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A NOTE ON THE UNION-CLOSED SETS CONJECTURE

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Abstract

It has been conjectured that for any union-closed set $\mathscr{A}$ there exists some element which is contained in at least half the sets in $\mathscr{A}$. It is shown that this conjecture is true if the number of sets in $\mathscr{A}$ is less than 25. Several conditions on a counterexample are also obtained,

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1. Introduction

A union-closed set $\mathscr{A}$ is defined as a non-empty finite collection of distinct, non-empty finite sets, closed under union (that is, if $S \in \mathscr{A}$ and $T \in \mathscr{A}$ then $S \cup T \in \mathscr{A}$).

The following conjecture is rephrased from [1].

CONJECTURE. *Let $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$ be a union-closed set. Then there exists an element which belongs to at least $\lceil \frac{n}{2} \rceil$ of the sets in $\mathscr{A}$, where*

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

In this paper we generalize results due to Sarvate and Renaud and to Norton and Sarvate. In particular we establish some inequalities involving the A_j's and n which must hold for any counterexample and prove that the conjecture is valid when $n \le 24$.

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2. Preliminaries and notations

We denote the union-closed set $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ by $\mathscr{A}(n)$. Assume, for convenience, that $|A_i| = w_i$, $w_1 \le w_2 \le \cdots \le w_n = q$, and $A_n = I_q = \{1, 2, \ldots, q\}$.

The support size of an $\mathscr{A}(n)$ is defined to be the number $q = w_n$. Let $\mathbb{A}(n, q)$ denote the set of possible $\mathscr{A}(n)$ with support size q. Theorems 2 and 3 in [4] show that the conjecture holds when $\mathscr{A}(n) \in \mathbb{A}(n, q)$, $q \le 6$.

Let $\mathscr{A}(n) \in \mathbb{A}(n, q)$ and let $x \in I_q$. Define $\mathscr{A}_x(n)$ to be the set of A_i in $\mathscr{A}(n)$ which contain x and let $|\mathscr{A}_x(n)| = d(x)$. Assume $\mathscr{C}_x(n)$ to be the set of A_j in $\mathscr{A}(n)$ not containing x and let $C_x = \cup\{A_j : A_j \in \mathscr{C}_x(n)\}$. Set $\mathscr{A}_x^* = \{A_i - \{x\} : A_i \in \mathscr{A}(n)\}$; it is clear that $\mathscr{A}_x^*$ is a union-closed set with support size $q - 1$.

3. Restrictions on the set sizes

Theorem 2 in [5] shows that the conjecture holds whenever $w_1 + w_2 \ge q$. This can be improved by the following result:

THEOREM 1. *The conjecture holds whenever*

(i) $$w_3 + w_{\frac{n+3}{2}} \ge q, \qquad \text{if } n \text{ is odd};$$

(ii) $$w_4 + w_{\frac{n+4}{2}} \ge q, \qquad \text{if } n \text{ is even}.$$

PROOF. (i) Suppose $d(x) \le (n-1)/2$ for all $x \in I_q$. This implies that

$$w_1 + w_2 + \cdots + w_{n-1} + q \le \frac{n-1}{2} q$$

and so

$$w_1 + w_2 + (w_3 + w_{\frac{n+3}{2}}) + (w_4 + w_{\frac{n+5}{2}}) + \cdots + (w_{\frac{n+1}{2}} + w_{n-1}) \le$$
$$\le w_1 + w_2 + \frac{n-3}{2}(w_3 + w_{\frac{n+3}{2}}) \le \frac{n-3}{2} q,$$

hence $w_3 + w_{\frac{n+3}{2}} < q$, a contradiction.

(ii) Theorem 1 in [4] shows the validity of the conjecture for odd n leads to its validity for $n + 1$, then the proof is similar to the previous case.

4. Smallest counterexample

Let n_0 be the minimum value of n taken over all the counterexamples to the union-closed conjecture. By Theorem 1 of [4], assume $n_0 = 2t + 1$. Let $\mathscr{A}(n_0) \in \mathbb{A}(n_0, q_0)$

be a counterexample to the conjecture with minimal support size q_0. We have the following:

THEOREM 2.

(i) $$|\mathscr{A}^*_x| < |\mathscr{A}(n_0)| \qquad \textit{for each } x \in I_{q_0}.$$

(ii) $$\mathscr{A}_x(n_0) \neq \mathscr{A}_y(n_0) \qquad \textit{for each } x, y \in I_{q_0}, x \neq y.$$

PROOF. (i) Suppose $\mathscr{A}^*_{x_0} \in \mathscr{A}_{x_0}(n_0, q_0 - 1)$ for some $x_0 \in I_{q_0}$. By the minimality of q_0 there exists an element z in $(t+1)$ sets of $\mathscr{A}^*_{x_0}$, hence in $(t+1)$ sets of $\mathscr{A}(n_0)$, a contradiction.

(ii) It is enough to consider that $\mathscr{A}_x(n_0) = \mathscr{A}_y(n_0)$ implies $|\mathscr{A}^*_x| = |\mathscr{A}^*_y| = |\mathscr{A}(n_0)|$.

THEOREM 3. *For any* $x \in I_{q_0}$ *there exists* $A_i \in \mathscr{A}(n_0)$ *such that* $A_i \in \mathscr{A}_x(n_0)$ *and* $A_i - \{x\} \in \mathscr{C}_x(n_0)$.

PROOF. Suppose, on the contrary, there is $x_0 \in I_{q_0}$ such that $A_i - \{x_0\} \notin \mathscr{A}(n_0)$ for every A_i containing x_0 in $\mathscr{A}(n_0)$. This implies $|\mathscr{A}^*_{x_0}| = |\mathscr{A}(n_0)|$ contradicting (i) of Theorem 2.

COROLLARY 1. *For any* $x \in I_{q_0}$, $C_x \cup \{x\} \in \mathscr{A}(n_0)$.

PROOF. Let $A_i \in \mathscr{A}_x(n_0)$ such that $A_i - \{x\} \in \mathscr{C}_x(n_0)$. Then $A_i - \{x\} \subseteq C_x$ and $C_x \cup A_i = C_x \cup \{x\} \in \mathscr{A}(n_0)$.

THEOREM 4. *If* $x, y \in I_{q_0}$, *then* $d(x) \leq d(y)$ *implies* $y \in C_x$.

PROOF. Suppose, on the contrary, $y \notin C_x$. For every $A_i \in \mathscr{A}_y(n_0)$ it follows that $x \in A_i$ (otherwise $y \in A_i \subseteq C_x$), hence $d(x) = d(y)$ and then $\mathscr{A}_x(n_0) = \mathscr{A}_y(n_0)$, contradicting (ii) of Theorem 2.

COROLLARY 2. *If* $d(x) = \mathrm{Min}\{d(y) : y \in I_{q_0}\}$ *then* $C_x = I_{q_0} - \{x\}$.

PROOF. Immediate from the theorem.

Let $x, y \in I_{q_0}$. By Theorem 4, $x \in C_y$ or $y \in C_x$ and so we have the following.

COROLLARY 3. $C_x \neq C_y$, *for every* $x, y \in I_{q_0}$.

PROOF. Assume $d(x) \leq d(y)$, then $y \in C_x$ and $y \notin C_y$.

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THEOREM 5. $n_0 \geq 2q_0 + 1$.

PROOF. In [2], Theorem 2.1, it is proved that, in any counterexample $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$ to the conjecture with n minimal, there are at least three distinct elements, each of which appears in exactly $(n-1)/2$ of the A_j's. Assume $d(q_0) = (n_0 - 1)/2$. By Theorem 3 $q_0 \in C_i$, for every $i = 1, 2, \dots, (q_0 - 1)$, and $q_0 \in I_{q_0} = A_{n_0}$. It follows that $q_0 \leq (n_0 - 1)/2$. This completes the proof.

THEOREM 6. *There are at least three distinct elements* $x_1, x_2, x_3 \in I_{q_0}$ *such that* $C_{x_i} = I_{q_0} - \{x_i\}$, $i = 1, 2, 3$.

PROOF. Let $d(x_1) = \min\{d(x) : x \in I_{q_0}\}$. Corollary 2 implies that $C_{x_1} = I_{q_0} - \{x_1\}$. It is easy to see that there exists an $A_i \in \mathscr{A}_{x_1}(n_0)$ such that $A_i \neq I_{q_0}$. Let $A_i = I_{q_0} - \{y_1, y_2, \dots, y_r\}$, then $x_1 \in A_i \subseteq C_{y_1} \subseteq I_{q_0} - \{y_1\}$. If $C_{y_1} = I_{q_0} - \{y_1\}$ we put $x_2 = y_1$ otherwise $C_{y_1} = I_{q_0} - \{y_1, z_1, z_2, \dots, z_s\}$ and so $x_1 \in A_i \subseteq C_{y_1} \cup \{y_1\} \subseteq C_{z_1}$. Obviously, continuing this process, we find an $x_2 \in I_{q_0}$ ($x_2 \neq x_1$) such that $C_{x_2} = I_{q_0} - \{x_2\}$. Let A'_{x_i} be a set of minimal cardinality containing x_i ($i = 1, 2$). Certainly $A'_{x_1} \cup A'_{x_2} \neq I_{q_0}$, (otherwise, since $\mathscr{B} = \{A_1, A_2, \dots, A_{n_0}\} - \{A'_{x_1}, A'_{x_2}\}$ is union-closed, there would exist an element z in at least t sets of $\mathscr{B}(n_0 - 2)$ and hence in at least $(t+1)$ sets of $\mathscr{A}(n_0)$). Arguing as above we can easily complete the proof.

Let $d(1) = \min\{d(x) : x \in I_{q_0}\}$ and let $\mathscr{B} = \mathscr{A}(n_0) - \mathscr{A}_1(n_0)$. Obviously $\mathscr{B}$ is a union-closed set. For each $z \in I_{q_0}$, put $|\{B \in \mathscr{B} : z \in B\}| = d^*(z)$. Let $x_1 = 1, x_2, x_3$ be as in Theorem 6. We have the following.

THEOREM 7. $d(1) \geq 5$.

PROOF. Obviously $d(1) \geq 3$. Suppose $d(1) = 4$. By Theorem 6, we have $\mathscr{A}_1(n_0) = \{I_{q_0}, I_{q_0} - \{x_2\}, I_{q_0} - \{x_3\}, B_1\}$ and so there exists a $z \in I_{q_0} - \{1\}$ with $d^*(z) \geq (n_0 - 3)/2$, hence $d(z) = (n_0 + 1)/2$, a contradiction.

THEOREM 8. $C_{x_4} \supseteq I_{q_0} - \{x_3, x_4\}$, *for some* $x_4 \in I_{q_0}$, $x_4 \notin \{x_1, x_2, x_3\}$.

PROOF. **Case (1)** Either $A_{x_1} \cup A_{x_2} \subset I_{q_0} - \{x_3\}$ or $A_{x_1} \cup A_{x_3} \subset I_{q_0} - \{x_2\}$ for some $A_{x_i} \in \mathscr{A}_{x_i}(n_0)$, $i = 1, 2, 3$. Without loss of generality, we may suppose that $A_{x_1} \cup A_{x_2} \subset I_{q_0} - \{x_3\}$. Proceeding as in Theorem 6, we get that $A_{x_1} \cup A_{x_2} \subseteq C_{x_4}$ and $C_{x_4} \supseteq I_{q_0} - \{x_3, x_4\}$.

Case (2) Note that $A_{x_1} \cup A_{x_2} \supseteq I_{q_0} - \{x_3\}$ and $A_{x_1} \cup A_{x_3} \supseteq I_{q_0} - \{x_2\}$ for all $A_{x_i} \in \mathscr{A}_{x_i}(n_0)$, $i = 1, 2, 3$. We prove that Case (2) is not possible. Let A'_{x_i} be a set of minimal cardinality containing x_i, $i = 1, 2, 3$. For $j = 2, 3$, let z_j be an element of I_{q_0} which belongs to at least $(n_0 - 1)/2$ of the sets in $\mathscr{A}(n_0) - \{A'_{x_1}, A'_{x_j}\}$. It is easy

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to see that $z_2 = x_3$ and $z_3 = x_2$ and so $d(x_2) = d(x_3) = (n_0 - 1)/2$. By Theorem 7 there exists $x_4 \notin \{x_1, x_2, x_3\}$ such that $x_4 \notin A'_{x_1}$. Since $A'_{x_1} \cup A_{x_2} \supseteq I_{q_0} - \{x_3\}$ for each $A_{x_2} \in \mathscr{A}_{x_2}(n_0)$, it follows that $\mathscr{A}_{x_2}(n_0) \subseteq \mathscr{A}_{x_4}(n_0)$, hence $\mathscr{A}_{x_2}(n_0) = \mathscr{A}_{x_4}(n_0)$ contradicting Theorem 2.

Let $I_{q_0} = \{1, 2, \ldots, q_0\}$ with $d(1) \le d(2) \le \cdots \le d(q_0)$. By Theorem 4 and Corollary 3, it follows that $|C_s| \in \{w_1, w_2, \ldots, w_{n_0}\}$, $|C_s| \ge q_0 - s$ and $w_{n_0-s} \ge q_0 - s$, for each $s \in I_{q_0}$. Theorems 6 and 8 say that $w_{n_0-4} \ge q_0 - 2$, $w_{n_0-3} = w_{n_0-2} = w_{n_0-1} = q_0 - 1$ and so, by a similar argument to that used in Theorem 1, we can prove the following.

COROLLARY 4. $w_{10} + w_{\frac{n_0+3}{2}} < q_0$.

PROOF. Note that

$$w_1 + w_2 + \cdots + w_9 + \cdots + w_{n_0-8} + (q_0-7) + (q_0-6) + (q_0-5) + \\ + (q_0-2) + 3(q_0-1) + q_0 \le \frac{n_0-1}{2} q_0.$$

Then

$$(w_1 + \cdots + w_9) + \frac{n_0-17}{2}(w_{10} + w_{\frac{n_0+3}{2}}) - 23 \le \frac{n_0-17}{2} q_0,$$

hence $w_{10} + w_{(n_0+3)/2} < q_0$.

THEOREM 9. $d(1) \ge 9$.

PROOF. By Theorem 7, we need to consider the following cases.

Case $d(1) = 5$. Let $\mathscr{A}_1(n_0) = \{I_{q_0}, I_{q_0} - \{x_2\}, I_{q_0} - \{x_3\}, B_1, B_2\}$, with $I_{q_0} - \{x_3, x_4\} \subseteq B_1 \subseteq I_{q_0} - \{x_4\}$, and let $z \in I_{q_0} - \{1\}$ such that $d^*(z) = (n_0 - 5)/2$. Necessarily, $B_1 = I_{q_0} - \{x_3, x_4\}$, $z = x_3$, $x_3 \notin B_2$ and $d(x_3) = (n_0 - 1)/2$. Since $x_4 \in (I_{q_0} - \{x_3, x_4\}) \cup A_{x_3}$, for each $A_{x_3} \in \mathscr{A}_{x_3}(n_0)$, it follows that $\mathscr{A}_{x_3}(n_0) \subseteq \mathscr{A}_{x_4}(n_0)$, hence $\mathscr{A}_{x_3}(n_0) = \mathscr{A}_{x_4}(n_0)$, contradicting Theorem 2.

Case $d(1) = 6$. Similar to the previous case.

Case $d(1) = 7$. Let $\mathscr{A}_1(n_0) = \{I_{q_0}, I_{q_0} - \{x_2\}, I_{q_0} - \{x_3\}, B_1, B_2, B_3, B_4\}$ with $I_{q_0} - \{x_3, x_4\} \subseteq B_1 \subseteq I_{q_0} - \{x_4\}$ and let $z \in I_{q_0} - \{1\}$ such that $d^*(z) = (n_0 - 7)/2$. Necessarily $z \in \{x_2, x_3, x_4\}$.

Suppose $B_1 = I_{q_0} - \{x_4\}$, then $z \notin B_2 \cup B_3 \cup B_4$. We can assume $z = x_2$. If there exists B_i $(i = 2, 3, 4)$ such that $y \notin B_i \cup \{x_2, x_3, x_4\}$, then $B_i \cup A_{x_2} \supseteq I_{q_0} - \{x_3\}$ or $B_i \cup A_{x_2} \supseteq I_{q_0} - \{x_4\}$ and so $y \in A_{x_2}$, hence $\mathscr{A}_y(n_0) = \mathscr{A}_{x_2}(n_0)$, a contradiction.

If $B_i \supseteq I_{q_0} - \{x_2, x_3, x_4\}$, for each $i = 2, 3, 4$, then

$$\mathscr{A}_1(n_0) = \big\{I_{q_0}, I_{q_0} - \{x_2\}, I_{q_0} - \{x_3\}, I_{q_0} - \{x_4\}, \\ I_{q_0} - \{x_2, x_3\}, I_{q_0} - \{x_2, x_4\}, I_{q_0} - \{x_2, x_3, x_4\}\big\}.$$

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Let A'_{x_2} be a set of minimal cardinality containing x_2. Obviously $\mathscr{B} = \mathscr{A}(n_0) - \{\mathscr{A}_1(n_0) \cup \{A'_{x_2}\}$ is a union-closed set and so there exists an r in $(n_0-7)/2$ sets of $\mathscr{B}$, a contradiction.

Suppose $B_1 = I_{q_0} - \{x_3, x_4\}$ and, obviously, $I_{q_0} - \{x_4\} \notin \mathscr{A}_1(n_0)$.

Let $z = x_2$. Necessarily $x_2 \notin B_2 \cup B_3 \cup B_4$, $d(x_2) = (n_0-1)/2$ and there exists a B_i $(i = 2, 3, 4)$ such that $y \notin B_i \cup \{x_2, x_3, x_4\}$, otherwise

$$\mathscr{A}_1(n_0)=\{I_{q_0}, I_{q_0}-\{x_2\}, I_{q_0}-\{x_3\}, I_{q_0}-\{x_3,x_4\}, I_{q_0}-\{x_2,x_3\}, I_{q_0}-\{x_2,x_4\}, I_{q_0}-\{x_2,x_3,x_4\}\}.$$

and so $(I_{q_0} - \{x_2, x_4\}) \cup (I_{q_0} - \{x_3, x_4\}) = I_{q_0} - \{x_4\} \notin \mathscr{A}_1(n_0)$. We again obtain $\mathscr{A}_y(n_0) = \mathscr{A}_{x_2}(n_0)$, which is a contradiction.

Let $z = x_4$. This is similar to the case $z = x_2$.

Let $z = x_3$. Necessarily $d(x_3) \geq (n_0-3)/2$. Since $I_{q_0} - \{x_3, x_4\} \cup A_3 = I_{q_0}$, it follows that $\mathscr{A}_{x_3} \subset \mathscr{A}_{x_4}$. If $d(x_3) = (n_0-1)/2$ then $\mathscr{A}_{x_3}(n_0) = \mathscr{A}_{x_4}(n_0)$, a contradiction. If $d(x_3) = (n_0-3)/2$ then $d(x_4) = (n_0-1)/2$ and $I_{q_0} - \{x_3\}$ is the only set of $\mathscr{A}(n_0)$ containing x_4 but not x_3 and so $|\mathscr{A}^*_{x_3}(n_0)| = n_0 - 1$. Let $r \in I_{q_0}$ such that r is contained in $(n_0-1)/2$ sets of $\mathscr{A}^*_{x_3}(n_0)$. Obviously $r \in I_{q_0} - \{x_3\}$, hence $d(r) = (n_0+1)/2$, a contradiction.

Case $d(1) = 8$. Similar to the previous case.

Let $d_r = |\{x \in I_{q_0} : d(x) = r\}|$. Obviously $\sum d_r = q_0$. For each element x_i counted in d_9 we have C_{x_i} containing $I_q - \{x_i\}$ by Theorem 4, hence $|C_{x_i}| = q_0 - 1$. For each x_i contained in d_{10} we have C_{x_i} containing all elements x_j such that $d(x_j) \geq 10$ except x_i itself, hence $|C_{x_i}| \geq q_0 - d_9 - 1$. Counting in this way, Theorem 2 of [4] and Theorem 9 lead to this inequality:

$$\begin{aligned} q_0 + d_9(q_0-1) + d_{10}(q_0-1-d_9) + d_{11}(q_0-1-d_9-d_{10}) + \cdots + \\ + d_{\frac{n_0-1}{2}}(q_0-1-d_9-d_{10}-\cdots-d_{\frac{n_0-3}{2}}) + 3(n_0-q_0-1) \leq \\ \leq 9d_9 + 10d_{10} + \cdots + \frac{n_0-1}{2} d_{\frac{n_0-1}{2}}, \end{aligned}$$

and then

$$(*) \qquad q_0^2 - 3q_0 + 3(n_0-1) - \sum d_i d_j \leq 9d_9 + 10d_{10} + \cdots + \frac{n_0-1}{2} d_{\frac{n_0-1}{2}}.$$

THEOREM 10. $n_0 \geq 25$.

PROOF. Theorem 4 in [5] shows that $n_0 \geq 19$.

For $n_0 = 19$, (*) leads to $q_0^2 - 12q_0 + 54 \leq 0$ which never holds.

For $n_0 = 21$, (*) gives $q_0^2 - (13 + d_9)q_0 + (d_9^2 + d_9 + 60) \leq 0$ which never holds.

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For $n_0 = 23$, (*) gives:

$$q_0^2 - (14 + d_9 + d_{10})q_0 + (d_9^2 + d_{10}^2 + d_9 d_{10} + 2d_9 + d_{10} + 66) \le 0$$

which never holds.

This completes the proof of the theorem.

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