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MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND PRINCIPAL BLOCKS

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1.

Let A be a matrix over a field Φ partitioned as follows

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix},$$

where A_{11} is $n\times n$ and A_{22} is $m\times m$. The objective of the present paper is to give further results on the problems mentioned in Section 1 of (**3**). Concretely we shall consider the following question: "we prescribe the characteristic polynomial $f(\lambda)=\lambda^{n+m}-c_1\lambda^{n+m-1}+\dots$ of A and the principal blocks A_{11}, A_{22}. Find a necessary and sufficient condition for the existence of A satisfying these prescribed conditions".

Problems like this have applications outside Mathematics and it seems that practical methods for constructing A when it exists would be important. See (**6, 7**).

Our previous best result is probably Theorem 2.2 of (**3**). We note that conditions for the fulfilment of condition iii) of this Theorem were studied in (**4**). We state below this result for the complex field after taking account of the results in (**4**).

Let $\rho_1,\dots,\rho_m$ be the characteristic values of A_{22} and let $\xi_1,\dots,\xi_{n+m}$ be the characteristic values prescribed for A. Choose $m-1$ of these complex numbers, $\xi_2,\dots,\xi_m$ (following the notation of (**3**)) and consider the set $\{\rho_2-\xi_2,\dots,\rho_m-\xi_m\}$. Let $P_1,\dots,P_n$ be a partition of this set in which we require n components and allow empty sets. Let s_i be the sum of the numbers in P_i setting $s_i=0$ whenever P_i is empty. Finally let $S=\operatorname{diag}(s_1,\dots,s_n)$. Of course, in general, there are many possibilities for S.

Theorem 1.1. *Assume the following conditions are satisfied*:

(a) $\operatorname{tr} A_{11}+\operatorname{tr} A_{22}=\sum_{i=1}^{n+m}\xi_i$.

(b) *The characteristic roots ρ_i of A_{22} are pairwise distinct.*

(c) *Either*

(α) *A_{11} is nonderogatory*

or

(β) *A_{11} is derogatory but there is a choice for S such that its principal elements are pairwise distinct.*

Then there exists a complex matrix A with characteristic values $\xi_1,\dots,\xi_{n+m}$ and prescribed principal blocks A_{11} and A_{22}.

The proof of this theorem can be easily obtained by combining results in (**3**) and (**4**). See the last sentence in (**4**). As noticed in (**4**), if $m \leq n-1$ it is not possible to choose S with pairwise distinct diagonal elements. Obviously the roles of A_{11} and A_{22} can be interchanged.

It is clear that condition (a) is a necessary condition and therefore cannot be removed. We think that partly due to Theorem 2.1 of (**3**) it will be difficult to remove condition (c) or, at least, that it will be difficult to replace it with a milder and simultaneously nice condition. Let us focus our attention on condition (b). It is certainly very severe and probably is the worst shortcoming of the theorem as there are many practical cases in which it is not satisfied. Therefore it would be important to remove it or to replace it with a milder condition. This is what we will do in the following sections.

2.

In order to achieve more generality we assume that the underlying field is arbitrary. We assume also that $n \geq m$. Let $J = J_1 \oplus \cdots \oplus J_p$, where

$$J_i = \begin{bmatrix} 0 & 0 & \cdots & \alpha_1^{(i)} \\ 1 & 0 & \cdots & \alpha_2^{(i)} \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & \alpha_{t_i}^{(i)} \end{bmatrix}, \quad i = 1, \ldots, p, \tag{2.1}$$

be the first (or second) natural normal form (**2**) of A_{22}.

Suppose that A_{22} is not diagonalizable, i.e., that at least one of $t_1, \ldots, t_p$ (t_1 say) is greater than 1 (the case $t_1 = \ldots = t_p = 1$ will be treated later). Suppose also that $f(\lambda) = \lambda^{n+m} - c_1\lambda^{n+m-1} + \ldots$ has a factor $g(\lambda)$, over Φ, of degree $m-1$. Let $P = [0] \oplus C$, where C is any matrix with characteristic polynomial $g(\lambda)$. In particular C may be the companion matrix of $g(\lambda)$. Let $B = (J-P) \oplus 0$ where 0 is the $(n-m)\times(n-m)$ zero block.

Theorem 2.1. *Assume the following conditions hold*:

(i) $c_1 = \operatorname{tr} A_{11} + \operatorname{tr} A_{22}$.

(ii) *There is a nonsingular* $n \times n$ *matrix* U *such that the minimal polynomial of the column* n*-vector* $[0, 1, 0, \ldots, 0]^T$ *relative to* $UA_{11}U^{-1} + B$ *has degree* n.

Then there exists a matrix, over Φ, *with characteristic polynomial* $f(\lambda)$ *and prescribed principal blocks* A_{11} *and* A_{22}.

Remark 1. Condition (ii) is the new condition that, in a certain sense, is the substitute for condition (b).

Remark 2. For the assumption (ii) to be satisfied the matrix $UA_{11}U^{-1} + B$ must be nonderogatory. This condition should not be considered too restrictive. In (**4**) it was shown that if one of A_{11} and B is nonderogatory and the other nonscalar, there exists

U such that $UA_{11}U^{-1}+B$ has n distinct characteristic values (provided the cardinality of Φ be $\geq n$) and therefore is nonderogatory.

Proof of the Theorem. Denote $UA_{11}U^{-1}+B$ by R. There exist a row x and $q \in \Phi$ such that

$$\begin{bmatrix} R & v \\ x & q \end{bmatrix},$$

where $v=[0,1,0,\dots,0]^T$, has characteristic polynomial $f_1(\lambda)=f(\lambda)/g(\lambda)$ (**5**), Lemma 1. Now if $-d_1$ is the coefficient of λ^n in $f_1(\lambda)$ (its degree is $n+1$) we have

$$d_1 = q + \operatorname{tr} R = q + \operatorname{tr} A_{11} + \operatorname{tr} J - \operatorname{tr} P.$$

Since $c_1 = \operatorname{tr} A_{11} + \operatorname{tr} A_{22}$, $c_1 = d_1 + \operatorname{tr} P$ and $\operatorname{tr} A_{22} = \operatorname{tr} J$ we conclude that $q=0$.

Let R be partitioned as follows

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where R_{11} is $m \times m$. Let X be an $m \times n$ matrix with first row x and remaining rows all zero which we assume to be partitioned as $X=[X_1\, X_2]$ with X_1 of type $m \times m$. Let

$$D = \begin{bmatrix} R_{11} & R_{12} & J-P \\ R_{21} & R_{22} & 0 \\ X_1 & X_2 & P \end{bmatrix}$$

and

$$E = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_{n-m} & 0 \\ I_m & 0 & I_m \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix. Clearly the characteristic polynomial of D is $f(\lambda)$. Consider now $D_1 = EDE^{-1}$ which also has $f(\lambda)$ as characteristic polynomial. The top left hand $n \times n$ corner of D_1 is $UA_{11}U^{-1}$ and its bottom right hand $m \times m$ corner is J. Let T be such that $TJT^{-1}=A_{22}$. Then $(U^{-1}\oplus T)D_1(U^{-1}\oplus T)^{-1}$ is a matrix that satisfies the requirements of our theorem.

3.

Now we examine the case in which A_{22} is diagonalizable over Φ. This case needs some modifications in the method of proof of the preceding theorem.

Assume A_{22} is similar to $J=\operatorname{diag}(\rho_1,\dots,\rho_m)$. As in Theorem 2.1 we shall assume that $f(\lambda)$ has a factor $g(\lambda)$, over Φ, of degree $m-1$. Let $P=[0]\oplus C$, where C is any matrix with characteristic polynomial $g(\lambda)$ and let $B=(J-P)\oplus 0$ with 0 the $(n-m)\times(n-m)$ zero matrix.

Theorem 3.1. *Assume the following conditions hold:*

(i) $c_1 = \operatorname{tr} A_{11} + \operatorname{tr} A_{22}$.

(ii) *There is a nonsingular* $n \times n$ *matrix* U *such that the minimal polynomial of the column* n*-vector* $v = [\rho_1, 0, \ldots, 0]^T$ *relative to* $R = UA_{11}U^{-1} + B$ *has degree* n.

Then there exists a matrix, over Φ, *with characteristic polynomial* $f(\lambda)$ *and prescribed principal blocks* A_{11} *and* A_{22}.

Proof. There is a row x and $q \in \Phi$ such that

$$\begin{bmatrix} R & v \\ x & q \end{bmatrix}$$

has characteristic polynomial $f_1(\lambda) = f(\lambda)/g(\lambda)$. From $c_1 = \mathrm{tr}\, A_{11} + \mathrm{tr}\, A_{22}$ we deduce that $q = 0$. Let X be an $m \times n$ matrix with first row x and remaining rows all zero and let us partition R and X as in the proof of Theorem 2.1. Now consider a matrix like the matrix D that appears in the proof of Theorem 2.1, etc. The rest of the proof is identical with the proof of Theorem 2.1 and thus we do not give further details.

4.

The strongest condition and most difficult to check in Theorem 2.1 is the condition that there exists U such that the minimal polynomial of $[0, 1, 0, \ldots, 0]^T$ relative to $UA_{11}U^{-1} + B$ be of degree n. If A_{22} has at least one characteristic root in Φ that is not a multiple root, this condition is not needed and can be replaced with a weaker condition of the type "$UA_{11}U^{-1} + G$ is nonderogatory". The matrix G will be defined below. We recall again that there is such a matrix U if one of A_{11} and G is nonderogatory and the other nonscalar and Φ has enough elements.

We assume, as before, that $f(\lambda)$ has a factor $g(\lambda)$, over Φ, of degree $m - 1$. Let $\rho_1 \in \Phi$ be a simple characteristic root of A_{22} and let $J = [\rho_1] \oplus J_1 \oplus \ldots \oplus J_q$ be one of its natural normal forms, where each J_i is of the form (2.1). We denote $J_1 \oplus \ldots \oplus J_q$ by J'. Let C be a matrix with characteristic polynomial $g(\lambda)$ and $G = (J' - C) \oplus 0$, where 0 is the $(n - m + 1) \times (n - m + 1)$ zero matrix.

Theorem 4.1. *Assume the following conditions hold:*

(i) $c_1 = \mathrm{tr}\, A_{11} + \mathrm{tr}\, A_{22}$.

(ii) *There is a nonsingular* $n \times n$ *matrix* U *such that* $UA_{11}U^{-1} + G$ *is nonderogatory.*

Then there exists a matrix over Φ *with characteristic polynomial* $f(\lambda)$ *and prescribed principal blocks* A_{11} *and* A_{22}.

Proof. Let $R = UA_{11}U^{-1} + G$. There is a row x, a column y and $\tau \in \Phi$ such that

$$R_1 = \begin{bmatrix} R & y \\ x & \tau \end{bmatrix}$$

has characteristic polynomial $f_1(\lambda) = f(\lambda)/g(\lambda)$ (**1**). From condition (i) it follows easily that $\tau = \rho_1$. Let us partition R_1 as follows

$$R_1 = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix},$$

where R_{11} is $(m-1)\times(m-1)$. Let

$$R_2=\left[\begin{array}{cc:c} R_{11} & R_{12} & J'-C \\ R_{21} & R_{22} & 0 \\ \hdashline \multicolumn{2}{c:}{0} & C \end{array}\right].$$

The characteristic polynomial of R_2 is $f(\lambda)$. Now let

$$E=\begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & I_{n-m+2} & 0 \\ I_{m-1} & 0 & I_{m-1} \end{bmatrix}.$$

The matrix $F=EDE^{-1}$ has characteristic polynomial $f(\lambda)$ and is of the following form

$$F=\left[\begin{array}{c:ccccc} UA_{11}U^{-1} & & & * & & \\ \hdashline & \rho_1 & 0 & 0 & \dots & 0 \\ & * & J_1 & 0 & \dots & 0 \\ * & * & 0 & J_2 & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots \\ & * & 0 & 0 & \dots & J_q \end{array}\right].$$

The elements below ρ_1 are, in general, different from zero. However, since ρ_1 is not a characteristic value of any J_i, the bottom right hand block is similar to A_{22}. Therefore F can be transformed by similarity into a matrix that satisfies the requirements of our theorem.

If A_{22}, having a simple characteristic root ρ_1, is diagonalizable the theorem remains valid and the proof is of course the same.

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