gamma(t) = (2\pi)^{-1} e^{-t^2/2} dt$, the normalized Gaussian measure.

The generalization due to D. Bakry and M. Emery [1] holds on a complete Riemannian manifold under the condition

$$\mathrm{Ric}(M) + \mathrm{Hess}(\Psi) \ge \rho I_m$$

with a strictly positive constant $\rho > 0$. Here, $\mathrm{Ric}(M)$ is the Ricci curvature and $\mathrm{Hess}(\Psi)$ is the Hessian of the smooth function Ψ inducing the reference Gibbs measure (replacing the Gaussian in (1.1)).

On the other hand, entropy, Fisher information and Gaussian measure have found their analogues in free probability and the central measure there is the semicircular law of compact support (see [10, 16, 17]). The first free LSI was discovered by

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Voiculescu [18] and in a specialized form it is given as

$$-\iint_{\mathbf{R}^2} \log|x-y|\, g(x)g(y)\,dxdy \le \frac{2\pi^2}{3}\|g\|_3^3 - \frac{1}{4} \tag{1.2}$$

when g is a probability density on $\mathbf{R}$ belonging to $L^3(\mathbf{R})$. A remark on the relation of inequalities (1.1) and (1.2) might be in order. The second one is not a formal extension of the first one, but the left-hand sides are entropy quantities and the right-hand sides are Fisher informations. Recall that the logarithmic integral is the main component of the rate function in a certain large deviation theorem while the third power of the L^3-norm functions is a kind of Fisher information.

Extending Voiculescu's result, Ph. Biane obtained in [3] another free probabilistic analogue of the LSI. He allowed a parameter function Q (in the role of Ψ), and the result is

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{2\rho}\Phi_Q(\mu) \quad \text{for } \mu \in \mathcal{M}(\mathbf{R}), \tag{1.3}$$

where the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ and the relative free Fisher information $\Phi_Q(\mu)$ were introduced earlier by Biane and Speicher [4] for $\mu \in \mathcal{M}(\mathbf{R})$, the probability measures on $\mathbf{R}$. To prove the inequality, Biane applied the classical LSI on the Euclidean space to the related self-adjoint random matrix ensembles and used the weak convergence of their mean eigenvalue distribution. For the details we refer to the original paper [3] and also to [11].

Our main aim here is to show a variant of Biane's free LSI for measures on the unit circle $\mathbf{T}$. In §2 of this paper we introduce the relative free entropy $\widetilde{\Sigma}_Q(\mu)$ and the relative free Fisher information $F_Q(\mu)$ for $\mu \in \mathcal{M}(\mathbf{T})$. In §4 we prove

$$\widetilde{\Sigma}_Q(\mu) \le \frac{1}{1+2\rho}F_Q(\mu) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

if Q is a C^1-function on $\mathbf{T}$ such that $Q(e^{it}) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ with a constant $\rho > -1/2$. To prove this, we apply Bakry and Emery's classical LSI on the special unitary group $\mathrm{SU}(n)$, a Riemannian manifold, to the related $n \times n$ special unitary random matrices and pass to the scaling limit as n goes to ∞. Here, we need the convergence of the empirical eigenvalue distribution of the random matrix not only in the mean but also in the almost sure sense that is a consequence of the corresponding large deviation principle. The proof of this large deviation for "special" unitary random matrices is sketched in §3 because it is a bit more complicated than that for unitary random matrices shown in [9].

In this way, we clarify the advantage of random matrix approximation procedure in studying free probabilistic analogues of certain classical theories involving relative entropy and/or Fisher information. Voiculescu's heuristics in [16] suggests that the classical entropy of random matrices, if suitably arranged, asymptotically converges to the free entropy of the limit distribution as the matrix size goes to infinity. Toward its rigorous realizations, this paper as well as our previous one [12] may be regarded as one more attempt subsequent to [2, 7] (see also §5.1 for more details).

2 Preliminaries

Let us begin by fixing some standard notations. Denote by $\mathcal{M}(\mathcal{X})$ the set of Borel probability measures on a certain Polish space $\mathcal{X}$. For $\mu, \nu \in \mathcal{M}(\mathcal{X})$ let $S(\mu, \nu)$ denote the relative entropy of μ with respect to ν. For an $n \times n$ complex matrix A, $\mathrm{Tr}_n(A)$ stands for the usual (non-normalized) trace of A and $\|A\|_{HS}$ for the Hilbert–Schmidt norm of A. The unitary group and the special unitary group of order n are denoted by $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, respectively.

Among extensive literature, Bakry and Emery gave a simple "local" criterion, the so-called Bakry and Emery criterion (or the Γ_2-criterion), for a given measure to satisfy a *logarithmic Sobolev inequality* (*LSI* for short). Their LSI is one of the key ingredients of the proof of our main theorem.

Let M be an m-dimensional smooth complete Riemannian manifold with the volume measure dx, and let $\mathrm{Ric}(M)$ denote the *Ricci curvature tensor* of M. For a real-valued C^2-function Ψ on M, the *Hessian* of Ψ is denoted by $\mathrm{Hess}(\Psi)$. The precise statement that Bakry and Emery established is as follows.

Theorem 2.1 (Bakry and Emery [1]) *Let $\Psi \in C^2(M)$, and set $d\nu(x) := \frac{1}{Z}e^{-\Psi(x)}dx$ with a normalization constant Z. Assume that the Bakry and Emery criterion*

$$\mathrm{Ric}(M) + \mathrm{Hess}(\Psi) \geq \rho I_m$$

holds with a constant $\rho > 0$. Then, for every $\mu \in \mathcal{M}(M)$ absolutely continuous with respect to ν, the inequality

$$S(\mu, \nu) \leq \frac{1}{2\rho} \int_M \left\| \nabla \log \frac{d\mu}{d\nu} \right\|^2 d\mu \tag{2.1}$$

holds whenever the density $d\mu/d\nu$ is smooth on M.

Recall that the left-hand side of (2.1) is the relative entropy, while the integral in the right-hand side can be recognized as the (classical) *relative Fisher information* of μ relative to ν.

For each $\mu \in \mathcal{M}(\mathbf{T})$, the *free entropy* $\Sigma(\mu)$ of μ is defined in the same manner as in the real line case:

$$\Sigma(\mu) := \iint_{\mathbf{T}^2} \log|\zeta - \eta| \, d\mu(\zeta) \, d\mu(\eta)$$

([8], [19, §10.7]). For its justification to be a right quantity, see [19, Proposition 10.8] in relation to the free Fisher information as well as [8, Proposition 1.4], [9] from the microstate approach or large deviation principle. As in the real line case, the *relative free entropy* $\widetilde{\Sigma}_Q(\mu)$ of $\mu \in \mathcal{M}(\mathbf{T})$ relative to a real-valued continuous function Q is defined based on the large deviation principle, which will be explained in the next section.

Assume that $\mu \in \mathcal{M}(\mathbf{T})$ has the density $p = d\mu/d\zeta$ with respect to the Haar probability measure $d\zeta = d\theta/2\pi$, $\zeta = e^{i\theta}$ with $\theta \in [-\pi, \pi)$ and further that p

belongs to the L^3-space $L^3(\mathbf{T}) := L^3(\mathbf{T}, d\zeta)$. The Hilbert transform of p

$$(2.2) \qquad (Hp)(e^{\mathrm{i}\theta}) := \lim_{\varepsilon \searrow 0} \int_{\varepsilon \le |t| < \pi} \frac{p(e^{\mathrm{i}(\theta - t)})}{\tan\left(\frac{t}{2}\right)} \frac{dt}{2\pi}$$

is important. The principal value limit in (2.2) exists for a.e. (as long as $p \in L^1(\mathbf{T})$), and it is known that $p \in L^q(\mathbf{T})$ implies $Hp \in L^q(\mathbf{T})$ as well for each $1 < q < \infty$. See [13, Chapter V] for detailed accounts on the Hilbert transform on $\mathbf{T}$. Following Voiculescu [19, §8.9] we call the quantity

$$F(\mu) := \int_{\mathbf{T}} ((Hp)(\zeta))^2 \, d\mu(\zeta) = \int_{\mathbf{T}} ((Hp)(\zeta))^2 p(\zeta) \, d\zeta$$

the *free Fisher information* of μ. When μ has no such density as above, $F(\mu)$ is defined to be $+\infty$. By [19, Corollary 8.8 and Definition 8.9] the free Fisher information can be written as

$$F(\mu) = \frac{1}{3}\Big(-1 + \int_{\mathbf{T}} p(\zeta)^3 \, d\zeta\Big).$$

When Q is a real-valued C^1-function on $\mathbf{R}$, the *relative free Fisher information* $\Phi_Q(\mu)$ of $\mu \in \mathcal{M}(\mathbf{R})$ was introduced by Biane and Speicher [4, §6] to be

$$(2.3) \qquad \Phi_Q(\mu) := 4 \int_{\mathbf{R}} \Big((Hp)(x) - \frac{1}{2} Q'(x) \Big)^2 \, d\mu(x)$$

if μ has the density $p = d\mu/dx$ belonging to $L^3(\mathbf{R})$, otherwise $+\infty$.

On the other hand, when Q is a real-valued C^1-function on $\mathbf{T}$, for each $\mu \in \mathcal{M}(\mathbf{T})$ we define the *relative free Fisher information* $F_Q(\mu)$ to be

$$(2.4) \qquad F_Q(\mu) := \int_{\mathbf{T}} \big((Hp)(\zeta) - Q'(\zeta)\big)^2 \, d\mu(\zeta) - \Big(\int_{\mathbf{T}} Q'(\zeta) \, d\mu(\zeta) \Big)^2$$

if μ has the density $p = d\mu/d\zeta$ belonging to $L^3(\mathbf{T})$, otherwise $+\infty$. Here, Q' means the derivative of $Q(e^{\mathrm{i}\theta})$ in θ, *i.e.*, $Q'(e^{\mathrm{i}\theta}) = \frac{d}{d\theta} Q(e^{\mathrm{i}\theta})$. The slight difference between the two formulas (2.3) and (2.4) is worth notice.

3 Large Deviations for Special Unitary Random Matrices

Let Q be a real-valued continuous function on $\mathbf{T}$. The weighted energy integral

$$-\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta) \, d\mu(\zeta) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

admits a unique minimizer $\mu_Q \in \mathcal{M}(\mathbf{T})$ or the *equilibrium measure* associated with Q (see [15, I.1.3]). Set $B(Q) := \Sigma(\mu_Q) - \int_{\mathbf{T}} Q(\zeta) \, d\mu_Q(\zeta)$. It is known [9] that the function

$$-\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta) \, d\mu(\zeta) + B(Q) \quad \text{for } \mu \in \mathcal{M}(\mathbf{T})$$

is the rate function of the large deviation for the empirical eigenvalue distribution of an $n \times n$ *unitary random matrix*

$$d\lambda_n^{\mathrm{U}}(Q)(U) := \frac{1}{Z_n^{\mathrm{U}}(Q)} \exp\bigl(-n \operatorname{Tr}_n(Q(U))\bigr)\, dU,$$

where dU is the Haar probability measure on $\mathrm{U}(n)$, $Q(U)$ is defined via functional calculus and $Z_n^{\mathrm{U}}(Q)$ is a normalization constant. Furthermore,

$$B(Q) = \lim_{n\to\infty} \frac{1}{n^2} \log \int \cdots \int_{\mathbf{T}^n} \exp\Bigl(-n \sum_{i=1}^{n} Q(\zeta_i)\Bigr) \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{n} d\zeta_i$$

where $d\zeta_i = d\theta_i/2\pi$ for $\zeta_i = e^{\mathrm{i}\theta_i}$. However, the above unitary random matrix $\lambda_n^{\mathrm{U}}(Q)$ is not suitable for our purpose as in [12], and thus we need to modify the above large deviation in the setup of $\mathrm{SU}(n)$.

The joint eigenvalue distribution on $\mathbf{T}^{n-1}$ of the Haar probability measure on $\mathrm{SU}(n)$ is known to have the explicit form

$$\text{(3.1)} \qquad \frac{1}{n!} \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{n-1} d\zeta_i \quad \text{with } \zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1},$$

or

$$\frac{1}{n!(2\pi)^{n-1}} \prod_{1\le i<j\le n} \bigl| e^{\mathrm{i}\theta_i} - e^{\mathrm{i}\theta_j} \bigr|^2 \prod_{i=1}^{n-1} d\theta_i \quad \text{with } \theta_n = -(\theta_1 + \cdots + \theta_{n-1}) \ (\mathrm{mod}\ 2\pi).$$

(See [12, §1.5] for a brief explanation of this standard fact.)

Let Q be a real-valued continuous function on $\mathbf{T}$. For each $n \in \mathbf{N}$ define $\lambda_n(Q) \in \mathcal{M}(\mathrm{SU}(n))$, the $n \times n$ *special unitary random matrix* associated with Q, by

$$\text{(3.2)} \qquad d\lambda_n^{\mathrm{SU}}(Q)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q)} \exp\bigl(-n \operatorname{Tr}_n(Q(U))\bigr)\, dU,$$

where dU is the Haar probability measure on $\mathrm{SU}(n)$ and $Z_n^{\mathrm{SU}}(Q)$ is a normalization constant. By the formula (3.1) the joint eigenvalue distribution on $\mathbf{T}^{n-1}$ of $\lambda_n^{\mathrm{SU}}(Q)$ is given as

$$d\tilde\lambda_n^{\mathrm{SU}}(Q)(\zeta_1,\dots,\zeta_{n-1}) = \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)} \exp\Bigl(-n \sum_{i=1}^{n} Q(\zeta_i)\Bigr) \prod_{1\le i<j\le n} |\zeta_i - \zeta_j|^2 \prod_{i=1}^{n-1} d\zeta_i,$$

with $\zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}$.

The next theorem is the large deviation principle for the empirical eigenvalue distribution of $\lambda_n^{\mathrm{SU}}(Q)$, whose proof based on the explicit form of the density of $\tilde\lambda_n^{\mathrm{SU}}(Q)$ just above will be sketched below.

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Theorem 3.1 *The finite limit* $B(Q) := \lim_{n\to\infty} 1/n^2 \log \widetilde{Z}_n^{\mathrm{SU}}(Q)$ *exists. When* $(\zeta_1,\dots,\zeta_{n-1})$ *is distributed on* $\mathbf{T}^{n-1}$ *according to* $\tilde{\lambda}_n^{\mathrm{SU}}(Q)$, *the empirical distribution* $\frac{1}{n}\big(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n}\big)$ *with* $\zeta_n=(\zeta_1\cdots\zeta_{n-1})^{-1}$ *satisfies the large deviation principle in the scale* $1/n^2$ *with the rate function*

$$\widetilde{\Sigma}_Q(\mu) := -\Sigma(\mu) + \int_{\mathbf{T}} Q(\zeta)\,d\mu(\zeta) + B(Q) \quad \textit{for } \mu\in\mathcal{M}(\mathbf{T}). \tag{3.3}$$

Furthermore, there exists a unique minimizer $\mu_Q\in\mathcal{M}(\mathbf{T})$ *of the rate function so that* $\widetilde{\Sigma}_Q(\mu_Q)=0$.

We call the rate function (3.3) the *relative free entropy* of μ with respect to Q, which is denoted by $\widetilde{\Sigma}_Q(\mu)$ as in the real line case in [4].

Sketch of the proof In the following let us keep the relation $\zeta_n=(\zeta_1\cdots\zeta_{n-1})^{-1}$. The proof below is essentially same as that in [9], though some modifications are needed. Set

$$F(\zeta,\eta) := -\log|\zeta-\eta| + \frac{1}{2}(Q(\zeta)+Q(\eta)).$$

As in [9] it suffices to prove the following inequalities:

(i)

$$\limsup_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q) \le -\inf_{\mu\in\mathcal{M}(\mathbf{T})}\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)d\mu(\eta).$$

(ii) For every $\mu\in\mathcal{M}(\mathbf{T})$,

$$\inf_G\Big[\limsup_{n\to\infty}\frac{1}{n^2}\log\tilde{\lambda}_n^{\mathrm{SU}}(Q)\Big\{\frac{1}{n}(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n})\in G\Big\}\Big]$$
$$\le -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)d\mu(\eta) - \liminf_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q),$$

where G runs over all neighborhoods of μ.

(iii) For every $\mu\in\mathcal{M}(\mathbf{T})$,

$$\liminf_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q) \ge -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)d\mu(\eta).$$

(iv) For every $\mu\in\mathcal{M}(\mathbf{T})$,

$$\inf_G\Big[\liminf_{n\to\infty}\frac{1}{n^2}\log\tilde{\lambda}_n^{\mathrm{SU}}(Q)\Big\{\frac{1}{n}(\delta_{\zeta_1}+\cdots+\delta_{\zeta_{n-1}}+\delta_{\zeta_n})\in G\Big\}\Big]$$
$$\ge -\iint_{\mathbf{T}^2} F(\zeta,\eta)\,d\mu(\zeta)\,d\mu(\eta) - \limsup_{n\to\infty}\frac{1}{n^2}\log\widetilde{Z}_n^{\mathrm{SU}}(Q),$$

where G is as in (ii).

The proofs of the first two are the same as in [9]. To prove (iii) and (iv), we may assume (see [9]) that μ has a continuous density $f > 0$ so that $\mu = f(e^{\mathrm{i}\theta})\,d\theta/2\pi$ and $\delta \le f(\zeta) \le \delta^{-1}$ on $\mathbf{T}$ for some $\delta > 0$. For each $n \in \mathbf{N}$ choose

$$0 = b_0^{(n)} < a_1^{(n)} < b_1^{(n)} < a_2^{(n)} < b_2^{(n)} < \cdots < a_n^{(n)} < b_n^{(n)} = 2\pi$$

such that

$$\frac{1}{2\pi}\int_0^{a_j^{(n)}} f(e^{\mathrm{i}\theta})\,d\theta = \frac{j-\frac{1}{2}}{n}, \quad \frac{1}{2\pi}\int_0^{b_j^{(n)}} f(e^{\mathrm{i}\theta})\,d\theta = \frac{j}{n};$$

hence

$$\frac{\pi\delta}{n} \le b_j^{(n)} - a_j^{(n)} \le \frac{\pi}{n\delta}, \quad \frac{\pi\delta}{n} \le a_j^{(n)} - b_{j-1}^{(n)} \le \frac{\pi}{n\delta} \tag{3.4}$$

for all $1 \le j \le n$. Define

$$\begin{aligned}
\Delta_n &:= \left\{ (e^{\mathrm{i}\theta_1}, \dots, e^{\mathrm{i}\theta_{n-1}}) : a_j^{(n)} \le \theta_j \le b_j^{(n)}, 1 \le j \le n-1 \right\},\\
\Theta_n &:= \left\{ (\theta_1, \dots, \theta_{n-1}) : a_j^{(n)} \le \theta_j \le b_j^{(n)}, 1 \le j \le n-1 \right\},\\
\xi_i^{(n)} &:= \max\left\{ Q(e^{\mathrm{i}\theta}) : a_i^{(n)} \le \theta \le b_i^{(n)} \right\} \quad \text{for } 1 \le i \le n-1,\\
d_{ij}^{(n)} &:= \min\left\{ |e^{\mathrm{i}s} - e^{\mathrm{i}t}| : a_i^{(n)} \le s \le b_i^{(n)},\ a_j^{(n)} \le t \le b_j^{(n)} \right\} \quad \text{for } 1 \le i, j \le n-1.
\end{aligned}$$

For every neighborhood G of μ, if n is sufficiently large, then we have

$$\Delta_n \subset \left\{ (\zeta_1, \dots, \zeta_{n-1}) \in \mathbf{T}^{n-1} : \frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}) \in G \right\}$$

so that with $\theta_n = -(\theta_1 + \cdots + \theta_{n-1})$

$$\begin{aligned}
\tilde{\lambda}_n^{\mathrm{SU}}(Q)&\left\{ \frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}) \in G \right\} \ge \tilde{\lambda}_n^{\mathrm{SU}}(Q)(\Delta_n)\\
&= \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)(2\pi)^{n-1}} \int \cdots \int_{\Theta_n} \exp\Big(-n \sum_{i=1}^n Q\big(e^{\mathrm{i}\theta_i}\big) \Big)\\
&\qquad \times \prod_{1 \le i < j \le n} \left| e^{\mathrm{i}\theta_i} - e^{\mathrm{i}\theta_j} \right|^2 d\theta_1 \cdots d\theta_{n-1}\\
&\ge \frac{1}{\widetilde{Z}_n^{\mathrm{SU}}(Q)(2\pi)^{n-1}} \exp\Big(-n \sum_{i=1}^{n-1} \xi_i^{(n)} \Big) e^{-nM} \prod_{1 \le i < j \le n-1} (d_{ij}^{(n)})^2\\
&\qquad \times \int \cdots \int_{\Theta_n} \prod_{i=1}^{n-1} \left| e^{\mathrm{i}\theta_i} - e^{-\mathrm{i}(\theta_1 + \cdots + \theta_{n-1})} \right|^2 d\theta_1 \cdots d\theta_{n-1},
\end{aligned}$$

where $M := \max\{Q(\zeta) : \zeta \in \mathbf{T}\}$. Notice

$$\{\theta_1 + \cdots + \theta_{n-1} : (\theta_1, \dots, \theta_{n-1}) \in \Theta_n\} = \Big[\sum_{i=1}^{n-1} a_i^{(n)}, \sum_{i=1}^{n-1} b_i^{(n)} \Big],$$

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and for n large enough,

$$\sum_{i=1}^{n-1} b_i^{(n)} - \sum_{i=1}^{n-1} a_i^{(n)} \geq \frac{n-1}{n}\pi\delta > \frac{3\pi}{n\delta}. \tag{3.5}$$

From (3.4) and (3.5) we can choose an interval $[\alpha,\beta] \subset \left[\sum_{i=1}^{n-1} a_i^{(n)}, \sum_{i=1}^{n-1} b_i^{(n)}\right]$ such that $\beta-\alpha = \pi\delta/n^2$ and

$$[-\beta,-\alpha] \subset \left[b_{k-1}^{(n)} + \frac{\pi\delta}{n^2}, a_k^{(n)} - \frac{\pi\delta}{n^2}\right] \pmod{2\pi}$$

for some $1 \leq k \leq n$. Then there exist subintervals $[\alpha_i,\beta_i] \subset \left[a_i^{(n)}, b_i^{(n)}\right]$, $1 \leq i \leq n-1$, such that

$$\beta_i - \alpha_i = \frac{\pi\delta}{n^2(n-1)}, \quad \sum_{i=1}^{n-1} \alpha_i = \alpha, \quad \sum_{i=1}^{n-1} \beta_i = \beta,$$

and hence

$$\begin{aligned}\int\cdots\int_{\Theta_n} \prod_{i=1}^{n-1} \left| e^{\mathrm{i}\theta_i} - e^{-\mathrm{i}(\theta_1+\cdots+\theta_{n-1})}\right|^2 d\theta_1\cdots d\theta_{n-1} \\ \geq \int_{\alpha_1}^{\beta_1}\cdots\int_{\alpha_{n-1}}^{\beta_{n-1}} \prod_{i=1}^{n-1} \left| e^{\mathrm{i}\theta_i} - e^{-\mathrm{i}(\theta_1+\cdots+\theta_{n-1})}\right|^2 d\theta_1\cdots d\theta_{n-1} \\ \geq \left(\frac{2\delta}{n^2}\right)^{2(n-1)} \left(\frac{\pi\delta}{n^2(n-1)}\right)^{n-1}.\end{aligned}$$

Therefore, for sufficiently large n, we get

$$\begin{aligned}\tilde{\lambda}_n^{\mathrm{SU}}(Q)\left\{\frac{1}{n}(\delta_{\zeta_1}+\cdots+\delta_{\zeta_n}) \in G\right\} \\ \geq \frac{(2\delta^3)^{n-1}}{\widetilde{Z}_n^{\mathrm{SU}}(Q) n^{7(n-1)}} \exp\Big(-n\sum_{i=1}^{n-1}\xi_i^{(n)}\Big) \prod_{1\leq i<j\leq n-1} \left(d_{ij}^{(n)}\right)^2.\end{aligned}$$

Since

$$\begin{aligned}\lim_{n\to\infty} \frac{2}{n^2} \sum_{1\leq i<j\leq n-1} \log d_{ij}^{(n)} &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f(e^{\mathrm{i}s})f(e^{\mathrm{i}t}) \log|e^{\mathrm{i}s}-e^{\mathrm{i}t}|\,ds dt \\ &= \iint_{\mathbf{T}^2} \log|\zeta-\eta|\,d\mu(\zeta)d\mu(\eta)\end{aligned}$$

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as well as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \xi_i^{(n)} = \frac{1}{2\pi} \int_0^{2\pi} Q(e^{\mathrm{i}s}) f(e^{\mathrm{i}s})\, ds = \int_{\mathbf{T}} Q(\zeta)\, d\mu(\zeta),$$

we have

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty} \frac{1}{n^2} \log \tilde{\lambda}_n^{\mathrm{SU}}(Q)\Big\{ \frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}) \in G \Big\} \\ &\geq - \iint_{\mathbf{T}^2} F(\zeta,\eta)\, d\mu(\zeta) d\mu(\eta) - \liminf_{n\to\infty} \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q) \end{aligned}$$

and

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n^2} \log \tilde{\lambda}_n^{\mathrm{SU}}(Q)\Big\{ \frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n}) \in G \Big\} \\ &\qquad\qquad \geq - \iint_{\mathbf{T}^2} F(\zeta,\eta)\, d\mu(\zeta)\, d\mu(\eta) - \limsup_{n\to\infty} \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q). \end{aligned}$$

These imply (iii) and (iv). ■

4 Free LSI for Measures on T

In this section, we will prove a free analogue of LSI for measures on **T**. The idea here is essentially same as Biane's work [3] (and also [12]). Namely, our free analogue arises as the scaling limit in the scale $1/n^2$ of the classical one (2.1) on the special unitary group $\mathrm{SU}(n)$.

Let us begin with some lemmas.

Lemma 4.1 *Let Q be a harmonic function on a neighborhood of the unit disk $\{\zeta \in \mathbf{C} : |\zeta| \leq 1\}$. For each $n \in \mathbf{N}$ and each $U \in \mathrm{SU}(n)$ define $Q(U)$ via the functional calculus and set $\Psi(U) := \mathrm{Tr}_n(Q(U))$. Then the following hold:*

(i) *The function $\Psi(U)$ on $\mathrm{SU}(n)$ is C^∞.*
(ii) $\nabla\Psi(U) = \mathrm{i}\big(Q'(U) - \frac{1}{n}\mathrm{Tr}_n(Q'(U))I_n\big)$.
(iii) *If $Q(e^{\mathrm{i}t}) - \frac{\rho}{2}t^2$ is convex on $\mathbf{R}$ for some constant $\rho > 0$, then* $\mathrm{Hess}(\Psi) \geq \rho I_{n^2-1}$.

Proof Assertions (i) and (iii) were shown in [12, Lemma 1.3]; thus we will prove only (ii). Set $f(t) := Q(e^{\mathrm{i}t})$ for $t \in \mathbf{R}$, and let $Y_k := \mathrm{i}X_k$ with $X_k = X_k^*$, $1 \leq k \leq n^2-1$, be a basis of the Lie algebra $\mathfrak{su}(n) = \{T \in M_n(\mathbf{C}) : T + T^* = 0, \mathrm{Tr}_n(T) = 0\}$ ($\cong \mathbf{R}^{n^2-1}$). For any $U_0 = e^{\mathrm{i}A_0} \in \mathrm{SU}(n)$ with $\mathrm{i}A_0 \in \mathfrak{su}(n)$ and for $x = (x_1, \ldots, x_{n^2-1}) \in \mathbf{R}^{n^2-1}$, we write

$$\Psi\Big(\exp\Big(\mathrm{i}A_0 + \sum_{k=1}^{n^2-1} x_k Y_k\Big)\Big) = \mathrm{Tr}_n\Big(f\Big(A_0 + \sum_{k=1}^{n^2-1} x_k X_k\Big)\Big).$$

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Thanks to [12, Lemma 1.2], we have

$$
\begin{aligned}
\nabla\Psi(U_0) &= \sum_{k=1}^{n^2-1} \operatorname{Tr}_n(f'(A_0)Y_k)Y_k \\
&= \sum_{k=1}^{n^2-1} \operatorname{Tr}_n\Big(\Big(f'(A_0)-\frac{1}{n}\operatorname{Tr}_n(f'(A_0))I_n\Big)Y_k\Big)Y_k \\
&= \sum_{k=1}^{n^2-1} \Big\langle \mathrm{i}\big(f'(A_0)-\frac{1}{n}\operatorname{Tr}_n(f'(A_0))I_n\big), Y_k\Big\rangle_{\operatorname{Tr}_n} Y_k \\
&= \mathrm{i}\big(f'(A_0)-\frac{1}{n}\operatorname{Tr}_n(f'(A_0))I_n\big) \\
&= \mathrm{i}\big(Q'(U_0)-\frac{1}{n}\operatorname{Tr}_n(Q'(U_0))I_n\big),
\end{aligned}
$$

implying (ii). ■

Lemma 4.2 *Assume that $\mu \in \mathcal{M}(\mathbf{T})$ has a continuous density $p = d\mu/d\zeta$ and that $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta-\eta|\,d\mu(\eta)$ is C^1 on $\mathbf{T}$. Then the following hold:*

(i) *$Q'_\mu(\zeta) = (Hp)(\zeta)$ for a.e. $\zeta \in \mathbf{T}$.*

(ii) *$\int_{\mathbf{T}}((Hp)(\zeta))p(\zeta)\,d\zeta = 0$.*

Proof (i) Let f be an arbitrary C^1-function on $\mathbf{T}$. Then we have

$$
\begin{aligned}
&\int_0^{2\pi} \frac{d}{d\theta}Q_\mu(e^{\mathrm{i}\theta})f(e^{\mathrm{i}\theta})\,\frac{d\theta}{2\pi} \\
&\quad= -\int_0^{2\pi} Q_\mu(e^{\mathrm{i}\theta})\frac{d}{d\theta}f(e^{\mathrm{i}\theta})\,\frac{d\theta}{2\pi} \\
&\quad= -\lim_{\varepsilon\searrow 0}\int_{|\theta-t|\ge\varepsilon} 2\log\big|e^{\mathrm{i}\theta}-e^{\mathrm{i}t}\big|\frac{d}{d\theta}f(e^{\mathrm{i}\theta})p(e^{\mathrm{i}t})\,\frac{d\theta\times dt}{(2\pi)^2} \\
&\quad= -\lim_{\varepsilon\searrow 0}\int_0^{2\pi}\Big(\int_{|\theta-t|\ge\varepsilon}\log\big(2(1-\cos(\theta-t))\big)\frac{d}{d\theta}f(e^{\mathrm{i}\theta})\,\frac{d\theta}{2\pi}\Big)p(e^{\mathrm{i}t})\,\frac{dt}{2\pi},
\end{aligned}
$$

where the second equality is due to the fact that $\log\big|e^{\mathrm{i}\theta}-e^{\mathrm{i}t}\big|\frac{d}{d\theta}f(e^{\mathrm{i}\theta})$ is bounded above. Integrating by parts we get

$$
\begin{aligned}
&\int_{|\theta-t|\ge\varepsilon}\log\big(2(1-\cos(\theta-t))\big)\frac{d}{d\theta}f(e^{\mathrm{i}\theta})\,\frac{d\theta}{2\pi} \\
&\quad= -\frac{\log(2(1-\cos\varepsilon))}{2\pi}\big(f(e^{\mathrm{i}(t+\varepsilon)})-f(e^{\mathrm{i}(t-\varepsilon)})\big)-\int_{|\theta-t|\ge\varepsilon}\frac{f(e^{\mathrm{i}\theta})}{\tan\left(\frac{\theta-t}{2}\right)}\,\frac{d\theta}{2\pi},
\end{aligned}
$$

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and hence

$$
\begin{aligned}
&\int_0^{2\pi} \frac{d}{d\theta} Q_\mu(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} \\
&\quad = \lim_{\varepsilon \searrow 0} \Big\{ \frac{\log(2(1-\cos\varepsilon))}{2\pi} \int_0^{2\pi} \big(f(e^{i(t+\varepsilon)}) - f(e^{i(t-\varepsilon)}) \big) p(e^{it}) \frac{dt}{2\pi} \\
&\qquad\qquad + \int_0^{2\pi} \Big(\int_{|\theta - t| \geq \varepsilon} \frac{f(e^{i\theta})}{\tan\left(\frac{\theta - t}{2}\right)} \frac{d\theta}{2\pi} \Big) p(e^{it}) \frac{dt}{2\pi} \Big\} \\
&\quad = \lim_{\varepsilon \searrow 0} \int_0^{2\pi} \Big(\int_{|\theta - t| \geq \varepsilon} \frac{p(e^{it})}{\tan\left(\frac{\theta - t}{2}\right)} \frac{dt}{2\pi} \Big) f(e^{i\theta}) \frac{d\theta}{2\pi} \\
&\quad = \int_0^{2\pi} (Hp)(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi}.
\end{aligned}
$$

In the above, the second equality comes from $\left| f(e^{i(t+\varepsilon)}) - f(e^{i(t-\varepsilon)}) \right| = O(\varepsilon)$ uniformly for $t \in [0, 2\pi)$, and since we have in particular $p \in L^2(\mathbf{T})$, the last one comes from the L^2-convergence of the involved principal value integral to Hp (see [5, 12.8.2 (2)]). Hence, the desired assertion follows since f is arbitrary.

(ii) is seen by taking the limit as $\varepsilon \searrow 0$ of

$$
\int_0^{2\pi} \Big(\int_{|t-\theta| \geq \varepsilon} \frac{p(e^{it})}{\tan\left(\frac{\theta - t}{2}\right)} \frac{dt}{2\pi} \Big) p(e^{i\theta}) \frac{d\theta}{2\pi}
= - \int_0^{2\pi} \Big(\int_{|\theta - t| \geq \varepsilon} \frac{p(e^{i\theta})}{\tan\left(\frac{t - \theta}{2}\right)} \frac{d\theta}{2\pi} \Big) p(e^{it}) \frac{dt}{2\pi},
$$

thanks to the L^2-convergence of the principal value integral as mentioned above. ■

The next theorem is the main result of the paper. One should note that the full power of the large deviation (especially, the almost sure convergence of the empirical eigenvalue distribution) is needed in the proof, while the weak convergence in the mean is enough in the proof of Biane's free LSI for measures on $\mathbf{R}$ in [3].

Theorem 4.3 *Let Q be a real-valued C^1-function on $\mathbf{T}$ such that $Q(e^{it}) - \frac{\rho}{2} t^2$ is convex on $\mathbf{R}$ with a constant $\rho > -1/2$. Then the inequality*

$$
\widetilde{\Sigma}_Q(\mu) \leq \frac{1}{1+2\rho} F_Q(\mu) \tag{4.1}
$$

holds for every $\mu \in \mathcal{M}(\mathbf{T})$.

In the special case where $Q \equiv 0$ and $\rho = 0$, the above (4.1) becomes

$$
-\Sigma(\mu) \leq F(\mu)
$$

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and the equilibrium measure μ_Q is the uniform distribution $d\zeta$.

In particular, the theorem implies that $F_Q(\mu) \geq 0$, *i.e.*,

$$\int_{\mathbf{T}} \big((Hp)(\zeta) - Q'(\zeta)\big)^2 d\mu(\zeta) \geq \Big(\int_{\mathbf{T}} Q'(\zeta)\, d\mu(\zeta)\Big)^2$$

for every $\mu \in \mathcal{M}(\mathbf{T})$ under the above assumption of Q. Also, suppose that the equilibrium measure μ_Q has a continuous density and its support is $\mathbf{T}$; then we have $Q(\zeta) = 2\int_{\mathbf{T}} \log|\zeta - \eta|\, d\mu_Q(\eta)$ for all $\zeta \in \mathbf{T}$ due to [15, Theorem I.3.1] so that Lemma 4.2 gives $F_Q(\mu_Q) = 0$.

Before passing to the proof, we should recall the following facts: the Ricci curvature tensor of $\mathrm{U}(n)$ is known to be degenerate, while that of $\mathrm{SU}(n)$ to be positive constant (see [14], a nice reference for the topic) and a straightforward computation shows that the Ricci curvature tensor of $\mathrm{SU}(n)$ with respect to the Riemannian structure associated with Tr_n is

$$\mathrm{Ric}(\mathrm{SU}(n)) = \frac{n}{2} I_{n^2-1}. \tag{4.2}$$

This is the reason why we have presented Theorem 3.1 with use of $\mathrm{SU}(n)$ instead of $\mathrm{U}(n)$.

Proof of Theorem 4.3 First, let us assume:

(a) Q is harmonic on a neighborhood of the unit disk;
(b) μ has a continuous density $p = d\mu/d\zeta$, and $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta - \eta|\, d\mu(\eta)$ is harmonic on a neighborhood of the unit disk.

For each $n \in \mathbf{N}$ define $n \times n$ special unitary random matrices $\lambda_n^{\mathrm{SU}}(Q)$ and $\lambda_n^{\mathrm{SU}}(Q_\mu)$ as in (3.2), *i.e.*,

$$d\lambda_n^{\mathrm{SU}}(Q)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q)} \exp(-n\,\mathrm{Tr}_n(Q(U)))\, dU,$$

$$d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) := \frac{1}{Z_n^{\mathrm{SU}}(Q_\mu)} \exp(-n\,\mathrm{Tr}_n(Q_\mu(U)))\, dU.$$

Let $\tilde\lambda_n^{\mathrm{SU}}(Q)$ and $\tilde\lambda_n^{\mathrm{SU}}(Q_\mu)$ be their joint eigenvalue distributions on $\mathbf{T}^{n-1}$ (see §3). Also, let $\hat\lambda_n^{\mathrm{SU}}(Q)$ and $\hat\lambda_n^{\mathrm{SU}}(Q_\mu)$ be their mean eigenvalue distributions defined by

$$\hat\lambda_n^{\mathrm{SU}}(Q) := \int \dots \int_{\mathbf{T}^{n-1}} \frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n})\, d\tilde\lambda_n^{\mathrm{SU}}(Q)(\zeta_1, \dots, \zeta_{n-1})$$

with $\zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}$ and similarly for $\hat\lambda_n^{\mathrm{SU}}(Q_\mu)$. According to Theorem 3.1, the empirical eigenvalue distribution of $\lambda_n^{\mathrm{SU}}(Q_\mu)$ satisfies the large deviation principle in the scale $1/n^2$ whose rate functions is $\widetilde\Sigma_{Q_\mu}(\mu)$. Moreover, note [15, Theorem I.3.1] that the equilibrium measure associated with Q_μ (or the minimizer of $\widetilde\Sigma_{Q_\mu}$) is the given μ. This large deviation principle guarantees the following facts (i) and (ii), which will be the key ingredients in our arguments below.

(i) $\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu) \to \mu$ weakly as $n \to \infty$;

(ii) the empirical distribution $\frac{1}{n}(\delta_{\zeta_1} + \cdots + \delta_{\zeta_n})$ weakly converges to μ almost surely as $n \to \infty$ when $(\zeta_1, \ldots, \zeta_{n-1})$ is distributed according to $\tilde{\lambda}_n^{\mathrm{SU}}(Q_\mu)$ and $\zeta_n = (\zeta_1 \cdots \zeta_{n-1})^{-1}$.

Set $\Psi_n(U) := n\,\mathrm{Tr}_n(Q(U))$ for $U \in \mathrm{SU}(n)$. Thanks to (a) above, Lemma 4.1(iii) together with (4.2) verifies the Bakry and Emery criterion

$$\mathrm{Ric}(\mathrm{SU}(n)) + \mathrm{Hess}(\Psi_n) \geq \Big(\frac{n}{2} + n\rho\Big) I_{n^2-1}.$$

Thus, by Theorem 2.1 due to Bakry and Emery we get

$$S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) \leq \frac{1}{2\big(\frac{n}{2} + n\rho\big)} \int_{\mathrm{SU}(n)} \Big\| \nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)} \Big\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu). \tag{4.3}$$

Notice

$$\frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U) = \frac{\widetilde{Z}_n^{\mathrm{SU}}(Q)}{\widetilde{Z}_n^{\mathrm{SU}}(Q_\mu)} \exp\big(-n\,\mathrm{Tr}_n(Q_\mu(U)) + n\,\mathrm{Tr}_n(Q(U))\big), \quad U \in \mathrm{SU}(n), \tag{4.4}$$

where $\widetilde{Z}_n^{\mathrm{SU}}(Q)$ and $\widetilde{Z}_n^{\mathrm{SU}}(Q_\mu)$ are the normalization constants of the joint eigenvalue distributions (see §3). Hence, we have

$$\begin{aligned}
&\frac{1}{n^2} S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) \\
&\quad = \frac{1}{n^2} \int_{\mathrm{SU}(n)} \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q) - \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q_\mu) \\
&\qquad - \int_{\mathrm{SU}(n)} \frac{1}{n} \mathrm{Tr}_n(Q_\mu(U))\, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) + \int_{\mathrm{SU}(n)} \frac{1}{n} \mathrm{Tr}_n(Q(U))\, d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\
&\quad = \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q) - \frac{1}{n^2} \log \widetilde{Z}_n^{\mathrm{SU}}(Q_\mu) \\
&\qquad - \int_{\mathbf{T}} Q_\mu(\zeta)\, d\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta) + \int_{\mathbf{T}} Q(\zeta)\, d\hat{\lambda}_n^{\mathrm{SU}}(Q_\mu)(\zeta),
\end{aligned}$$

and therefore, thanks to (b) and (i) above,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^2} S\big(\lambda_n^{\mathrm{SU}}(Q_\mu), \lambda_n^{\mathrm{SU}}(Q)\big) &= B(Q) - B(Q_\mu) - \int_{\mathbf{T}} Q_\mu(\zeta)\, d\mu(\zeta) + \int_{\mathbf{T}} Q(\zeta)\, d\mu(\zeta) \\
&= \widetilde{\Sigma}_Q(\mu),
\end{aligned} \tag{4.5}$$

where the last equality comes from the fact that μ is the minimizer with $\widetilde{\Sigma}_{Q_\mu}(\mu) = 0$, *i.e.*,

$$\int_{\mathbf{T}} Q_\mu(\zeta)\, d\mu(\zeta) + B(Q_\mu) = \Sigma(\mu).$$

Therefore, the scaling limit in the scale $1/n^2$ of the left-hand side of (4.3) becomes the relative free entropy $\widetilde{\Sigma}_Q(\mu)$. We will seek the scaling limit in the scale $1/n^2$ of the right-hand side of (4.3). By (4.4) and Lemma 4.1(ii), we have

$$\begin{aligned}\nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U) &= -n\nabla\big(\mathrm{Tr}_n(Q_\mu(U)) - \mathrm{Tr}_n(Q(U))\big) \\ &= -\mathrm{i}\big\{ n\big(Q_\mu'(U) - Q'(U)\big) - \big(\mathrm{Tr}_n(Q_\mu'(U) - Q'(U))\big) I_n\big\}\end{aligned}$$

so that

$$\begin{aligned}&\Big\| \nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\Big\|_{HS}^2 \\ &\qquad = n^2\,\mathrm{Tr}_n\Big(\big(Q_\mu'(U) - Q'(U)\big)^2\Big) - n\Big(\mathrm{Tr}_n\big(Q_\mu'(U) - Q'(U)\big)\Big)^2.\end{aligned}$$

Thus, we get

$$\begin{aligned}&\frac{1}{n^2}\cdot\frac{1}{2\big(\frac{n}{2}+n\rho\big)}\int_{\mathrm{SU}(n)}\Big\| \nabla \log \frac{d\lambda_n^{\mathrm{SU}}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\Big\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\ &\quad = \frac{1}{1+2\rho}\Big\{\int_{\mathrm{SU}(n)} \frac{1}{n}\mathrm{Tr}_n\Big(\big(Q_\mu'(U) - Q'(U)\big)^2\Big)\,d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\ &\qquad\qquad - \int_{\mathrm{SU}(n)} \frac{1}{n^2}\Big(\mathrm{Tr}_n\big(Q_\mu'(U) - Q'(U)\big)\Big)^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U)\Big\}.\end{aligned}$$

The above-mentioned fact (i) implies that

$$\begin{aligned}&\int_{\mathrm{SU}(n)} \frac{1}{n}\mathrm{Tr}_n\Big(\big(Q_\mu'(U) - Q'(U)\big)^2\Big)\,d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\ &\qquad = \int_{\mathbf{T}} \big(Q_\mu'(\zeta) - Q'(\zeta)\big)^2\,d\hat\lambda_n^{\mathrm{SU}}(Q_\mu)(\zeta) \to \int_{\mathbf{T}} \big(Q_\mu'(\zeta) - Q'(\zeta)\big)^2\,d\mu(\zeta)\end{aligned}$$

as $n \to \infty$, while the above fact (ii) implies that, with $\zeta_n := (\zeta_1 \cdots \zeta_{n-1})^{-1}$,

$$\begin{aligned}&\int_{\mathrm{SU}(n)} \frac{1}{n^2}\Big(\mathrm{Tr}_n\big(Q_\mu'(U) - Q'(U)\big)\Big)^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U) \\ &\qquad = \int_{\mathbf{T}^{n-1}} \Big(\frac{1}{n}\sum_{i=1}^n \big(Q_\mu'(\zeta_i) - Q'(\zeta_i)\big)\Big)^2 d\tilde\lambda_n^{\mathrm{SU}}(Q_\mu)(\zeta_1,\dots,\zeta_{n-1}) \\ &\qquad\qquad \to \Big(\int_{\mathbf{T}} \Big(Q_\mu'(\zeta) - Q'(\zeta)\Big)\,d\mu(\zeta)\Big)^2 \quad \text{as } n \to \infty.\end{aligned}$$

Thanks to the assumption (b), Lemma 4.2 implies that

$$\begin{aligned}\Big(\int_{\mathbf{T}}\big(Q'_\mu(\zeta)-Q'(\zeta)\big)\,d\mu(\zeta)\Big)^2 &= \Big(\int_{\mathbf{T}}((Hp)(\zeta))p(\zeta)\,d\zeta-\int_{\mathbf{T}}Q'(\zeta)\,d\mu(\zeta)\Big)^2\\ &= \Big(\int_{\mathbf{T}}Q'(\zeta)\,d\mu(\zeta)\Big)^2\end{aligned}$$

so that we get

(4.6)
$$\lim_{n\to\infty}\frac{1}{n^2}\cdot\frac{1}{2\left(\frac{n}{2}+n\rho\right)}\int_{\mathrm{SU}(n)}\Big\|\nabla\log\frac{d\lambda_n^{SU}(Q_\mu)}{d\lambda_n^{\mathrm{SU}}(Q)}(U)\Big\|_{HS}^2 d\lambda_n^{\mathrm{SU}}(Q_\mu)(U)=\frac{1}{1+2\rho}F_Q(\mu).$$

By (4.3), (4.5) and (4.6) we have shown the desired inequality (4.1) under the assumptions (a) and (b).

Next, let us deal with a general Q as stated in the theorem. Let $\mu\in\mathcal{M}(\mathbf{T})$ with a density $p=d\mu/d\zeta\in L^3(\mathbf{T})$. For each $0<r<1$, we consider the Poisson integrals Q_r and p_r of Q and p, respectively; that is,

$$\begin{aligned}Q_r(e^{\mathrm{i}\theta})&:=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-t)Q(e^{\mathrm{i}t})\,dt,\\ p_r(e^{\mathrm{i}\theta})&:=\frac{1}{2\pi}\int_0^{2\pi}P_r(\theta-t)p(e^{\mathrm{i}t})\,dt\end{aligned}$$

with the Poisson kernel $P_r(\theta):=(1-r^2)/(1-2r\cos\theta+r^2)$. Define $\mu_r\in\mathcal{M}(\mathbf{T})$ by $d\mu_r(\zeta):=p_r(\zeta)\,d\zeta$. In the same way as in [12, Theorem 2.7], it is easy to see that

$$\widetilde{\Sigma}_{Q_r}(\mu_r)\le\frac{1}{1+2\rho}F_{Q_r}(\mu_r) \tag{4.7}$$

by what we have already shown, and also that

$$\lim_{r\nearrow 1}\widetilde{\Sigma}_{Q_r}(\mu_r)=\widetilde{\Sigma}_Q(\mu).$$

Notice that $\|p_r-p\|_{L^3}\to 0$ and hence $\|Hp_r-Hp\|_{L^3}\to 0$ as $r\nearrow 1$. Since Q is a C^1-function, Q'_r becomes the Poisson integral of Q' so that $\|Q'_r-Q'\|_\infty\to 0$ as $r\nearrow 1$ as well. These imply that

$$\begin{aligned}\lim_{r\nearrow 1}F_{Q_r}(\mu_r)&=\lim_{r\nearrow 1}\Big\{\int_{\mathbf{T}}\big((Hp_r)(\zeta)-Q'_r(\zeta)\big)^2d\mu_r(\zeta)-\Big(\int_{\mathbf{T}}Q'_r(\zeta)\,d\mu_r(\zeta)\Big)^2\Big\}\\ &=\int_{\mathbf{T}}\big((Hp)(\zeta)-Q'(\zeta)\big)^2d\mu(\zeta)-\Big(\int_{\mathbf{T}}Q'(\zeta)\,d\mu(\zeta)\Big)^2=F_Q(\mu).\end{aligned}$$

Hence, the desired inequality (4.1) follows by taking the limit of (4.7). ■

5 Supplementary Remarks

5.1 Scaling Limit Formulas for Relative Free Entropy and Relative Free Fisher Information

It seems worthwhile to state in a separate proposition some scaling limit formulas given in the proof of the main theorem. In fact, the formula for relative free entropy was essentially obtained in [7]. In the special unitary case, the proof of (4.5) gives Proposition 5.1(1), while that of (4.6) gives Proposition 5.1(2), because the derivative formula in Lemma 4.1(ii) is still valid for any $U \in \mathrm{SU}(n)$ when Q is a real-valued C^1-function on $\mathbf{T}$. The unitary cases are similar.

Proposition 5.1

(1) *Let Q be a real-valued continuous function on* $\mathbf{T}$, *and* $\mu \in \mathcal{M}(\mathbf{T})$. *If* $Q_\mu(\zeta) := 2\int_{\mathbf{T}} \log|\zeta - \eta|\,d\mu(\eta)$ *is finite and continuous on* $\mathbf{T}$, *then*

$$\widetilde{\Sigma}_Q(\mu) = \lim_{n\to\infty} \frac{1}{n^2} S\big(\lambda_n^{SU}(Q_\mu), \lambda_n^{SU}(Q)\big) = \lim_{n\to\infty} \frac{1}{n^2} S\big(\lambda_n^{\mathrm{U}}(Q_\mu), \lambda_n^{\mathrm{U}}(Q)\big).$$

(2) *In addition, if μ has a continuous density $d\mu/d\zeta$ and both Q and Q_μ are C^1-functions on* $\mathbf{T}$, *then*

$$\begin{aligned} F_Q(\mu) &= \lim_{n\to\infty} \frac{1}{n^3} \int_{SU(n)} \Big\| \nabla \log \frac{d\lambda_n^{SU}(Q_\mu)}{d\lambda_n^{SU}(Q)}(U) \Big\|_{HS}^2 d\lambda_n^{SU}(Q_\mu)(U) \\ &= \lim_{n\to\infty} \frac{1}{n^3} \int_{\mathrm{U}(n)} \Big\| \nabla \log \frac{d\lambda_n^{\mathrm{U}}(Q_\mu)}{d\lambda_n^{\mathrm{U}}(Q)}(U) \Big\|_{HS}^2 d\lambda_n^{\mathrm{U}}(Q_\mu)(U). \end{aligned}$$

Similar limit formulas are given also in the real line case. The details are left to the reader (see [12, (2.7)] for instance).

5.2 The Optimality Question of Free LSI's

We examine, by computing particular examples of measures, whether or not Biane's free LSI for measures on $\mathbf{R}$ as well as our free LSI for measures on $\mathbf{T}$ are optimal. First, consider the real line case. Let $Q(x) := \rho x^2/2$ on $\mathbf{R}$ with $\rho > 0$. The equilibrium measure associated with Q is known to be the $(0, 1/\rho)$-semicircular measure $\gamma_{0,2/\sqrt{\rho}}$, where we write $\gamma_{0,r}$ for the semicircular measure with mean 0 and variance $r^2/4$:

$$d\gamma_{0,r}(x) := \frac{2}{\pi r^2}\sqrt{r^2 - x^2}\,\chi_{[-r,r]}(x)dx.$$

For each $\alpha > 0$ we have

$$\widetilde{\Sigma}_Q(\gamma_{0,2/\sqrt{\alpha}}) = \frac{\rho}{2\alpha} + \frac{1}{2}\log\alpha - \frac{1}{2}\log\rho - \frac{1}{2},$$

$$\Phi_Q(\gamma_{0,2/\sqrt{\alpha}}) = \frac{(\alpha-\rho)^2}{\alpha}.$$

Therefore, we get

$$\lim_{\alpha\to 0}\frac{\widetilde{\Sigma}_Q(\gamma_{0,2/\sqrt{\alpha}})}{\Phi_Q(\gamma_{0,2/\sqrt{\alpha}})}=\lim_{\alpha\to 0}\frac{\rho+\alpha\log\alpha-\alpha(\log\rho+1)}{2(\alpha-\rho)^2}=\frac{1}{2\rho},$$

which shows the following:

Proposition 5.2 *The bound $1/2\rho$ in Biane's free LSI for measures on* $\mathbf{R}$ ([3] *or* (1.3)) *is best possible.*

Next, consider the unit circle case. For each $2\le\lambda\le\infty$, the equilibrium measure associated with $Q(\zeta) := -2\operatorname{Re}\zeta/\lambda$ on $\mathbf{T}$ is known to be

$$\nu_\lambda := \Bigl(1+\frac{2}{\lambda}\cos\theta\Bigr)\frac{d\theta}{2\pi}\qquad\Bigl(\text{with } \nu_\infty := \frac{d\theta}{2\pi}\Bigr),$$

and its free entropy to be $\Sigma(\nu_\lambda) = -1/\lambda^2$ (see [10, 5.3.10]). When $4<\lambda\le\infty$, since $Q\left(e^{it}\right)+\frac{1}{\lambda}t^2 = \frac{2}{\lambda}\left(\frac{t^2}{2}-\cos t\right)$ is convex on $\mathbf{R}$, the free LSI (4.1) holds with $1/(1+2\rho)=\lambda/(\lambda-4)$. For example, for $2\le\alpha\le\infty$ we compute

$$\widetilde{\Sigma}(\nu_\alpha)=\Bigl(\frac{1}{\alpha}-\frac{1}{\lambda}\Bigr)^2,\quad F_Q(\nu_\alpha)=2\Bigl(\frac{1}{\alpha}-\frac{1}{\lambda}\Bigr)^2,$$

but the optimality of the bound $1/(1+2\rho)$ in (4.1) is currently unknown to us. This situation is the same as in the free transportation cost inequality for measures on $\mathbf{T}$ (see [12, §3.2]).

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