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BASES OF T-EQUIVARIANT COHOMOLOGY OF BOTT–SAMELSON VARIETIES

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Abstract

We construct combinatorial bases of the T-equivariant cohomology $H^\bullet_T(\Sigma, k)$ of the Bott–Samelson variety Σ under some mild restrictions on the field of coefficients k. These bases allow us to prove the surjectivity of the restrictions $H^\bullet_T(\Sigma, k) \to H^\bullet_T(\pi^{-1}(x), k)$ and $H^\bullet_T(\Sigma, k) \to H^\bullet_T(\Sigma\backslash\pi^{-1}(x), k)$, where $\pi : \Sigma \to G/B$ is the canonical resolution. In fact, we also construct bases of the targets of these restrictions by picking up certain subsets of certain bases of $H^\bullet_T(\Sigma, k)$ and restricting them to $\pi^{-1}(x)$ or $\Sigma\backslash\pi^{-1}(x)$ respectively. As an application, we calculate the cohomology of the costalk-to-stalk embedding for the direct image $\pi_*\underline{k}_\Sigma$. This algorithm avoids division by 2, which allows us to re-establish 2-torsion for parity sheaves in Braden's example, Braden and Williamson ['Modular intersection cohomology complexes on flag varieties', *Math. Z.* **272**(3–4) (2012), 697–727].

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1. Introduction

Let Σ be a Bott–Samelson variety for a connected semisimple complex group G. In this paper, we study the T-equivariant cohomology $H^\bullet_T(\Sigma, k)$, where T is a maximal torus in G and k is a principal ideal domain. The direction of our research is mainly determined by Härterich's preprint [11]. However, this preprint uses Arabia's difficult results [1, 2], which, as explicitly stated, are valid for the ring of coefficients $\mathbb{Q}$. Therefore, we prefer not to use geometrical bases (coming from Białynicki–Birula cells) and construct combinatorial bases instead. If the sequence of simple reflections determining Σ has length r, then we define in total 2^{2^r-1} bases B_ρ of $H^\bullet_T(\Sigma, k)$ under some mild restriction on the characteristic of k (Theorem 4.9 and Lemma 6.1).

Let $\pi : \Sigma \to G/B$ be the canonical resolution and $x \in G/B$ be an arbitrary T-fixed point. Using the previously constructed bases of $H^\bullet_T(\Sigma, k)$, we can construct a basis of $H^\bullet_T(\pi^{-1}(x), k)$ as follows (Theorem 4.11, Remark 4.13 and Lemma 6.2):

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(1) choose an index ρ;
(2) choose a subset $M \subset B_\rho$;
(3) consider the restrictions $\{f|_{\pi^{-1}(x)} \mid f \in M\}$.

This fact implies that the restriction $H_T^\bullet(\Sigma, k) \to H_T^\bullet(\pi^{-1}(x), k)$ is surjective.

One may naturally ask what happens if we consider the complement $\Sigma \backslash \pi^{-1}(x)$ instead of $\pi^{-1}(x)$? It turns out that there exists a basis $H_T^\bullet(\Sigma \backslash \pi^{-1}(x), k)$ that can be constructed from a basis B_ρ of $H_T^\bullet(\Sigma, k)$ by steps similar to steps (1–3) above (Theorem 5.6, Remark 5.7 and Lemma 6.3).

A plausible motivation to consider the T-equivariant cohomology of $\Sigma \backslash \pi^{-1}(x)$ is to calculate the decomposition of the direct image $\pi_* \underline{k}_\Sigma$ into a direct sum of parity sheaves introduced in [15]. It was noted by the authors of this paper that the natural map $i_\lambda^! \mathcal{F} \to i_\lambda^* \mathcal{F}$ plays a decisive role in determining such a decomposition at least when k is a field (see [15, Proposition 2.26]). Here, i_λ is the embedding of a (closed) stratum. In this paper, we address the following question.

Problem 1.1. *Let $\pi : \Sigma \to G/B$ be a Bott–Samelson resolution and $x \in G/B$ be a T-fixed point. Denote by $i_x : \{x\} \hookrightarrow G/B$ the natural embedding. How is the map $\mathbb{H}_T^\bullet(\{x\}, i_x^! \pi_* \underline{k}_\Sigma) \to \mathbb{H}_T^\bullet(\{x\}, i_x^* \pi_* \underline{k}_\Sigma)$ to be calculated?*

It is answered in this paper by Corollary 6.5. Note that, unlike [15], this problem does not involve any stratifications. However, we can apply its solution to parity sheaves by considering the stratification $G/B = \bigsqcup_{x \in W} BxB/B$ and dividing by the T-equivariant Euler classes of the natural embeddings $\{x\} \hookrightarrow BxB/B$. The corresponding construction is given in Section 6.6.

Our algorithm is similar to the one described in [17]. It uses the same construction of the transition matrix as in [17, Theorem 4.10.3], which was previously used by Fiebig for his upper bound for Lusztig's conjecture [8] and which originally comes from the same Härterich's preprint [11]. The advantage of our approach here compared with [17] is that we do not divide by 2 when we compute products of the basis elements of $H_T^\bullet(\pi^{-1}(x), k)$ (cf. formula (4.20) of this paper and [17, Lemma 4.8.3]). This allows us to re-establish the 2-torsion for hexagonal permutations in Braden's example [5, Appendix A].

The paper is organized as follows. In Section 2, we explain how to adjust Brion's proofs [6] of localization theorems to our situation of coefficients different from $\mathbb{C}$ and of noncompact spaces. We use some ideas from [9], where the authors also prove localization theorems additionally assuming finiteness of T-curves (which is not the case for Bott–Samelson varieties). It is important to notice that we do need some form of GKM-restriction ((C3) in Corollary 2.5) to prove the intersection formula for the image. To ensure this condition, we first work with coefficients $\mathbb{Z}' = \mathbb{Z}$ or $\mathbb{Z}' = \mathbb{Z}[1/2]$ if the root system contains a component of type C_n and then change coefficients to a principal ideal domain in Section 6.1.

In Section 3, we introduce the main characters of the paper: the Bott–Samelson variety, combinatorial galleries, load-bearing walls, orders $\lhd$ and $<$, tree analogs of combinatorial galleries, and so on.

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In Section 4, we construct bases of $H^\bullet_T(\Sigma,\mathbb{Z}')$ and $H^\bullet_T(\Sigma_x,\mathbb{Z}')$. We use here the criteria proved by Härterich [11, Theorems 6.2 and 6.3]. The proofs of these results use only the smooth case of $SL_2(\mathbb{C})$ or $PSL_2(\mathbb{C})$, which can be handled by [3]. We develop here the main combinatorial tool of this paper: operators of copy Δ and concentration ∇_t, which construct elements of $H^\bullet_T(\Sigma,\mathbb{Z}')$ from elements of $H^\bullet_T(\Sigma',\mathbb{Z}')$, where Σ' is the Bott–Samelson variety for the truncated sequence. Finally, we construct bases of $H^\bullet_T(\Sigma_x,\mathbb{Z}')$ in a way similar to [17]. Our new product of basis elements (4.20) does not include division, which is a definite advantage.

Operators of copy and concentration in a less deterministic form already appeared in Härterich's preprint [11] (see the arguments after Corollary 8.2). However, Härterich's operators are restricted to the cohomology of the fibre only, which involves taking arbitrary lifts of projections of basis elements. The latter are often hard to find. We resolve this problem by constructing bases of the cohomology of the whole Bott–Samelson variety and then restricting them to the fibres.

In Section 5, we construct a basis of $H^\bullet_T(\Sigma\backslash\pi^{-1}(x),\mathbb{Z}')$. To achieve this goal, we need to prove the localization theorem for $\Sigma\backslash\pi^{-1}(x)$ and a criterion for $H^\bullet_T(\Sigma\backslash\pi^{-1}(x),\mathbb{Z}')$ (Proposition 5.2) similar to Härterich's criteria [11, Theorems 6.2 and 6.3].

Section 6 is devoted to applications of the obtained results. We begin with the change of coefficients in Section 6.1, which allows us to obtain bases of $H^\bullet_T(\Sigma,k)$, $H^\bullet_T(\Sigma_x,k)$ and $H^\bullet_T(\Sigma\backslash\pi^{-1}(x),k)$ for any principal ideal domain k of characteristic not 2 if the root system contains a component of type C_n. Then we solve Problem 1.1 and in Section 6.6 show how this information can be used to decompose the direct image $\pi_*\underline{k}_\Sigma[r]$ to a direct sum of indecomposable parity sheaves. As an example, we show in Section 6.7 that this decomposition may depend on the characteristic of k (Theorem 6.11) by considering a hexagonal permutation as in Braden's example [5, Appendix A].

Finally, we note that all the above results are valid in the affine setting [14] with the corresponding restriction on the characteristic, as we use only local techniques. The reader may consult, for example, [10] about affine pavings.

2. Localization theorems

2.1. Generalities. We denote the fact that N is a subset of M, including the case $N=M$, by $N\subset M$, reserving the notation $N\subsetneq M$ for the proper inclusion. We write $i_{M,N}:N\hookrightarrow M$ for the natural inclusion map. We sometimes write $r^\bullet_{N,M}$ for the map $H^\bullet_G(M,k)\to H^\bullet_G(N,k)$ induced by a G-equivariant embedding $i_{M,N}:N\hookrightarrow M$. We denote by $|X|$ the cardinality of a finite set X and by $\mathrm{Map}(X,Y)$ the set of all maps from X to Y. For a set S with an equivalence relation $\sim$, we denote by $\mathrm{rep}(S,\sim)$ any set of representatives of $\sim$-equivalence classes.

It this paper, we consider the bounded equivariant derived category $D^b_T(X,k)$ for a commutative ring k and a topological group T acting continuously on a topological space X, which is called a *T-space* in that case. For any object $\mathscr{F}$ of this category, one can define the T-equivariant hypercohomology $\mathbb{H}^\bullet_T(X,\mathscr{F})$. The basic definitions and

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properties of this category and T-equivariant cohomologies can be found in [4]. We shall also use the functors f_*, f^*, $f_!$, $f^!$ between equivariant derived categories defined in [4].

In particular, we can consider the T-equivariant cohomology $H_T^\bullet(X,k)=\mathbb{H}_T^\bullet(X,\underline{k}_X)$ with coefficients in the constant sheaf. It also admits the following description via the ordinary cohomology:

$$H_T^\bullet(X,k)=H^\bullet((X\times E_T)/T,k),$$

where E_T is a universal principal T-bundle. We often write $r_{N,M}^\bullet$ for the map $H_T^\bullet(M,k)\to H_T^\bullet(N,k)$ induced by a T-equivariant embedding $i_{M,N}:N\hookrightarrow M$.

2.2. Isomorphism of localizations of modules. We want to formulate here a simple lemma from commutative algebra whose proof is left to the reader.

Let S be a (unitary) commutative ring, M and N be S-modules and $q\in S$. Consider the ring of quotients $S'=S[q^{-1}]$ and S'-modules of quotients $M'=M[q^{-1}]$ and $N'=N[q^{-1}]$. Any homomorphism of S-modules $f:M\to N$ gives rise to the homomorphism $f':M'\to N'$ of S'-modules that maps m/q^k to $f(m)/q^k$.

Lemma 2.1. *Suppose that for some integers $a,b\geqslant 0$ the following conditions hold:*

(1) $q^aN\subset \operatorname{im} f$;

(2) $q^b\ker f=0$.

Then $f':M'\to N'$ is an isomorphism of S'-modules.

2.3. The equivariant Mayer–Vietoris sequence for open subsets. Remember the following well-known result.

Proposition 2.2 (Mayer–Vietoris sequence). *Let X be a T-space. For any open T-stable subsets U, V and an object $\mathscr{F}\in D_T^b(X,k)$, we have the following exact sequence:*

$$\begin{aligned}\cdots\to\mathbb{H}_T^{i-1}(U\cap V,\mathscr{F}|_{U\cap V})&\to\mathbb{H}_T^i(U\cup V,\mathscr{F}|_{U\cup V})\\ &\to\mathbb{H}_T^i(U,\mathscr{F}|_U)\oplus\mathbb{H}_T^i(V,\mathscr{F}|_V)\to\mathbb{H}_T^i(U\cap V,\mathscr{F}|_{U\cap V})\\ &\to\mathbb{H}_T^{i+1}(U\cup V,\mathscr{F}|_{U\cup V})\to\cdots.\end{aligned}$$

In the proofs of the localization theorems, this proposition is applied as follows. Suppose that $X=U\cup V$, where X is a T-space and U, V are its open T-stable subspaces. Suppose additionally that there exist elements $u\in H_T^n(\mathrm{pt},k)$ and $v\in H_T^m(\mathrm{pt},k)$ such that u annihilates $\mathbb{H}_T^\bullet(O_U,\mathscr{F}_U)$ and v annihilates $\mathbb{H}_T^\bullet(O_V,\mathscr{F}_V)$ for any open T-stable subsets $O_U\subset U$, $O_V\subset V$ and any objects $\mathscr{F}_U\in D_T^b(O_U,k)$, $\mathscr{F}_V\in D_T^b(O_V,k)$. Then Proposition 2.2 implies that u^2v and uv^2 annihilate $\mathbb{H}_T^\bullet(O,\mathscr{F})$ for any open T-stable $O\subset X$ and object $\mathscr{F}\in D_T^b(O,k)$. Indeed, let $O_U=O\cap U$, $O_V=O\cap V$ and $f\in\mathbb{H}_T^i(O,\mathscr{F})$. Then uvf is mapped to 0 by the following part of the Mayer–Vietoris sequence:

$$\mathbb{H}_T^{i+n+m}(O,\mathscr{F})\to\mathbb{H}_T^{i+n+m}(O_U,\mathscr{F}|_U)\oplus\mathbb{H}_T^{i+n+m}(O_V,\mathscr{F}|_V).$$

By exactness, uvf comes from $\mathbb{H}_T^{i+n+m-1}(O_U\cap O_V,\mathscr{F})$. Thus, multiplying by u (respectively by v), we prove that $u^2vf=0$ (respectively $uv^2f=0$).

Another trivial corollary of the Mayer–Vietoris sequence is as follows.

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COROLLARY 2.3. *Let X be a T-space, $X = \bigsqcup_{i\in I} X_i$, each X_i be open and T-stable and I be finite. Suppose that $Y \subset X$ is another T-subspace. We write $Y_i = Y \cap X_i$. An element $f \in H^\bullet(Y,k)$ belongs to the image of the restriction $H^\bullet_T(X,k) \to H^\bullet_T(Y,k)$ if and only if each $f|_{Y_i}$ belongs to image of the restriction $H^\bullet_T(X_i,k) \to H^\bullet_T(Y_i,k)$.*

PROOF. Induction together with the finiteness of I reduces the problem to the case $I = \{1,2\}$. As $X_1 \cap X_2 = \varnothing$ and the Mayer–Vietoris sequence is compatible with restrictions, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^\bullet_T(X,k) & \longrightarrow & H^\bullet_T(X_1,k)\oplus H^\bullet_T(X_2,k) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^\bullet_T(Y,k) & \longrightarrow & H^\bullet_T(Y_1,k)\oplus H^\bullet_T(Y_2,k) & \longrightarrow & 0 \end{array}$$

Hence the required result follows. □

The above arguments apply to any topological group T not necessarily a torus. In this paper, we are however interested only in the case of a torus $T \simeq (\mathbb{C}^\times)^n$ and use the following notation:

$$S_k = H^\bullet_T(\mathrm{pt},k) \simeq S(X(T)\otimes_{\mathbb{Z}} k),$$

where $X(T)$ is the character group of T and S in the right-hand side means taking the symmetric algebra. This is a $\mathbb{Z}$-graded algebra such that $S_k^2 = X(T)\otimes_{\mathbb{Z}} k$. Finally, note that in the next section we need to consider the compact subtorus $K = (S^1)^n$ of $T \simeq \mathbb{C}^n$. We can replace T-equivariant cohomology with K-equivariant cohomology if necessary.

2.4. Localization. We prove here some localization theorems, closely following [6] (see also [9] for the case of coefficients different from $\mathbb{C}$).

THEOREM 2.4. *Let $\Gamma < T$ be a closed subgroup of T and X be a paracompact T-space that has an open covering $X = \bigcup_{i\in I} Y^{(i)}$ such that for any $i \in I$:*

- *$Y^{(i)}$ is open and T-equivariant;*
- *there exists a T-equivariant embedding of $Y^{(i)}$ in a finite dimensional rational representation $V^{(i)}$ of T.*

Denote by Λ_Γ the set of all weights of T occurring as weights of some $V^{(i)}$ and having nontrivial restriction to Γ.

Then the natural restriction morphism $H^\bullet_T(X,k) \to H^\bullet_T(X^\Gamma,k)$ becomes an isomorphism after inverting all elements of $\Lambda_\Gamma \otimes_{\mathbb{Z}} k$.

PROOF. For simplicity of notation, we assume that $Y^{(i)}$ is a subset of $V^{(i)}$. Let us write

$$V^{(i)} = \mathbb{C}_{\lambda_1^{(i)}} \oplus \cdots \oplus \mathbb{C}_{\lambda_{n_i}^{(i)}},$$

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where $\mathbb{C}_\lambda$ is the representation of T with weight $\lambda \in X(T)$. Let U be an open K-invariant neighbourhood of X^Γ in X. Then the set $Y^{(i)}\backslash U$ does not have Γ-fixed points, which we prefer to write as

$$(Y_i\backslash U) \cap (V^{(i)})^\Gamma = \varnothing. \tag{2.1}$$

Without loss of generality, we can assume that $\lambda_j^{(i)}$ restricts trivially to Γ if and only if $j \leqslant m_i$. Then $(V^{(i)})^\Gamma$ consists of the points of the form $(c_1, \ldots, c_{m_i}, 0, \ldots, 0)$. Consider the open subsets $W_j^{(i)} = \{(c_1, \ldots, c_{n_i}) \in V^{(i)} \mid c_j \neq 0\}$ of $V^{(i)}$. It follows from (2.1) that

$$Y_i\backslash U \subset \bigcup_{j=m_i+1}^{n_i} W_j^{(i)}.$$

For any set of the union in the right-hand side, there exists a T-equivariant map $W_j^{(i)} \to \mathbb{C}^\times_{\lambda_j^{(i)}} \cong T/\ker \lambda_j^{(i)}$, which is the projection to the jth coordinate.

Let us take an open K-invariant subset $O \subset (Y^{(i)}\backslash U) \cap W_j^{(i)}$ for $j > m_i$. Then composition of maps $O \hookrightarrow W_j^{(i)} \to \mathbb{C}^\times_{\lambda_j^{(i)}} \to \mathrm{pt}$ gives rise to the following sequence of cohomologies:

$$H_K^2(\mathrm{pt}, k) \to H_K^2(\mathbb{C}^\times_{\lambda_j^{(i)}}, k) \to H_K^2(W_j^{(i)}, k) \to H_K^2(O, k).$$

Identifying T-equivariant and K-equivariant cohomologies, we obtain that the image of the first Chern class $c_1(\lambda_j^{(i)}) \otimes k$ is zero (already for the first map as follows from [13, 1.9(1)]). Writing this Chern class as $\lambda_j^{(i)} \otimes k$, we get that it annihilates $\mathbb{H}_K^\bullet(O, \mathscr{F})$ for any $\mathscr{F} \in D_K^b(O, k)$.

Gluing all subsets $(Y^{(i)}\backslash U) \cap W_j^{(i)}$ by the Mayer–Vietoris sequence for open subsets by the method described in Section 2.3, we get the following property:

$$\begin{gathered}\textit{there exist naturals } a_i \textit{ such that } q = \prod_{i\in I} \prod_{j=m_i+1}^{n_i} (\lambda_j^{(i)} \otimes k)^{a_i} \\ \textit{annihilates } \mathbb{H}_K^\bullet(X\backslash U, \mathscr{F}) \textit{ for any object } \mathscr{F} \in D_K^b(X\backslash U, k).\end{gathered} \tag{2.2}$$

Consider the following direct limit $L^n(k) := \varinjlim_{U\supset X^\Gamma} H_K^n(U, k)$ that runs over all K-invariant open neighbourhoods U of X^Γ. Denote by $\alpha_U^n : H_K^n(U, k) \to L^n(k)$ its natural morphisms. We define $L^\bullet(k) = \bigoplus_{n\in\mathbb{Z}} L^n(k)$. It is an S_k-module and we get homomorphisms $\alpha_U^\bullet : H_K^\bullet(U, k) \to L^\bullet(k)$ of S_k-modules.

We are going to apply Lemma 2.1 to prove that $\alpha_X^\bullet$ becomes an isomorphism after inverting q. We know that $q \in S_k^{2t}$ for some $t \in \mathbb{Z}$.

Let us check condition (1) of Lemma 2.1. Let $u \in L^n(k)$. By the definition of the direct limit, $u = \alpha_U^n(\bar{u})$ for some $\bar{u} \in H_K^n(U, k)$ and some K-invariant open U containing X^Γ. We have the exact sequence

$$H_K^\bullet(X, k) \xrightarrow{r_{U,X}^\bullet} H_K^\bullet(U, k) \xrightarrow{\partial^\bullet} \mathbb{H}_K^{\bullet+1}(X\backslash U, i^! \underline{k}_X),$$

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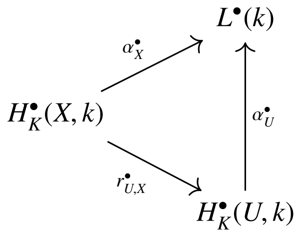
where $i : X\backslash U \hookrightarrow X$ is the natural embedding. Hence and from (2.2), we get $\partial^{n+2t}(q\bar{u}) = q\partial^n(\bar{u}) = 0$. The exactness of the above sequence yields $q\bar{u} = r_{U,X}^{n+2t}(v)$ for some $v \in H_K^{n+2t}(X, k)$. It remains to recall the commutative diagram



from the definition of the direct limit and write

$$qu = q\alpha_U^n(\bar{u}) = \alpha_U^{n+2t}(q\bar{u}) = \alpha_U^{n+2t} \circ r_{U,X}^{n+2t}(v) = \alpha_X^{n+2t}(v).$$

Let us check now condition (2) of Lemma 2.1. Take some $v \in H_K^n(X, k)$ such that $\alpha_X^n(v) = 0$. By the definition of the direct limit, we get $v|_U = 0$ for some K-invariant open U containing X^Γ. Consider the distinguished triangle

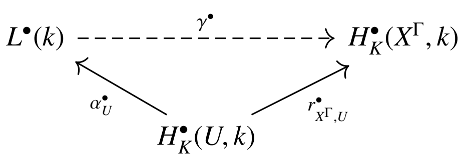
$$i_* i^! \underline{k}_X \to \underline{k}_X \to j_* j^* \underline{k}_X \xrightarrow{+1},$$

where $j : U \hookrightarrow X$ and $i : X\backslash U \hookrightarrow X$ are the natural embeddings. It yields the exact sequence

$$\mathbb{H}_K^\bullet(X\backslash U, i^! \underline{k}_X) \xrightarrow{\beta_U^\bullet} H_K^\bullet(X, k) \xrightarrow{r_{U,X}^\bullet} H_K^\bullet(U, k).$$

Hence, $v = \beta_U^n(w)$ for some $w \in \mathbb{H}_K^\bullet(X\backslash U, i^! \underline{k}_X)$. Multiplying by q and applying (2.2), we get $qv = \beta_U^{n+2t}(qw) = 0$.

The universal mapping property for direct limits yields the (unique) morphism $\gamma^\bullet$ such that the diagram



is commutative for any open U containing X^Γ. By [16, (1.9)], $\gamma^\bullet$ is an isomorphism. It is obviously an isomorphism of S_k-modules. Considering the case $U = X$ and applying the fact that $\alpha_X^\bullet$ becomes an isomorphism after inverting q, we get that $r_{X^\Gamma,X}^\bullet$ also becomes an isomorphism after inverting q and moreover after inverting all elements of $\Lambda_\Gamma \otimes_{\mathbb{Z}} k$. □

As our next step, we explain how to adjust [6, Theorem 6 from Brion's paper] to the case of arbitrary coefficients.

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Corollary 2.5 (Cf. [6, Theorem 6]). *Let $H_T^\bullet(X,k)$ be a free S_k-module. Under the hypothesis of Theorem 2.4 with the additional assumption that $\Lambda_T \otimes \mathbf{1}_k$ does not contain zero divisors of S_k, the restriction*

$$i^*_{X,X^T} : H_T^\bullet(X,k) \to H_T^\bullet(X^T,k)$$

is an embedding.

Moreover, if $H_T^\bullet(X^T,k)$ does not have S_k-torsion (for example, X^T is finite) and the following conditions hold:

(C1) *k is a unique factorization domain;*
(C2) *$\lambda \otimes \mathbf{1}_k$ is prime in $S(X(T)\otimes_{\mathbb{Z}} k)$ for any $\lambda \in \Lambda_T$;*
(C3) *$\lambda \otimes \mathbf{1}_k \notin \Lambda_{\ker\lambda} \otimes_{\mathbb{Z}} k$ for any $\lambda \in \Lambda_T$,*

then we have

$$\operatorname{im} i^*_{X,X^T} = \bigcap_{\lambda\in\Lambda_T} \operatorname{im} i^*_{X^{\ker\lambda},X^T}.$$

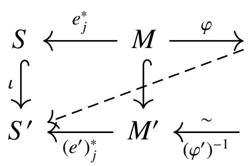
Proof. Let $M = H_T^\bullet(X,k)$, $N = H_T^\bullet(X^T,k)$, $R = \Lambda_T \otimes_{\mathbb{Z}} k$, $S = S_k$, $S' = R^{-1}S$, $M' = R^{-1}M$, $N' = R^{-1}N$, $\varphi = i^*_{X,X^T}$ and φ' be the morphism from M' to N' induced by φ. By Theorem 2.4, φ' is an isomorphism.

Let $\{e_j\}_{j\in J}$ be an S-basis of M. Then $\{e_j/1\}_{j\in J}$ is an S'-basis of M'. Suppose that $\varphi(\alpha_1 e_{j_1} + \cdots + \alpha_k e_{j_k}) = 0$ for $\alpha_1,\ldots,\alpha_k \in S$ and mutually distinct indices $j_1,\ldots,j_k \in J$. We get

$$\varphi'\Big(\frac{\alpha_1}{1}\cdot\frac{e_{j_1}}{1} + \cdots + \frac{\alpha_k}{1}\cdot\frac{e_{j_k}}{1}\Big) = \varphi'\Big(\frac{\alpha_1 e_{j_1} + \cdots + \alpha_k e_{j_k}}{1}\Big) = \frac{\varphi(\alpha_1 e_{j_1} + \cdots + \alpha_k e_{j_k})}{1} = 0.$$

Hence, $\alpha_1/1 = \cdots = \alpha_k/1 = 0$ in S'. Therefore $\alpha_1 = \cdots = \alpha_k = 0$, as R does not contain zero divisors.

Now let us prove the second statement. Let $e_j^* : M \to S$ and $(e')_j^* : M' \to S'$ be the jth coordinate functions for M and M', respectively. Consider the following commutative diagram:



Denoting the dashed arrow by f_j, we get the following relation:

$$f_j \circ \varphi = \iota \circ e_j^*. \tag{2.3}$$

Note that all functions f_j uniquely define elements of N:

$$f_j(u) = f_j(u') \quad \forall j \in J \Longrightarrow u = u'. \tag{2.4}$$

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Let us take $u \in \bigcap_{\lambda \in \Lambda_T} \operatorname{im} i^*_{X^{\ker\lambda}, X^T}$. Consider the coefficients $f_j(u) \in S'$. If they all belong to $\iota(S)$, then in view of (2.3), the following calculation is possible:

$$f_j \circ \varphi\Big(\sum_{j\in J} \iota^{-1}(f_j(u))e_j\Big) = \iota \circ e_j^*\Big(\sum_{j\in J} \iota^{-1}(f_j(u))e_j\Big) = f_j(u).$$

Now (2.4) implies that $u = \varphi\big(\sum_{j\in J} f_j(u)e_j\big) \in \operatorname{im} i^*_{X,X^T}$.

It only remains to prove that $f_j(u) \in S$ for all $j \in J$. Suppose the contrary holds. By (C2), in this case, $f_j(u)$ contains an uncancellable prime denominator $\lambda \otimes \mathbf{1}_k$ for some $j \in J$ and $\lambda \in \Lambda_T$.

To proceed, let us introduce the following notation: $\Gamma = \ker\lambda$, $N_\lambda = H^\bullet_T(X^\Gamma, k)$, $R_\lambda = \Lambda_\Gamma \otimes_{\mathbb{Z}} k$, $S'_\lambda = R_\lambda^{-1}S$, $M'_\lambda = R_\lambda^{-1}M$, $N'_\lambda = R_\lambda^{-1}N_\lambda$, $\varphi_\lambda = i^*_{X,X^\Gamma}$ and φ'_λ is the morphism from M'_λ to N'_λ induced by φ_λ. By Theorem 2.4, φ'_λ is an isomorphism.

As $u \in \operatorname{im} i^*_{X^\Gamma, X^T}$, we can write $u = i^*_{X^\Gamma,X^T}(v)$ for some $v \in N_\lambda$. Similarly to the diagram above, we have the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\varphi_\lambda} & N_\lambda \\ \big\downarrow & & \big\downarrow \\ M'_\lambda & \xleftarrow[(\varphi'_\lambda)^{-1}]{\sim} & N'_\lambda \end{array}$$

There exists some product $\mathcal{P}_\lambda$ of elements of R_λ such that $(\mathcal{P}_\lambda/1)(\varphi'_\lambda)^{-1}(v/1) = m/1$ for some $m \in M$. Applying φ'_λ to this equality, we get $\mathcal{P}_\lambda v/1 = (\varphi'_\lambda)(m/1) = \varphi_\lambda(m)/1$, which is an equality in N'_λ. Therefore, there exists another product $\mathcal{P}'_\lambda$ of elements of R_λ such that

$$\mathcal{P}'_\lambda \mathcal{P}_\lambda v = \mathcal{P}'_\lambda \varphi_\lambda(m) = \varphi_\lambda(\mathcal{P}'_\lambda m).$$

Applying $i^*_{X^\Gamma,X^T}$ to both sides of this equality,

$$\mathcal{P}'_\lambda \mathcal{P}_\lambda u = i^*_{X^\Gamma,X^T}(\mathcal{P}'_\lambda \mathcal{P}_\lambda v) = i^*_{X^\Gamma,X^T} \circ \varphi_\lambda(\mathcal{P}'_\lambda m) = \varphi(\mathcal{P}'_\lambda m).$$

Finally, applying f_j,

$$(\mathcal{P}'_\lambda \mathcal{P}_\lambda/1)f_j(u) = f_j(\mathcal{P}'_\lambda \mathcal{P}_\lambda u) = f_j \circ \varphi(\mathcal{P}'_\chi m) = e_j^*(\mathcal{P}'_\chi m)/1 \in \iota(S).$$

This is a contradiction, as $\mathcal{P}'_\lambda \mathcal{P}_\lambda$ by our GKM-restriction (C3) does not have factors proportional to $\lambda \otimes \mathbf{1}_k$. □

3. Bott–Samelson variety

Let G be a connected semisimple complex algebraic group, T be its maximal torus and B be its Borel subgroup containing T. We denote by W, Φ, Φ^+, Π the Weyl group, the set of all roots, the set of positive roots and the set of simple roots respectively.

Let α be a root. We denote by s_α and U_α the simple reflection and the unipotent subgroup corresponding to α respectively. Let G_α be the subgroup of G generated U_α and $U_{-\alpha}$. This subgroup is isomorphic to either $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{PSL}_2(\mathbb{C})$. We set

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$B_\alpha = G_\alpha \cap B$. Let P_α be the parabolic subgroup of G corresponding to α. If α is simple, then $P_\alpha = B \cup Bs_\alpha B$. We denote by $x_\alpha : \mathbb{C} \to U_\alpha$ the canonical homomorphism.

Throughout the paper, we fix a sequence $s = (s_1, s_2, \ldots, s_r)$ of simple reflections, where $s_i = s_{\alpha_i}$ for some $\alpha_i \in \Pi$, and consider the *Bott–Samelson* variety

$$\Sigma = P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_r}/B^r,$$

where B^r acts as follows:

$$(p_1, p_2, \ldots, p_r) \cdot (b_1, b_2, \ldots, b_r) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, b_{r-1}^{-1}p_rb_r).$$

We denote by $[p_1, \ldots, p_r]$ the point of Σ corresponding to $(p_1, \ldots, p_r)$. It is well known that Σ is a smooth complex variety of dimension r.

Let $\pi : \Sigma \to G/B$ be the map $\pi([p_1, \ldots, p_r]) = p_1 \cdots p_rB/B$. For any $x \in G/B$, we fix the notation $\Sigma_x = \pi(x)$ and $\bar{\Sigma}_x = \Sigma \backslash \Sigma_x$. We can also view Σ as a closed subvariety of $(G/B)^r$ via the embedding $\iota : \Sigma \hookrightarrow (G/B)^r$ defined by

$$\iota([p_1, \ldots, p_r]) = (p_1B, p_1p_2B, \ldots, p_1p_2 \cdots p_rB). \tag{3.1}$$

This map is an isomorphism for $G = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{PSL}_2(\mathbb{C})$.

Each point of G/B fixed by T can be written uniquely as wB for some $w \in W$. So, abusing notation, we will denote this point simply by w. Consider the following set:

$$\Gamma = \{(\gamma_1, \ldots, \gamma_r) \mid \gamma_i = s_i \text{ or } \gamma_i = e\}.$$

The elements of this set are called *combinatorial galleries*. We make T act on Σ by $t \cdot [p_1, p_2, \ldots, p_r] = [tp_1, p_2, \ldots, p_r]$. Then Γ can be thought of as the set of all T-fixed points of Σ if we identify $(\gamma_1, \ldots, \gamma_r)$ with $[\gamma_1, \ldots, \gamma_r]$. The embedding ι defined above is clearly T-equivariant. Moreover, Σ is covered by open T-equivariant subsets

$$U^\gamma = \{[x_{\gamma_1(-\alpha_1)}(c_1)\gamma_1, x_{\gamma_2(-\alpha_2)}(c_2)\gamma_2, \ldots, x_{\gamma_r(-\alpha_r)}(c_r)\gamma_r] \mid c_1, c_2, \ldots, c_r \in \mathbb{C}\},$$

where γ runs through Γ.

For each $\gamma = (\gamma_1, \ldots, \gamma_r) \in \Gamma$ and $i = 0, \ldots, r$, we write $\gamma^i = \gamma_1 \cdots \gamma_i$. So we get $\gamma^0 = e$. If additionally $i > 0$, then we write $\boldsymbol{\beta}_i(\gamma) = \gamma^i(-\alpha_i)$ and $\widetilde{\boldsymbol{\beta}}_i(\gamma) = \gamma^{i-1}(-\alpha_i)$. If $\boldsymbol{\beta}_i(\gamma) > 0$, then we say that i is *load-bearing* for γ or that the wall corresponding to $\boldsymbol{\beta}_i(\gamma)$ is *load-bearing*. For any $A \subset W$, we write

$$\Gamma_A = \{\gamma \in \Gamma \mid \pi(\gamma) \in A\}, \quad \bar{\Gamma}_A = \Gamma \backslash \Gamma_A.$$

If $A = \{x\}$, then we use the simplified notation $\Gamma_x = \Gamma_{\{x\}}$ and $\bar{\Gamma}_x = \bar{\Gamma}_{\{x\}}$.

For $\alpha \in \Phi^+$ and $\gamma \in \Gamma$, we set

$$J(\gamma) = \{i \mid \boldsymbol{\beta}_i(\gamma) > 0\}, \quad M_\alpha(\gamma) = \{i \mid \boldsymbol{\beta}_i(\gamma) = \pm\alpha\},$$
$$J_\alpha(\gamma) = \{i \mid \boldsymbol{\beta}_i(\gamma) = \alpha\} = J(\gamma) \cap M_\alpha(\gamma),$$
$$D(\gamma) = \{i \mid \widetilde{\boldsymbol{\beta}}_i(\gamma) > 0\}, \quad D_\alpha(\gamma) = \{i \mid \widetilde{\boldsymbol{\beta}}_i(\gamma) = \alpha\} = D(\gamma) \cap M_\alpha(\gamma).$$

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Note that $\boldsymbol{\beta}_i(\gamma) > 0 \Leftrightarrow \gamma^i s_i < \gamma^i$ and $\widetilde{\boldsymbol{\beta}}_i(\gamma) > 0 \Leftrightarrow \gamma^{i-1} s_i < \gamma^{i-1}$. Using these subsets, we introduce the following equivalence relation on Γ:

$$\gamma \sim_\alpha \delta \Longleftrightarrow \gamma_i = \delta_i \quad \text{unless } \boldsymbol{\beta}_i(\gamma) = \pm\alpha.$$

One can easily check that $M_\alpha(\gamma)$ depends only on the $\sim_\alpha$-equivalence class of γ.

We will use the following two relations on Γ:

$$\delta \lhd \gamma \Longleftrightarrow \delta^0 = \gamma^0, \ldots, \delta^{i-1} = \gamma^{i-1}, \delta^i < \gamma^i \quad \text{for some } i = 0, \ldots, r;$$
$$\delta < \gamma \Longleftrightarrow \delta^i < \gamma^i, \delta^{i+1} = \gamma^{i+1}, \ldots, \delta^r = \gamma^r \quad \text{for some } i = 0, \ldots, r.$$

Clearly, $\lhd$ is a total order on Γ, whereas $<$ in general becomes a total order only when restricted to some Γ_x. As usual, we set $\delta \unlhd \gamma \Leftrightarrow \delta \lhd \gamma$ or $\delta = \gamma$ and $\delta \leqslant \gamma \Leftrightarrow \delta < \gamma$ or $\delta = \gamma$. Note that $\delta \sim_\alpha \gamma$ and $J_\alpha(\delta) \subset J_\alpha(\gamma)$ imply $\delta \unlhd \gamma$. We recall the following lemma from [11].

PROPOSITION 3.1. *Let $M_\alpha(\gamma) = \{i_1 < \cdots < i_\ell\}$. Then for $1 \leqslant j < \ell$,*

$$i_j \in J_\alpha(\gamma) \Longleftrightarrow i_{j+1} \in D_\alpha(\gamma)$$

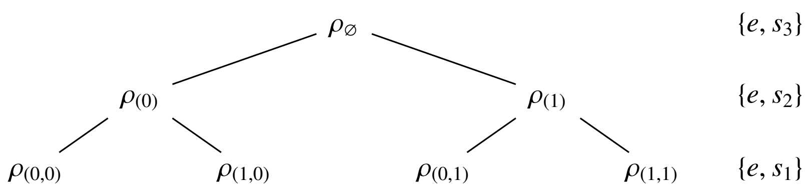
and $i_\ell \in J_\alpha(\gamma)$ if and only if $s_\alpha \pi(\gamma) < \pi(\gamma)$. In particular, if $\gamma \sim_\alpha \delta$ and $\pi(\gamma) = \pi(\delta)$, then $J_\alpha(\delta) \subset J_\alpha(\gamma) \Leftrightarrow D_\alpha(\delta) \subset D_\alpha(\gamma)$.

We use the symbol $\cdot$ to denote the addition of a new entry to a sequence: $(a_1, \ldots, a_n) \cdot b = (a_1, \ldots, a_n, b)$. Conversely, for a nonempty sequence $a = (a_1, \ldots, a_n)$, we denote by $a' = (a_1, \ldots, a_{n-1})$ its truncation. In what follows, we define $|a| = n$ to be the length of a sequence $a = (a_1, \ldots, a_n)$.

For any integer $r \geqslant 0$, let Tr_r denote the binary tree that consists of all sequences (including the empty one) with entries 0 or 1 of length less than r. We obviously have $|\mathrm{Tr}_r| = 2^r - 1$ and $\mathrm{Tr}_r = \mathrm{Tr}_{r-1} \cdot 0 \sqcup \mathrm{Tr}_{r-1} \cdot 1 \sqcup \{\varnothing\}$ for $r > 0$. To construct bases of the T-equivariant cohomology of the Bott–Samelson varieties, we consider the set

$$\Upsilon = \{\rho : \mathrm{Tr}_r \to \{s_1, \ldots, s_r\} \mid \rho_u = e \text{ or } \rho_u = s_{r-|u|}\}.$$

Elements of this set are thus tree analogs of combinatorial galleries. For example, for $r = 3$, we draw Υ as



where the right column shows the sets to which the elements of the corresponding rows belong.

Our notation above implicitly referred to the sequence $s = (s_1, \ldots, s_r)$ and the group G. If, for the sake of induction, we want to consider the same objects for the shorter

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sequence $s' = (s_1, \ldots, s_{r-1})$, we add $'$ to our symbols: Σ', Γ', $\bar{\Gamma}'_x$, and so on. For example, let $r > 0$ and $\rho \in \Upsilon$. For any $\varepsilon \in \{0, 1\}$, we consider its *ε-truncation* $\rho'_\varepsilon \in \Upsilon'$ defined by $(\rho'_\varepsilon)_{u'} = \rho_{u'\cdot\varepsilon}$ (at the picture above, ρ'_0 and ρ'_1 are the left and right subtrees respectively).

In Section 5, we consider the cases $G = \mathrm{SL}_2(\mathbb{C})$ and $G = \mathrm{PSL}_2(\mathbb{C})$. The sequence s is then characterized only by its length r. We denote by Σ^2_r and Γ^2_r the Bott–Samelson variety and the set of combinatorial galleries respectively.

We also consider the Bott–Samelson variety corresponding to the empty sequence ($r = 0$). This is the one-point variety $\Sigma = \Gamma = \{\varnothing\}$.

In what follows, we shall always consider a ring of coefficients k which is a principal ideal domain of characteristic not equal to 2 if the root system contains a component of type C_n. As the ordinary cohomology $H^\bullet(\Sigma, k)$ vanishes in odd degrees and is a free k-module in each degree, the degeneracy of the Leray spectral sequence at the E_2-term implies

$$H^\bullet_T(\Sigma, k) \simeq H^\bullet(\Sigma, k) \otimes_k S_k.$$

Therefore, we can apply the first part of Corollary 2.5 to prove that the restriction morphism $H^\bullet_T(\Sigma, k) \to H^\bullet_T(\Gamma, k)$ is an embedding. We denote its image by $\mathcal{X}(k)$.

Similarly, $H^\bullet_T(\Sigma_x, k) \simeq H^\bullet(\Sigma_x, k) \otimes_k S_k$ and we can apply Corollary 2.5 to prove that the restriction morphism $H^\bullet_T(\Sigma_x, k) \to H^\bullet_T(\Gamma_x, k)$ is an embedding. We denote its image by $\mathcal{X}_x(k)$.

In order to ensure conditions (C1)–(C3) of Corollary 2.5, we want to fix the ring $\mathbb{Z}'$ for each root system as follows: $\mathbb{Z}' = \mathbb{Z}[1/2]$ if the root system contains a component of type C_n and $\mathbb{Z}' = \mathbb{Z}$ otherwise. This choice automatically guarantees that Theorem 2.4 and Corollary 2.5 hold for $k = \mathbb{Z}'$, since $\Lambda_T \subset \Phi$ in these assertions.

Therefore, from now on, we will assume that the cohomologies (ordinary and equivariant) are taken with coefficients $\mathbb{Z}'$ unless otherwise explicitly stated. We also set $S = S_{\mathbb{Z}'}$, $\mathcal{X} = \mathcal{X}(\mathbb{Z}')$ and $\mathcal{X}_x = \mathcal{X}_x(\mathbb{Z}')$.

Note that all the above constructions are also valid for the Kac–Moody groups [14, 6.1.16]. These groups have standard Borel subgroups, standard maximal tori and standard parabolic subgroups [14, 6.17, 6.18], which can be used to define the Bott–Samelson varieties (also called Bott–Samelson–Demazure–Hansen varieties) similarly to how they were defined at the beginning of this section [14, 7.1.3]. We therefore prefer to carry out our calculations in the finite case, implying that they are all true in the affine case as well.

4. Bases of the images $\mathcal{X}$ and $\mathcal{X}_x$

4.1. Härterich's localization theorems. We formulate here the following two results due to Härterich [11]. It is important to note that one needs to be more careful with the ring of coefficients when applying the localization theorems in the proofs of these results. For our ring of coefficients $\mathbb{Z}'$, one can apply Theorem 2.4 and Corollary 2.5.

Proposition 4.1 [11, Theorem 6.2]. *An element $f \in H_T^\bullet(\Gamma)$ belongs to the image $\mathcal{X}$ of the restriction $i^*_{\Sigma,\Gamma} : H_T^\bullet(\Sigma) \to H_T^\bullet(\Gamma)$ if and only if*

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \mod \alpha^{|J_\alpha(\gamma)|}$$

for any positive root α and gallery $\gamma \in \Gamma$.

Proposition 4.2 [11, Theorem 6.3]. *An element $f \in H_T^\bullet(\Gamma_x)$ belongs to the image $\mathcal{X}_x$ of the restriction $i^*_{\Sigma_x,\Gamma_x} : H_T^\bullet(\Sigma_x) \to H_T^\bullet(\Gamma_x)$ if and only if*

$$\sum_{\delta\in\Gamma_x,\delta\sim_\alpha\gamma,D_\alpha(\delta)\subset D_\alpha(\gamma)} (-1)^{|D_\alpha(\delta)|} f(\delta) \equiv 0 \mod \alpha^{|D_\alpha(\gamma)|}$$

for any positive root α and gallery $\gamma \in \Gamma_x$.

The reader can either find the proofs of these results in Härterich's original preprint [11] or derive them from the proof of Proposition 5.2 by similarity.

4.2. Copy and concentration. In this section, we describe two ways to get elements of $\mathcal{X}$ from elements of $\mathcal{X}'$. Suppose that $r > 0$. For $f' \in H_T^\bullet(\Gamma')$, we define its *copy* $\Delta f' \in H_T^\bullet(\Gamma)$ by $\Delta f'(\gamma) = f'(\gamma')$ for any $\gamma \in \Gamma$. Clearly, Δ is an S-linear operation.

Lemma 4.3. *It holds that $\Delta f' \in \mathcal{X}$ if $f' \in \mathcal{X}'$.*

Proof. By Proposition 4.1, we must prove that

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f'(\delta') \equiv 0 \mod \alpha^{|J_\alpha(\gamma)|} \tag{4.1}$$

for any $\gamma \in \Gamma$ and $\alpha \in \Phi^+$.

Case 1. $r \notin M_\alpha(\gamma)$. In this case, $\delta \sim_\alpha \gamma$ implies $\delta_r = \gamma_r$. Therefore, we can rewrite (4.1) as

$$\sum_{\delta'\in\Gamma',\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \mod \alpha^{|J_\alpha(\gamma')|}, \tag{4.2}$$

which holds by Proposition 4.1 applied to $f' \in \mathcal{X}'$.

Case 2. $r \in M_\alpha(\gamma)\backslash J_\alpha(\gamma)$. Choosing in (4.1) the gallery δ so that $r \notin J_\alpha(\delta)$, we can rewrite this equivalence as (4.2).

Case 3. $r \in J_\alpha(\gamma)$. Consider the following equivalence relation on the set $\{\delta \in \Gamma \mid \delta \sim_\alpha \gamma\}$: $\delta \equiv \tau \Leftrightarrow \delta' = \tau'$. Clearly, every equivalence class of this relation consists of exactly two elements. Therefore, the sum in (4.1) can be broken into a sum of the following subsums:

$$(-1)^{|J_\alpha(\delta)|} f'(\delta') + (-1)^{|J_\alpha(\tau)|} f'(\tau')$$

for different $\delta \equiv \tau$. As $|J_\alpha(\delta)|$ and $|J_\alpha(\tau)|$ have different parities, the above sum equals zero. □

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For $f' \in H_T^\bullet(\Gamma')$ and $t \in \{e, s_r\}$, we define $\nabla_t f' \in H_T^\bullet(\Gamma)$, called the *concentration* of f' at t, by

$$\nabla_t f'(\gamma) = \begin{cases} \beta_r(\gamma) f'(\gamma') & \text{if } \gamma_r = t, \\ 0 & \text{otherwise,} \end{cases}$$

for any $\gamma \in \Gamma$. Clearly, ∇_t is an S-linear operation.

Lemma 4.4. *It holds that $\nabla_t f' \in \mathcal{X}$ if $f' \in \mathcal{X}'$.*

Proof. We shall give the proof for $\nabla_e f'$, the proof for $\nabla_{s_r} f'$ being similar.

By Proposition 4.1, we must prove that

$$\sum_{\delta \in \Gamma, \delta \sim_\alpha \gamma, \delta_r = e, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} \beta_r(\delta) f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|} \tag{4.3}$$

for any $\gamma \in \Gamma$ and $\alpha \in \Phi^+$. Clearly, it suffices to consider the case $J_\alpha(\gamma) \neq \emptyset$. We shall use the notation $M_\alpha(\gamma) = \{i_1 < \cdots < i_\ell\}$ and $x = \pi(\gamma)$.

Case 1. $r \notin M_\alpha(\gamma)$. In this case, $\delta \sim_\alpha \gamma$ implies $\delta_r = \gamma_r$. Thus, it suffices to consider the case $\gamma_r = e$, as otherwise our sum is equal to zero. We can rewrite (4.3) as

$$\sum_{\delta \in \Gamma_x, \delta \sim_\alpha \gamma, \delta_r = e, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} x(-\alpha_r) f'(\delta') \\ + \sum_{\delta \in \Gamma_{s_\alpha x}, \delta \sim_\alpha \gamma, \delta_r = e, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} s_\alpha x(-\alpha_r) f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}.$$

As $s_\alpha x(-\alpha_r) \equiv x(-\alpha_r) \bmod \alpha$, it suffices to prove that

$$\sum_{\delta \in \Gamma, \delta \sim_\alpha \gamma, \delta_r = e, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|} \tag{4.4}$$

and

$$\sum_{\delta \in \Gamma_x, \delta \sim_\alpha \gamma, \delta_r = e, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|-1}. \tag{4.5}$$

We can rewrite (4.4) as

$$\sum_{\delta' \in \Gamma', \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma')|}.$$

It holds by Proposition 4.1 applied to $f' \in \mathcal{X}'$. Noting that $J_\alpha(\delta) \subset J_\alpha(\gamma)$ is equivalent to $D_\alpha(\delta') \subset D_\alpha(\gamma')$ in (4.5) by Proposition 3.1, we can rewrite (4.5) as

$$\sum_{\delta' \in \Gamma_x, \delta' \sim_\alpha \gamma', D_\alpha(\delta') \subset D_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|-1}.$$

By Proposition 3.1,

$$|D_\alpha(\gamma')| \geqslant |J_\alpha(\gamma')| - 1 = |J_\alpha(\gamma)| - 1 \tag{4.6}$$

and $|J_\alpha(\delta')| = |D_\alpha(\delta')|$ for $s_\alpha x > x$ and $|J_\alpha(\delta')| = |D_\alpha(\delta')| + 1$ for $s_\alpha x < x$. Therefore, the above equivalence follows from Proposition 4.2 applied to the element $f'|_{\Gamma'_x}$, which

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belongs to $\mathcal{X}'_x$ as is easy to see from the following commutative diagram:

$$\begin{array}{ccc} H^\bullet_T(\Sigma') & \longrightarrow & H^\bullet_T(\Sigma'_x) \\ \downarrow & & \downarrow \\ H^\bullet_T(\Gamma') & \longrightarrow & H^\bullet_T(\Gamma'_x) \end{array}$$

Case 2. $r \in M_\alpha(\gamma)\backslash J_\alpha(\gamma)$. In this case, $i_\ell = r$, $|D_\alpha(\gamma)| = |J_\alpha(\gamma)|$ and $s_\alpha x > x$ by Proposition 3.1. If δ belonged to $\Gamma_{s_\alpha x}$ in (4.3), we would get by Proposition 3.1 that $r \in J_\alpha(\delta)$ and thus the inclusion $J_\alpha(\delta) \subset J_\alpha(\gamma)$ would not hold. On the other hand, for any $\delta \in \Gamma_x$ such that $\delta \sim_\alpha \gamma$, we have $r \in M_\alpha(\delta)\backslash J_\alpha(\delta)$, whence $\boldsymbol{\beta}_r(\delta) = -\alpha$. Therefore, it suffices to prove that

$$\sum_{\delta' \in \Gamma_x, \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|-1}. \tag{4.7}$$

If $\gamma' \in \Gamma_x$, then by Proposition 3.1 the summation runs over $\delta' \in \Gamma_x$ such that $\delta' \sim_\alpha \gamma'$ and $D_\alpha(\delta') \subset D_\alpha(\gamma')$. Therefore, (4.7) follows from (4.6) and Proposition 4.2 applied to $f'|_{\Gamma'_x}$.

We assume now that $\gamma' \in \Gamma_{xs_r} = \Gamma_{s_\alpha x}$. Note that $M_\alpha(\gamma') = \{i_1 < \cdots < i_{\ell-1}\}$. This set is not empty (that is, $\ell > 1$), as $s_\alpha \pi(\gamma') = x < s_\alpha x = \pi(\gamma')$, whence $i_{\ell-1} \in J_\alpha(\gamma')$. Consider the gallery $\widetilde{\gamma}'$ that is obtained from γ' by replacing $\gamma_{i_{\ell-1}}$ with $\gamma_{i_{\ell-1}} s_{i_{\ell-1}}$. We clearly have

$$\widetilde{\gamma}' \sim_\alpha \gamma', \quad \widetilde{\gamma}' \in \Gamma_x, \quad J_\alpha(\widetilde{\gamma}') = J_\alpha(\gamma')\backslash\{i_{\ell-1}\}, \quad D_\alpha(\widetilde{\gamma}') = D_\alpha(\gamma').$$

Finally it remains to note that in (4.7), we have $s_\alpha \pi(\delta') = s_\alpha x > x = \pi(\delta')$, whence $i_{\ell-1} \notin J_\alpha(\delta')$. Thus $J_\alpha(\delta') \subset J_\alpha(\gamma')$ is equivalent to $J_\alpha(\delta') \subset J_\alpha(\widetilde{\gamma}')$ and hence by Proposition 3.1 to $D_\alpha(\delta') \subset D_\alpha(\widetilde{\gamma}')$. Thus we can rewrite (4.7) as follows

$$\sum_{\delta' \in \Gamma_x, \delta' \sim_\alpha \widetilde{\gamma}', D_\alpha(\delta') \subset D_\alpha(\widetilde{\gamma}')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|-1}.$$

This equivalence again follows from (4.6) and Proposition 4.2 applied to $f'|_{\Gamma'_x}$.

Case 3. $r \in J_\alpha(\gamma)$. In this case, $i_\ell = r$ and $s_\alpha x < x$. We can rewrite (4.3) as

$$\sum_{\delta' \in \Gamma_x, \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|+1} \alpha f'(\delta') - \sum_{\delta' \in \Gamma_{s_\alpha x}, \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} \alpha f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}.$$

It suffices to prove that

$$\sum_{\delta' \in \Gamma_x, \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') + \sum_{\delta' \in \Gamma_{s_\alpha x}, \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|-1},$$

which follows from Proposition 4.1, as $|J_\alpha(\gamma)| - 1 = |J_\alpha(\gamma')|$. □

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For notational purposes, its convenient to define

$$\widetilde{\nabla}_t f'(\gamma) = \begin{cases} \widetilde{\beta}_r(\gamma) f'(\gamma') & \text{if } \gamma_r = t, \\ 0 & \text{otherwise.} \end{cases}$$

COROLLARY 4.5. *It holds that* $\widetilde{\nabla}_t f' \in \mathcal{X}$ *if* $f' \in \mathcal{X}'$.

PROOF. The result follows from $\widetilde{\nabla}_e f' = \nabla_e f'$ and $\widetilde{\nabla}_{s_r} f' = -\nabla_{s_r} f'$. □

4.3. Folding the ends. For $r > 0$, we define the automorphism $\gamma \mapsto \dot{\gamma}$ of Γ by

$$\dot{\gamma}_i = \begin{cases} \gamma_i & \text{if } i < r, \\ s_r \gamma_r & \text{if } i = r. \end{cases}$$

It satisfies the following properties:

- $M_\alpha(\dot{\gamma}) = M_\alpha(\gamma)$;
- $\dot{\delta} \sim_\alpha \dot{\gamma} \iff \delta \sim_\alpha \gamma$;
- if $r \notin M_\alpha(\gamma)$, then $J_\alpha(\dot{\gamma}) = J_\alpha(\gamma)$. If $r \in M_\alpha(\gamma)$, then $J_\alpha(\dot{\gamma}) = J_\alpha(\gamma) \triangle \{r\}$, where $\triangle$ stands for the symmetric difference;
- $D_\alpha(\dot{\gamma}) = D_\alpha(\gamma)$;
- $\dot{\gamma} \in \Gamma_x \Leftrightarrow \gamma \in \Gamma_{xs_r}$,

whose proofs are left to the reader.

This automorphism of Γ induces an automorphism of $H_T^\bullet(\Gamma)$ by $\dot{f}(\gamma) = f(\dot{\gamma})$. Clearly, these automorphisms are of order 2.

LEMMA 4.6. *It holds that* $\dot{\mathcal{X}} = \mathcal{X}$, $\dot{\mathcal{X}}_x = \mathcal{X}_{xs_r}$.

PROOF. Actually we only have to prove that $\dot{\mathcal{X}} \subset \mathcal{X}$. Take any $f \in \mathcal{X}$. By Proposition 4.1, we must check the equivalence

$$\sum_{\delta \in \Gamma, \delta \sim_\alpha \gamma, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|J_\alpha(\gamma)|} \tag{4.8}$$

for arbitrary $\gamma \in \Gamma$ and $\alpha \in \Phi^+$.

Case 1. $r \notin M_\alpha(\gamma)$. In this case, we can rewrite (4.8) as

$$\sum_{\delta \in \Gamma, \dot{\delta} \sim_\alpha \dot{\gamma}, J_\alpha(\dot{\delta}) \subset J_\alpha(\dot{\gamma})} (-1)^{|J_\alpha(\dot{\delta})|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|J_\alpha(\dot{\gamma})|}.$$

It holds by Proposition 4.1.

Case 2. $r \in M_\alpha(\gamma) \backslash J_\alpha(\gamma)$. In this case, $\gamma \sim_\alpha \dot{\gamma}$ and $J_\alpha(\dot{\gamma}) = J_\alpha(\gamma) \sqcup \{r\}$. We can rewrite (4.8) as[1]

$$\sum_{\delta \in \Gamma, \dot{\delta} \sim_\alpha \dot{\gamma}, r \in J_\alpha(\dot{\delta}), J_\alpha(\dot{\delta}) \subset J_\alpha(\dot{\gamma})} (-1)^{|J_\alpha(\dot{\delta})|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|J_\alpha(\gamma)|}. \tag{4.9}$$

[1]If $r \notin A$, then $B \subset A$ if and only if $r \in B \triangle \{r\}$ and $B \triangle \{r\} \subset A \triangle \{r\}$.

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To prove it, let us write the two equivalences

$$\sum_{\delta\in\Gamma,\dot{\delta}\sim_\alpha\dot{\gamma},J_\alpha(\dot{\delta})\subset J_\alpha(\dot{\gamma})}(-1)^{|J_\alpha(\dot{\delta})|}f(\dot{\delta})\equiv 0 \bmod \alpha^{|J_\alpha(\dot{\gamma})|},$$
$$\sum_{\delta\in\Gamma,\dot{\delta}\sim_\alpha\gamma,J_\alpha(\dot{\delta})\subset J_\alpha(\gamma)}(-1)^{|J_\alpha(\dot{\delta})|}f(\dot{\delta})\equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|},$$

which hold by Proposition 4.1. Subtracting the latter from the former and considering everything modulo $\alpha^{|J_\alpha(\gamma)|}$, we get (4.9).

Case 3. $r\in J_\alpha(\gamma)$. In this case, $\gamma\sim_\alpha\dot{\gamma}$ and we can rewrite (4.8) as[1]

$$-\sum_{\delta\in\Gamma,\dot{\delta}\sim_\alpha\gamma,J_\alpha(\dot{\delta})\subset J_\alpha(\gamma)}(-1)^{|J_\alpha(\dot{\delta})|}f(\dot{\delta})\equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}.$$

It holds by Proposition 4.1. □

4.4. Fixing the ends. Let $r>0$. Consider the natural embedding $\iota:\Sigma'\hookrightarrow\Sigma$ defined by $[p_1,\dots,p_{r-1}]\mapsto[p_1,\dots,p_{r-1},e]$. This is a B-equivariant hence also a T-equivariant embedding. We get the following commutative diagram for restrictions:

$$\begin{array}{ccc} H_T^\bullet(\Sigma) & \longrightarrow & H_T^\bullet(\Sigma') \\ \downarrow & & \downarrow \\ H_T^\bullet(\Gamma) & \longrightarrow & H_T^\bullet(\Gamma') \end{array}$$

Let f be an element of $\mathcal{X}$ (that is, in the image of the left arrow). It follows from the commutativity of the above diagram that the composition $f'=f\circ\iota$ belongs to $\mathcal{X}'$ (that is, to the image of the right arrow).

LEMMA 4.7. *Let $f\in\mathcal{X}$, $r>0$ and $t\in\{e,s_r\}$. We define $f'\in H_T^\bullet(\Gamma')$ by $f'(\gamma')=f(\gamma'\cdot t)$. Then $f'\in\mathcal{X}'$.*

PROOF. The argument preceding the formulation of this lemma proves the claim for $t=e$. Now let $t=s_r$. By Lemma 4.6, we get $\dot{f}\in\mathcal{X}$. Then by the case $t=e$, we get $\dot{f}\circ\iota\in\mathcal{X}'$. The result follows from

$$\dot{f}\circ\iota(\gamma')=\dot{f}(\gamma'\cdot e)=f(\gamma'\cdot s_r)=f'(\gamma').$$ □

LEMMA 4.8. *Let $f\in\mathcal{X}$, $r>0$ and $t\in\{e,s_r\}$. Suppose that $f(\gamma)=0$ for all γ such that $\gamma_r\neq t$. Then $f(\gamma)$ is divisible in S by $\boldsymbol{\beta}_r(\gamma)$ for any $\gamma\in\Gamma$. Moreover, the function $\gamma'\mapsto f(\gamma'\cdot t)/\boldsymbol{\beta}_r(\gamma'\cdot t)$, where $\gamma'\in\Gamma'$, belongs to $\mathcal{X}'$.*

PROOF. We shall prove the first claim by induction with respect to $\trianglelefteq$. Suppose that $f(\delta)$ is divisible by $\boldsymbol{\beta}_r(\delta)$ for any $\delta\vartriangleleft\gamma$. We must prove that $f(\gamma)$ is divisible by $\boldsymbol{\beta}_r(\gamma)$. Clearly, we need only to consider the case $\gamma_r=t$.

[1]If $r\in A$, then $B\subset A$ if and only if $B\triangle\{r\}\subset A$.

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We take for α the positive of the two roots $\boldsymbol{\beta}_r(\gamma)$ and $-\boldsymbol{\beta}_r(\gamma)$. Thus, $r \in M_\alpha(\gamma)$.

Case 1. $r \in J_\alpha(\gamma)$. In this case, $|J_\alpha(\gamma)| > 0$. Thus, by Proposition 4.1,

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha.$$

As $\delta \sim_\alpha \gamma$ and $J_\alpha(\delta) \subset J_\alpha(\gamma)$ imply $\delta \trianglelefteq \gamma$, the claim follows.

Case 2. $r \notin J_\alpha(\gamma)$. In this case, $r \in J_\alpha(\dot\gamma)$, whence $|J_\alpha(\dot\gamma)| > 0$. Moreover, $\dot\gamma \sim_\alpha \gamma$. By Proposition 4.1,

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,\delta_r=t,J_\alpha(\delta)\subset J_\alpha(\dot\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha. \tag{4.10}$$

We claim that

$$\delta \sim_\alpha \gamma, \delta_r = t, J_\alpha(\delta) \subset J_\alpha(\dot\gamma) \Longrightarrow \delta \trianglelefteq \gamma. \tag{4.11}$$

We get $\delta \lhd \dot\gamma$ as $\delta_r = t \neq \dot\gamma_r$. Thus, there exists some i_0 such that $\delta^{i_0} < \dot\gamma^{i_0}$ and $\delta_i = \gamma_i$ for $i < i_0$. If $i_0 < r$ then $\delta^{i_0} < \gamma^{i_0}$ and thus $\delta \lhd \gamma$. On the other hand, if $i_0 = r$ then $\gamma = \delta$, since $\delta_r = t = \gamma_r$.

Now it follows from (4.10), (4.11) and the inductive hypothesis that $f(\gamma)$ is divisible by α.

Let us prove the second claim. We denote by f' the function under consideration: $f'(\gamma') = f(\gamma' \cdot t)/\boldsymbol{\beta}_r(\gamma' \cdot t)$. By Proposition 4.1, we must check the equivalence

$$\sum_{\delta'\in\Gamma',\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma')|} \tag{4.12}$$

for any $\gamma' \in \Gamma'$ and $\alpha \in \Phi^+$. Clearly, we can assume that $J_\alpha(\gamma') \neq \varnothing$. We set $\gamma := \gamma' \cdot t$. Let us fix the notation

$$M_\alpha(\gamma) = \{i_1 < \cdots < i_\ell\}, \quad y = \pi(\gamma').$$

By Proposition 4.1,

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}. \tag{4.13}$$

Case 1. $r \notin M_\alpha(\gamma)$. We can rewrite (4.13) as

$$\begin{aligned} & yt(-\alpha_r) \sum_{\delta\in\Gamma_{yt},\delta\sim_\alpha\gamma,\delta_r=t,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f'(\delta') \\ & \quad + s_\alpha yt(-\alpha_r) \sum_{\delta\in\Gamma_{s_\alpha yt},\delta\sim_\alpha\gamma,\delta_r=t,J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}. \end{aligned}$$

We have $s_\alpha yt(-\alpha_r) = yt(-\alpha_r) + c\alpha$ for some $c \in \mathbb{Z}$. Thus, the above equivalence takes the form

$$\begin{aligned} & s_\alpha yt(-\alpha_r) \sum_{\delta'\in\Gamma',\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \\ & \quad - c\alpha \sum_{\delta'\in\Gamma'_y,\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \bmod \alpha^{|J_\alpha(\gamma')|}. \end{aligned} \tag{4.14}$$

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Our aim is to get rid of the second line. As the restriction $f|_{\Gamma_{yt}}$ belongs to $\mathcal{X}^{yt}$, Proposition 4.2 implies that

$$\sum_{\delta\in\Gamma_{yt},\delta\sim_\alpha\gamma,\delta_r=t,D_\alpha(\delta)\subset D_\alpha(\gamma)}(-1)^{|D_\alpha(\delta)|}f(\delta)\equiv 0 \mod \alpha^{|D_\alpha(\gamma)|},$$

which can be written as

$$\pm yt(-\alpha_r)\sum_{\delta'\in\Gamma_y,\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')}(-1)^{|J_\alpha(\delta)|}f'(\delta')\equiv 0 \mod \alpha^{|D_\alpha(\gamma)|},$$

where + is taken if $s_\alpha yt > yt$ and − is taken otherwise (see Proposition 3.1). Moreover, $yt(-\alpha_r) = \gamma^r(-\alpha_r)$ is not proportional to α by the hypothesis of the current case. Hence, it follows from the above equivalence that

$$c\alpha\sum_{\delta'\in\Gamma_y,\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')}(-1)^{|J_\alpha(\delta)|}f'(\delta')\equiv 0 \mod \alpha^{|D_\alpha(\gamma)|+1}.$$

It remains to note that $|J_\alpha(\gamma')| = |J_\alpha(\gamma)| \geqslant |D_\alpha(\gamma)| + 1$, add the above equivalence to (4.14) and note that $s_\alpha yt(-\alpha_r)$ is also not proportional to α.

Case 2. $r \in M_\alpha(\gamma)\backslash J_\alpha(\gamma)$. In this case, $i_\ell = r$, $\gamma\sim_\alpha\dot\gamma$ $J_\alpha(\dot\gamma) = J_\alpha(\gamma)\cup\{r\}$. By Proposition 4.1,

$$\sum_{\delta\in\Gamma,\delta\sim_\alpha\gamma,\delta_r=t,J_\alpha(\delta)\subset J_\alpha(\dot\gamma)}(-1)^{|J_\alpha(\delta)|}\boldsymbol{\beta}_r(\delta)f'(\delta')\equiv 0 \mod \alpha^{|J_\alpha(\dot\gamma)|}.$$

As $r \in J_\alpha(\dot\gamma)$, this equivalence can be rewritten as

$$\sum_{\delta'\in\Gamma',\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')}(-1)^{|J_\alpha(\delta'\cdot t)|}\boldsymbol{\beta}_r(\delta'\cdot t)f'(\delta')\equiv 0 \mod \alpha^{|J_\alpha(\gamma')|+1}. \tag{4.15}$$

Considering separately the cases $\delta' \in \Gamma'_y$ and $\delta' \in \Gamma'_{s_\alpha y}$,

$$(-1)^{|J_\alpha(\delta'\cdot t)|}\boldsymbol{\beta}_r(\delta'\cdot t)f'(\delta') = (-1)^{|J_\alpha(\delta')|}yt(-\alpha_r).$$

We know that $yt(-\alpha_r) = \boldsymbol{\beta}_r(\gamma) = -\alpha$. Thus, dividing (4.15) by $-\alpha$, we get (4.12).

Case 3. $r \in J_\alpha(\gamma)$. In this case, $i_\ell = r$ and (4.13) can be rewritten as

$$\sum_{\delta'\in\Gamma',\delta'\sim_\alpha\gamma',J_\alpha(\delta')\subset J_\alpha(\gamma')}(-1)^{|J_\alpha(\delta'\cdot t)|}\boldsymbol{\beta}_r(\delta'\cdot t)f'(\delta')\equiv 0 \mod \alpha^{|J_\alpha(\gamma')|+1}. \tag{4.16}$$

Considering separately the cases $\delta' \in \Gamma'_y$ and $\delta' \in \Gamma'_{s_\alpha y}$,

$$(-1)^{|J_\alpha(\delta'\cdot t)|}\boldsymbol{\beta}_r(\delta'\cdot t)f'(\delta') = -(-1)^{|J_\alpha(\delta')|}yt(-\alpha_r).$$

We know that $yt(-\alpha_r) = \boldsymbol{\beta}_r(\gamma) = \alpha$. Thus, dividing (4.16) by $-\alpha$, we get (4.12). □

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4.5. Bases for $\mathcal{X}$. For any $\rho \in \Upsilon$, we construct the subset B_ρ of $H^\bullet(\Sigma)$ inductively by

$$B_\varnothing = \{1\}, \quad B_\rho = \Delta(B_{\rho'_0}) \cup \nabla_{\rho_\varnothing}(B_{\rho'_1}).$$

By Lemmas 4.3 and 4.4, we get $B_\rho \subset \mathcal{X}$.

Theorem 4.9. *The set B_ρ is an S-basis of $\mathcal{X}$.*

Proof. We apply induction on r. This result is clearly true for $r = 0$. Therefore, we assume that $r > 0$ and that $B_{\rho'_0}$ and $B_{\rho'_1}$ are bases of $\mathcal{X}'$.

Let $f \in \mathcal{X}$. By Lemma 4.7, the function $f' \in H^\bullet_T(\Gamma')$ defined by $f'(\delta') = f(\delta' \cdot \rho_\varnothing s_r)$ belongs to $\mathcal{X}'$. We have

$$(f - \Delta(f'))(\delta' \cdot \rho_\varnothing s_r) = f(\delta' \cdot \rho_\varnothing s_r) - \Delta(f')(\delta' \cdot \rho_\varnothing s_r) = f'(\delta') - f'(\delta') = 0$$

for any $\delta' \in \Gamma'$. Let us define

$$h'(\delta') = \frac{(f - \Delta(f'))(\delta' \cdot \rho_\varnothing)}{\boldsymbol{\beta}_r(\delta' \cdot \rho_\varnothing)}$$

for any $\delta' \in \Gamma'$. By Lemma 4.8, h' is a well-defined function of $\mathcal{X}'$. The above formulas show that $f - \Delta(f') = \nabla_{\rho_\varnothing}(h')$, whence $f = \Delta(f') + \nabla_{\rho_\varnothing}(h')$. By the inductive hypothesis and the linearity of Δ and $\nabla_{\rho_\varnothing}$, the function f belongs to the S-span of B_ρ.

It remains to prove the S-linear independence of elements of B_ρ. Let $B_{\rho'_0} = \{b^{(0)}_1, \dots, b^{(0)}_{n_0}\}$ and $B_{\rho'_1} = \{b^{(1)}_1, \dots, b^{(1)}_{n_1}\}$. Suppose that

$$\sum_{i=1}^{n_0} \alpha^{(0)}_i \Delta(b^{(0)}_i) + \sum_{i=1}^{n_1} \alpha^{(1)}_i \nabla_{\rho_\varnothing}(b^{(1)}_i) = 0$$

for some $\alpha^{(0)}_i, \alpha^{(1)}_i \in S$. Consider the decomposition $\Gamma = \Gamma' \cdot \rho_\varnothing \sqcup \Gamma' \cdot \rho_\varnothing s_r$. Restricting the above equality to $\Gamma' \cdot \rho_\varnothing s_r$, we get $\sum_{i=1}^{n_0} \alpha^{(0)}_i b^{(0)}_i = 0$. Hence all $\alpha^{(0)}_i = 0$ and $\sum_{i=1}^{n_1} \alpha^{(1)}_i \nabla_{\rho_\varnothing}(b^{(1)}_i) = 0$. Thus, $\sum_{i=1}^{n_1} \alpha^{(1)}_i \boldsymbol{\beta}_r(\delta) b^{(1)}_i(\delta') = 0$ for any $\delta \in \Gamma' \cdot \rho_\varnothing s_r$. Cancelling a nonzero element $\boldsymbol{\beta}_r(\delta)$, we get that $\sum_{i=1}^{n_1} \alpha^{(1)}_i b^{(1)}_i(\delta') = 0$ for any $\delta' \in \Gamma'$. Hence all $\alpha^{(1)}_i = 0$. □

4.6. Basis for $\mathcal{X}_x$. For any gallery $\gamma \in \Gamma$, we define

$$\mathbf{a}(\gamma) = \prod_{i \in D(\gamma)} \widetilde{\boldsymbol{\beta}}_i(\gamma) = \prod_{\alpha \in \Phi^+} \alpha^{|D_\alpha(\gamma)|},$$

$$\mathbf{b}_\varnothing = 1, \quad \mathbf{b}_\gamma = \begin{cases} \Delta(\mathbf{b}_{\gamma'}) & \text{if } r \notin D(\gamma), \\ \widetilde{\nabla}_{\gamma_r}(\mathbf{b}_{\gamma'}) & \text{if } r \in D(\gamma). \end{cases}$$

By Lemma 4.3 and Corollary 4.5, we get $\mathbf{b}_\gamma \in \mathcal{X}$.

Lemma 4.10. *Let $\gamma \in \Gamma_x$. Then $\mathbf{b}_\gamma(\gamma) = \mathbf{a}(\gamma)$ and $\mathbf{b}_\gamma(\delta) = 0$ for any $\delta \in \Gamma_x$ such that $\delta < \gamma$.*

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PROOF. The first formula follows directly from the definition of $\mathbf{b}_\gamma$ and $\widetilde{\nabla}_t$. Let us prove the second claim inductively. From $\delta < \gamma$ it follows that there exists some $i_0 = 1, \dots, r-1$ such that $\delta^{i_0} < \gamma^{i_0}$ and $\delta^i = \gamma^i$ for $i > i_0$. Clearly $\delta' < \gamma'$. Assume that $\delta_r = \gamma_r$. Then $\delta', \gamma' \in \Gamma_{x\gamma_r}$. Thus by induction, $\mathbf{b}_\gamma(\delta) = \mathbf{b}_{\gamma'}(\delta') = 0$ if $r \notin D(\gamma)$ and $\mathbf{b}_\gamma(\delta) = \widetilde{\boldsymbol{\beta}}_r(\delta)\mathbf{b}_{\gamma'}(\delta') = 0$ if $r \in D(\gamma)$. Now assume on the contrary that $\delta_r \neq \gamma_r$. However, $\delta^r = \gamma^r = x$. Hence $\delta^{r-1} \neq \gamma^{r-1}$. This means that $i_0 = r-1$ and $\gamma^{r-1}s_r = \delta^{r-1} < \gamma^{r-1}$. Hence $r \in D(\gamma)$. Therefore, $\mathbf{b}_\gamma(\delta) = \widetilde{\nabla}_{\gamma_r}(\mathbf{b}_{\gamma'})(\delta) = 0$. □

THEOREM 4.11. *The set* $\{\mathbf{b}_\gamma|_{\Gamma_x} \mid \gamma \in \Gamma_x\}$ *is an* S*-basis of* $\mathcal{X}_x$. *In particular, the restriction* $\mathcal{X} \to \mathcal{X}_x$ *is surjective.*

PROOF. This set is S-linearly independent by Lemma 4.10, as $<$ is a total order on Γ_x and $\mathbf{a}(\gamma) \neq 0$. Let us prove that any element $f \in \mathcal{X}_x$ is representable as an S-linear combination. We apply induction on the cardinality of the set

$$C(f) = \{\delta \in \Gamma_x \mid \text{ there exists } \gamma \in \Gamma_x \text{ such that } \delta \geqslant \gamma \text{ and } f(\gamma) \neq 0\},$$

the upper closure of the support of f. If $C(f) = \varnothing$, then $f = 0$ and the result follows. Suppose now that $C(f) \neq \varnothing$ and let γ be its minimal element with respect to $<$. As

$$\delta \in \Gamma_x, \quad \delta \sim_\alpha \gamma, \quad D_\alpha(\delta) \subset D_\alpha(\gamma) \Longrightarrow \delta \leqslant \gamma, \tag{4.17}$$

Proposition 4.2 implies that $f(\gamma)$ is divisible by $\prod_{\alpha \in \Phi^+} \alpha^{|D_\alpha(\gamma)|} = \mathbf{a}(\gamma)$. Consider the difference $h = f - f(\gamma)/\mathbf{a}(\gamma)\mathbf{b}_\gamma$. By Lemma 4.10, we get $C(h) \subset \{\delta \in \Gamma_x \mid \delta > \gamma\} \subsetneq C(f)$. By induction, h belongs to the S-span of our set. Thus, so does f. □

COROLLARY 4.12. *The restrictions* $H^\bullet_T(\Sigma) \to H^\bullet_T(\Sigma_x)$ *and* $H^\bullet(\Sigma) \to H^\bullet(\Sigma_x)$ *are surjective.*

PROOF. The surjectivity of the first morphism follows from Theorem 4.11. As Σ and Σ_x are equivariantly formal, the second morphism is obtained from the first one by applying $? \otimes_S \mathbb{Z}'$ (where $S^i\mathbb{Z}' = 0$ for $i > 0$). Hence it is also surjective. □

REMARK 4.13. We describe how to construct the tree $\rho_r(x) \in \Upsilon$ by an element $x \in W$. If $r = 0$ then $\rho_r(x)$ is the empty tree. Now assume that $r > 0$. By a property of the Bruhat order, we have either $x > xs_r$ or $x < xs_r$. We set $\rho_r(x)_\varnothing = e$ in the former case and $\rho_r(x)_\varnothing = s_r$ in the latter case. This choice of $\rho_r(x)_\varnothing$ is actually defined by

$$x\rho_r(x)_\varnothing > x\rho_r(x)_\varnothing s_r. \tag{4.18}$$

If $r = 1$ then our algorithm stops. If $r > 1$ then we define the left subtree $\rho_r(x)'_0$ and the right subtree $\rho_r(x)'_1$ inductively by

$$\rho_r(x)'_0 = \rho_{r-1}(x\rho_r(x)_\varnothing s_r), \quad \rho_r(x)'_1 = \rho_{r-1}(x\rho_r(x)_\varnothing). \tag{4.19}$$

For example, let $r = 3$, $s = (s_1, s_2, s_1)$ and $x = s_2$, where $s_1 = s_{\alpha_1}$, $s_2 = s_{\alpha_2}$ and α_1, α_2 are simple roots of the root system of type A_2. Calculating according to the above algorithm, we obtain that $\rho_3(x)$ is the following tree:

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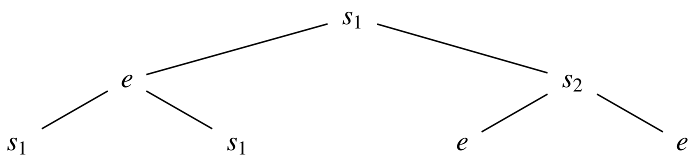

Here the left subtree is $\rho_2(x)$, the right subtree is $\rho_2(xs_1)$ and the elements of the bottom row read from left to right are $\rho_1(xs_2), \rho_1(x), \rho_1(xs_1), \rho_1(xs_1s_2)$.

By induction it is easy to prove that, up to sign, all elements $\mathbf{b}_\gamma$ with $\gamma \in \Gamma_x$ belong to $B_{\rho_r(x)}$. Indeed, this is obvious for $r = 0$. Let $r > 0$ and $\gamma \in \Gamma_x$. By induction, $\mathbf{b}_{\gamma'}$ up to sign belongs to $B_{\rho_{r-1}(x\gamma_r)}$. First, assume that $r \notin D(\gamma)$. Then $\mathbf{b}_\gamma = \Delta(\mathbf{b}_{\gamma'})$ and $\gamma^{r-1}s_r > \gamma^{r-1}$. As $\gamma^{r-1} = x\gamma_r$, we get $x\gamma_r s_r > x\gamma_r$. Hence, by (4.18), we get $\rho_r(x)_\varnothing = \gamma_r s_r$. By (4.19), we have $\rho_r(x)'_0 = \rho_{r-1}(x\rho_r(x)_\varnothing s_r) = \rho_{r-1}(x\gamma_r)$. Thus, $\mathbf{b}_\gamma$ up to sign belongs to $\Delta(B_{\rho_r(x)'_0})$, which is a subset of $B_{\rho_r(x)}$. Now assume that $r \in D(\gamma)$. Then $\mathbf{b}_\gamma = \widetilde{\nabla}_{\gamma_r}(\mathbf{b}_{\gamma'}) = \pm\nabla_{\gamma_r}(\mathbf{b}_{\gamma'})$ and $\gamma^{r-1}s_r < \gamma^{r-1}$. By (4.18) and (4.19), we get $\rho_r(x)_\varnothing = \gamma_r$ and $\rho_r(x)'_1 = \rho_{r-1}(x\rho_r(x)_\varnothing) = \rho_{r-1}(x\gamma_r)$, respectively. Thus, $\mathbf{b}_\gamma$ up to sign belongs to $\nabla_{\rho_r(x)_\varnothing}(B_{\rho_r(x)'_1})$, which is a subset of $B_{\rho_r(x)}$.

Finally, we write down the exact inductive formula for the values of the basis functions:

$$\mathbf{b}_\gamma(\delta) = \begin{cases} \mathbf{b}_{\gamma'}(\delta') & \text{if } r \notin D(\gamma), \\ (\delta')^{r-1}(-\alpha_r)\mathbf{b}_{\gamma'}(\delta') & \text{if } r \in D(\gamma) \text{ and } \delta_r = \gamma_r, \\ 0 & \text{if } r \in D(\gamma) \text{ and } \delta_r \neq \gamma_r. \end{cases} \tag{4.20}$$

5. Basis of the image $\bar{X}_x$

5.1. Localization for $\bar{\Sigma}_x$. Let k be a principal ideal domain with invertible 2 if the root system contains a component of type C_n. We are going to consider the complement $\bar{\Sigma}_x = \Sigma \backslash \pi^{-1}(x)$ to the fibre of the map $\pi : \Sigma \to G/B$, where x is a T-fixed point of G/B (see Section 3). As $\bar{\Sigma}_x$ is just a T-subspace of Σ, we can apply Theorem 2.4 to it as well. However, it is more difficult to apply Corollary 2.5. Actually, the only problem to overcome is to prove that $H^\bullet_T(\bar{\Sigma}_x, k)$ is a free S_k-module. Unfortunately, we can not solve this problem in the same way as for Σ: we do not know if $\bar{\Sigma}_x$ has an affine paving.

Consider the natural embeddings $i : \Sigma_x \hookrightarrow \Sigma$ and $j : \bar{\Sigma}_x \hookrightarrow \Sigma$. From the non-equivariant distinguished triangle

$$j_! j^* \underline{k}_\Sigma \to \underline{k}_\Sigma \to i_* i^* \underline{k}_\Sigma \xrightarrow{+1}, \tag{5.1}$$

we get the exact sequence

$$H^{2m}(\Sigma, k) \to H^{2m}(\Sigma_x, k) \to \mathbb{H}^{2m+1}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) \to H^{2m+1}(\Sigma, k) = 0.$$

The left morphism is surjective by Corollary 4.12 and the following corollary of the projection formula (cf. [12, VI.5.1]).

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Proposition 5.1. *Let $\mathbb{Z}' \to k$ be the natural ring homomorphism. For any topological space X, we get the exact sequence*

$$0 \to H_c^i(X) \otimes_{\mathbb{Z}'} k \to H_c^i(X, k) \to \mathrm{Tor}_1(H_c^{i+1}(X), k) \to 0.$$

Hence $\mathbb{H}^{2m+1}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) = 0$. Since Σ is compact,

$$0 = \mathbb{H}^{2m+1}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) = \mathbb{H}_c^{2m+1}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) = H_c^{2m+1}(\bar{\Sigma}_x, k).$$

The Poincaré duality in the form [7, Theorem 3.3.3] yields the following exact sequence:

$$0 = \mathrm{Ext}^1_{k\text{-mod}}(H_c^{2\dim\Sigma - 2m+1}(\bar{\Sigma}_x, k), k) \to H^{2m}(\bar{\Sigma}_x, k) \\ \to \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma - 2m}(\bar{\Sigma}_x, k), k) \to 0.$$

Hence,

$$H^{2m}(\bar{\Sigma}_x, k) \simeq \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma-2m}(\bar{\Sigma}_x, k), k). \tag{5.2}$$

From (5.1), we get the exact sequence

$$0 = H^{2m-1}(\Sigma_x, k) \to \mathbb{H}^{2m}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) \to H^{2m}(\Sigma, k).$$

The right cohomology is a finitely generated free k-module. We get that its submodule $\mathbb{H}^{2m}(\Sigma, j_! \underline{k}_{\bar{\Sigma}_x}) = H_c^{2m}(\bar{\Sigma}_x, k)$ is a finitely generated free k-module. Hence and from (5.2), it follows that $H^{2m}(\bar{\Sigma}_x, k)$ is also a finitely generated free k-module.

Now, again applying the Poincaré duality in the form [7, Theorem 3.3.3],

$$0 = \mathrm{Ext}^1_{k\text{-mod}}(H_c^{2\dim\Sigma-2m}(\bar{\Sigma}_x, k), k) \to H^{2m+1}(\bar{\Sigma}_x, k) \\ \to \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma-2m-1}(\bar{\Sigma}_x, k), k) = 0.$$

Hence we get $H^{2m+1}(\bar{\Sigma}_x, k) = 0$.

Now the degeneracy of the Leray spectral sequence at the E_2-term implies

$$H_T^\bullet(\bar{\Sigma}_x, k) \simeq H^\bullet(\bar{\Sigma}_x) \otimes_k S_k.$$

This module is therefore a free S_k-module.

Let $\bar{X}_x(k)$ denote the image of the restrictions $H_T^\bullet(\bar{\Sigma}_x, k) \to H_T^\bullet(\bar{\Gamma}_x, k)$, which is injective by the first part of Corollary 2.5.

5.2. Review of Härterich's constructions. We shall briefly sketch Härterich's constructions, in order to be able to apply them to the cohomology of the difference $\bar{\Sigma}_x$ in Section 5.3 and prove the criterion (Proposition 5.2) for the image $\bar{X}_x$ of the restriction $i^*_{\bar{\Sigma}_x, \bar{\Gamma}_x} : H_T^\bullet(\bar{\Sigma}_x) \to H_T^\bullet(\bar{\Gamma}_x)$.

Let α be a positive root and $\gamma \in \Gamma$. We set $T_\alpha := \ker\alpha$ and $M_\alpha(\gamma) = \{i_1 < \cdots < i_\ell\}$. Härterich [11, Section 4] constructs the embedding $\nu_\gamma^\alpha : (G_\alpha/B_\alpha)^\ell \hookrightarrow \Sigma$ by requiring that its composition with the map $\iota : \Sigma \hookrightarrow (G/B)^r$ defined by (3.1) be equal to

$$(g_1, \ldots, g_\ell) \xmapsto{\iota\circ\nu_\gamma^\alpha} (\gamma^1_{\min}, \ldots, \gamma^{i_1-1}_{\min}, g_1\gamma^{i_1}_{\min}, \ldots, g_1\gamma^{i_2-1}_{\min}, \ldots, g_\ell\gamma^{i_\ell}_{\min}, \ldots, g_\ell\gamma^r_{\min}),$$

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where $\gamma_{\min}$ is the minimal element with respect to $\lhd$ in the $\sim_\alpha$-equivalence class of γ (that is, the unique element of this class having no load-bearing α-walls). Here and in what follows we write g instead of gB_α or gB if it is clear from the context that we consider an element of G_α/B_α or G/B respectively. Note that for $\ell = 0$, the map v_γ^α takes $(G_\alpha/B_\alpha)^\ell$ isomorphically to $\{\gamma\}$.

Clearly, v_γ^α depends only on the $\sim_\alpha$-equivalence class of γ. Corollary 4.4 from [11] states

$$\Sigma^{T_\alpha} = \bigsqcup_{\gamma\in\mathrm{rep}(\Gamma,\sim_\alpha)} \mathrm{im}\, v_\gamma^\alpha. \tag{5.3}$$

We note that to ensure the T-equivariance of v_γ^α, we must define an appropriate T-action on $(G_\alpha/B_\alpha)^\ell$. This can be done as follows:

$$t\cdot(g_1,\ldots,g_\ell) = (tg_1t^{-1},\ldots,tg_\ell t^{-1}).$$

Similarly, we can consider the Bott–Samelson variety Σ_ℓ^2 for the subgroup G_α of G (generated by the unipotent root subgroups U_α and $U_{-\alpha}$) using the sequence $(\alpha,\ldots,\alpha)$ of length ℓ. Recall that we denote by Γ_ℓ^2 the set points of Σ_ℓ^2 fixed by the maximal torus $G_\alpha\cap T$ of G_α. We identify this set with the set of combinatorial galleries $(\gamma_1,\ldots,\gamma_\ell)$, where $\gamma_i = e$ or $\gamma_i = s_\alpha$.

The isomorphism $\iota:\Sigma_\ell^2 \xrightarrow{\sim} G_\alpha/B_\alpha$ becomes an isomorphism of T-spaces if we define the following T-action on Σ_ℓ^2:

$$t\cdot[p_1,\ldots,p_\ell] := [tp_1t^{-1},\ldots,tp_\ell t^{-1}].$$

The set of T-fixed points of Σ_ℓ^2 is again Γ_ℓ^2.

It is clear that $(\mathrm{im}\, v_\gamma^\alpha)^T = \{\delta\in\Gamma \mid \delta\sim_\alpha\gamma\}$. We can compute the preimage of each point of the last set with respect to the map $v_\alpha^\gamma\circ\iota:\Sigma_\ell^2\to\Sigma$. Indeed, without loss of generality, it suffices to compute $(v_\alpha^\gamma\circ\iota)^{-1}(\gamma)$. Let us define

$$g_j = \begin{cases} s_\alpha & \text{if } i_j\in J_\alpha(\gamma),\\ e & \text{otherwise.}\end{cases}$$

This definition and [11, Remark preceeding (4.1)] ensures $v_\gamma^\alpha(g_1,\ldots,g_\ell)=\gamma$. We define $\bar g := [g_1, g_1^{-1}g_2,\ldots,g_{\ell-1}^{-1}g_\ell]$ as an element of $(G_\alpha/B_\alpha)^\ell$. Hence $v_\alpha^\gamma\circ\iota(\bar g)=\gamma$. The equivalence

$$j\in J(\bar g)\Leftrightarrow \bar g^j s_\alpha < \bar g^j \Leftrightarrow g_j = \bar g^j = s_\alpha \Leftrightarrow i_j\in J_\alpha(\gamma)$$

proves that

$$i_{J(\delta)} = J_\alpha(v_\gamma^\alpha\circ\iota(\delta)) \tag{5.4}$$

for any $\delta\in\Gamma_\ell^2$.

Now we are going to explain how to compute the intersection $\mathrm{im}\, v_\gamma^\alpha\cap\pi^{-1}(x)$. Suppose that $\ell>0$ and $v_\gamma^\alpha(g_1,\ldots,g_\ell)\in\pi^{-1}(x)$. Consider the following commutative diagram:

$$\begin{array}{ccccc} (G_\alpha/B_\alpha)^\ell & \xrightarrow{\nu_\gamma^\alpha} & \Sigma & \xrightarrow{\iota} & (G/B)^r \\ & & & \searrow^{\pi} & \downarrow {\scriptstyle \mathrm{pr}_r} \\ & & & & G/B \end{array}$$

Recalling the definition of $\iota \circ \nu_\gamma^\alpha$ (see Härterich [11, (4.1)]),

$$g_\ell \gamma_{\min}^r B = xB. \tag{5.5}$$

If $g_\ell \in B_\alpha$, then (5.5) together with the Bruhat decomposition yields $x = \gamma_{\min}^r$. On the contrary, if the last equality holds, then (5.5) is true for any $g_\ell \in B_\alpha$.

If $g_\ell \in U_\alpha s_\alpha B_\alpha$, then $g_\ell = x_\alpha(c)s_\alpha b$ for some $c \in \mathbb{C}$ and $b \in B_\alpha$. Applying [11, (4.2)], we can rewrite (5.5) as

$$g_\ell \gamma_{\min}^r B = x_\alpha(c)s_\alpha b \gamma_{\min}^r B = x_\alpha(c)s_\alpha \gamma_{\min}^r B. \tag{5.6}$$

This representation is already canonical in the sense of [13, 1.13], as

$$(s_\alpha \gamma_{\min}^r)^{-1}(\alpha) = (\gamma_{\min}^r)^{-1} s_\alpha(\alpha) = (\gamma_{\min}^r)^{-1}(-\alpha) = -(\gamma_{\min}^r)^{-1}(\alpha) < 0.$$

Now, comparing (5.6) with (5.5) by the Bruhat decomposition, we get $x = s_\alpha \gamma_{\min}^r$ and $c = 0$. This analysis proves the following formulas:

$$\operatorname{im} \nu_\gamma^\alpha \cap \pi^{-1}(x) = \begin{cases} \varnothing & \text{if } \gamma \notin \Gamma_{\{x, s_\alpha x\}}, \\ \nu_\gamma^\alpha((G_\alpha/B_\alpha)^{\ell-1} \times \{e\}) & \text{if } x = \pi(\gamma_{\min}), \\ \nu_\gamma^\alpha((G_\alpha/B_\alpha)^{\ell-1} \times \{s_\alpha\}) & \text{if } x = s_\alpha \pi(\gamma_{\min}) \end{cases} \tag{5.7}$$

if $\ell > 0$ and

$$\operatorname{im} \nu_\gamma^\alpha \cap \pi^{-1}(x) = \begin{cases} \varnothing & \text{if } \gamma \notin \Gamma_x, \\ \{\gamma\} & \text{if } \gamma \in \Gamma_x \end{cases} \tag{5.8}$$

if $\ell = 0$.

5.3. Description of $\bar{X}_x$. We will prove the following analog of Propositions 4.1 and 4.2.

PROPOSITION 5.2. *An element $f \in H_T^\bullet(\bar{\Gamma}_x)$ belongs to the image $\bar{X}_x$ of the restriction $i_{\bar{\Sigma}_x,\bar{\Gamma}_x}^* : H_T^\bullet(\bar{\Sigma}_x) \to H_T^\bullet(\bar{\Gamma}_x)$ if and only if*

$$\sum_{\delta \in \Gamma, \delta \sim_\alpha \gamma, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|} \tag{5.9}$$

for any positive root α and gallery $\gamma \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ and

$$\sum_{\delta \in \Gamma_{s_\alpha x}, \delta \sim_\alpha \gamma, D_\alpha(\delta) \subset D_\alpha(\gamma)} (-1)^{|D_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|D_\alpha(\gamma)|} \tag{5.10}$$

for any positive root α and gallery $\gamma \in \Gamma_{s_\alpha x}$.

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Proof. Subtracting $\pi^{-1}(x)$ from (5.3),

$$\bar{\Sigma}_x^{T_\alpha} = \Sigma^{T_\alpha}\backslash\pi^{-1}(x) = \bigsqcup_{\gamma\in\mathrm{rep}(\Gamma,\sim_\alpha)} \mathrm{im}\, \nu_\gamma^\alpha\backslash\pi^{-1}(x).$$

Note that $(\mathrm{im}\, \nu_\gamma^\alpha\backslash\pi^{-1}(x))^T = \{\delta\in\bar{\Gamma}_x \mid \delta\sim_\alpha\gamma\}$.

By Corollary 2.3, an element $f\in H_T^\bullet(\bar{\Gamma}_x)$ belongs to the image of $H_T^\bullet(\bar{\Sigma}_x^{T_\alpha})\to H_T^\bullet(\bar{\Gamma}_x)$ if and only if each restriction $f^\gamma := f|_{\{\delta\in\bar{\Gamma}_x\mid\delta\sim_\alpha\gamma\}}$ belongs to the image of $H_T^\bullet(\mathrm{im}\, \nu_\gamma^\alpha\backslash\pi^{-1}(x))\to H_T^\bullet(\{\delta\in\bar{\Gamma}_x\mid\delta\sim_\alpha\gamma\})$. Clearly, it suffices to consider only the case $\{\delta\in\bar{\Gamma}_x\mid\delta\sim_\alpha\gamma\}\neq\emptyset$. We fix such a $\gamma\in\Gamma$ and consider the set $M_\alpha(\gamma)=\{i_1<\cdots<i_\ell\}$.

Case 1. $\gamma\notin\Gamma_{\{x,s_\alpha x\}}$. By (5.7) or (5.8), we get that $\mathrm{im}\, \nu_\gamma^\alpha\backslash\pi^{-1}(x)=\mathrm{im}\, \nu_\gamma^\alpha$ and $\{\delta\in\bar{\Gamma}_x\mid\delta\sim_\alpha\gamma\}=\{\delta\in\Gamma\mid\delta\sim_\alpha\gamma\}$. We have the commutative diagram

$$\begin{array}{ccccc}
 & & \Gamma_\ell^2 & \hookrightarrow & \Sigma_\ell^2 \\
 & & \iota\downarrow\wr & & \wr\downarrow\iota \\
 & & \{e,s_\alpha\}^\ell & \hookrightarrow & (G_\alpha/B_\alpha)^\ell \\
 & & \nu_\gamma^\alpha\downarrow\wr & & \wr\downarrow\nu_\gamma^\alpha \\
\{\delta\in\Gamma\mid\delta\sim_\alpha\gamma\} & = & (\mathrm{im}\, \nu_\gamma^\alpha)^T & \hookrightarrow & \mathrm{im}\, \nu_\gamma^\alpha
\end{array}$$

and hence we get the following commutative diagram for cohomologies:

$$\begin{array}{ccc}
H_T^\bullet(\Gamma_\ell^2) & \longleftarrow & H_T^\bullet(\Sigma_\ell^2) \\
(\nu_\gamma^\alpha\circ\iota)^*\uparrow\wr & & \wr\uparrow(\nu_\gamma^\alpha\circ\iota)^* \\
H_T^\bullet(\{\delta\in\Gamma\mid\delta\sim_\alpha\gamma\}) & \longleftarrow & H_T^\bullet(\mathrm{im}\, \nu_\gamma^\alpha)
\end{array}$$

Thus f^γ belongs to the image of the bottom arrow if and only if $f\circ\nu_\gamma^\alpha\circ\iota$ belongs to the image of the top arrow. By [11, Proposition 5.4(a)] this is equivalent to

$$\sum_{\delta\in\Gamma_\ell^2,\ J(\delta)\subset J(\tau)} (-1)^{|J(\delta)|} f\circ\nu_\gamma^\alpha\circ\iota(\delta)\equiv 0 \bmod \alpha^{|J(\tau)|} \tag{5.11}$$

for any $\tau\in\Gamma_\ell^2$. By (5.4), we have the equivalences

$$|J(\delta)| = |J_\alpha(\nu_\gamma^\alpha\circ\iota(\delta))|,\quad J(\delta)\subset J(\tau)\Leftrightarrow J_\alpha(\nu_\gamma^\alpha\circ\iota(\delta))\subset J_\alpha(\nu_\gamma^\alpha\circ\iota(\tau)).$$

So (5.11) can be rewritten as

$$\sum_{\delta\in\Gamma_\ell^2,\ J_\alpha(\nu_\gamma^\alpha\circ\iota(\delta))\subset J_\alpha(\nu_\gamma^\alpha\circ\iota(\tau))} (-1)^{|J_\alpha(\nu_\gamma^\alpha\circ\iota(\delta))|} f\circ\nu_\gamma^\alpha\circ\iota(\delta)\equiv 0 \bmod \alpha^{|J_\alpha(\nu_\gamma^\alpha\circ\iota(\tau))|}.$$

Replacing $\nu_\gamma^\alpha\circ\iota(\delta)$ and $\nu_\gamma^\alpha\circ\iota(\tau)$ with δ and γ respectively, we get the final version (5.9).

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Case 2. $x = \pi(\gamma_{\min})$. The condition $\{\delta \in \bar{\Gamma}_x \mid \delta \sim_\alpha \gamma\} \neq \varnothing$ implies $\ell > 0$. Let $Y = (G_\alpha/B_\alpha)^{\ell-1}$ and $Z = \{e, s_\alpha\}^{\ell-1}$ be its set of T-fixed points. By (5.7),

$$\operatorname{im} v_\gamma^\alpha \backslash \pi^{-1}(x) = v_\gamma^\alpha(Y \times ((G_\alpha/B_\alpha)\backslash\{e\})).$$

Consider the following maps:

- the natural inclusions $\tilde{\imath} : \Gamma^2_{\ell-1} \hookrightarrow \Sigma^2_{\ell-1}$, $i_{Y,Z} : Z \hookrightarrow Y$ and $\hat{\imath} : \{s_\alpha\} \hookrightarrow (G_\alpha/B_\alpha)\backslash\{e\}$;
- $a : Y \to Y \times ((G_\alpha/B_\alpha)\backslash\{e\})$ and $b : Z \to Z \times \{s_\alpha\}$ that add the point s_α to the last position;
- the projection $p : Y \times ((G_\alpha/B_\alpha)\backslash\{e\}) \to Y$ to the first $\ell - 1$ coordinates.

By definition, $p \circ a = \mathrm{id}$. Thus $a^* \circ p^* = \mathrm{id}$ on the level of cohomology. In particular, a^* is surjective. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & \Gamma^2_{\ell-1} & \overset{\tilde{\imath}}{\hookrightarrow} & \Sigma^2_{\ell-1} \\
 & & {\scriptstyle\iota}\downarrow\wr & & \wr\downarrow{\scriptstyle\iota} \\
 & & Z & \overset{i_{Y,Z}}{\hookrightarrow} & Y \\
 & & {\scriptstyle b}\downarrow\wr & & \downarrow{\scriptstyle a} \\
 & & Z \times \{s_\alpha\} & \overset{i_{Y,Z}\times\hat{\imath}}{\hookrightarrow} & Y \times ((G_\alpha/B_\alpha) \setminus \{e\}) \\
 & & {\scriptstyle v_\gamma^\alpha}\downarrow\wr & & \wr\downarrow{\scriptstyle v_\gamma^\alpha} \\
\{\delta \in \Gamma_{s_\alpha x} \mid \delta \sim_\alpha \gamma\} & = & (\operatorname{im} v_\gamma^\alpha \setminus \pi^{-1}(x))^T & \hookrightarrow & \operatorname{im} v_\gamma^\alpha \setminus \pi^{-1}(x)
\end{array}$$

Hence we get the following commutative diagram for cohomologies:

$$\begin{array}{ccc}
H_T^\bullet(\Gamma^2_{\ell-1}) & \overset{\tilde{\imath}^*}{\longleftarrow} & H_T^\bullet(\Sigma^2_{\ell-1}) \\
{\scriptstyle\iota^*}\uparrow\wr & & \wr\uparrow{\scriptstyle\iota^*} \\
H_T^\bullet(Z) & \overset{i^*_{Y,Z}}{\longleftarrow} & H_T^\bullet(Y) \\
{\scriptstyle b^*}\uparrow\wr & & \uparrow{\scriptstyle a^*} \\
H_T^\bullet(Z \times \{s_\alpha\}) & \overset{(i_{Y,Z}\times\hat{\imath})^*}{\longleftarrow} & H_T^\bullet(Y \times ((G_\alpha/B_\alpha) \setminus \{e\})) \\
{\scriptstyle (v_\gamma^\alpha)^*}\uparrow\wr & & \wr\uparrow{\scriptstyle (v_\gamma^\alpha)^*} \\
H_T^\bullet(\{\delta \in \Gamma_{s_\alpha x} \mid \delta \sim_\alpha \gamma\}) & \longleftarrow & H_T^\bullet(\operatorname{im} v_\gamma^\alpha \setminus \pi^{-1}(x))
\end{array}$$

The surjectivity of a^* proves that b^* maps isomorphically $\operatorname{im}(i_{Y,Z} \times \hat{\imath})^*$ onto $\operatorname{im} i^*_{Y,Z}$. Therefore the same is true about the whole left vertical column of the above diagram: the image of the bottom arrow is mapped isomorphically onto $\operatorname{im} \tilde{\imath}^*$. Thus f^γ belongs to the image of the bottom arrow if and only if $f \circ v_\alpha^\gamma \circ b \circ \iota \in \operatorname{im} \tilde{\imath}^*$. By [11, Proposition 5.4(a)] this is equivalent to

$$\sum_{\delta \in \Gamma^2_{\ell-1},\, J(\delta) \subset J(\tau)} (-1)^{|J(\delta)|} f \circ v_\gamma^\alpha \circ b \circ \iota(\delta) \equiv 0 \bmod \alpha^{|J(\tau)|} \tag{5.12}$$

for any $\tau \in \Gamma^2_{\ell-1}$. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma^2_{\ell-1} & \xrightarrow{\iota} & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle b} \\ \Gamma^2_{\ell} & \xrightarrow[\iota]{} & Z \times \{s_\alpha\} \end{array}$$

where $q(\delta_1, \ldots, \delta_{\ell-1}) = (\delta_1, \ldots, \delta_{\ell-1}, \delta_{\ell-1} \cdots \delta_2\delta_1 s_\alpha)$. One easily notes that $v^\alpha_\gamma \circ b \circ \iota(\delta)$ runs over the set $\{\delta \in \Gamma_{s_\alpha x} \mid \delta \sim_\alpha \gamma\}$ as δ runs over $\Gamma^2_{\ell-1}$. By (5.4),

$$J_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta)) = J_\alpha(v^\alpha_\gamma \circ \iota \circ q(\delta)) = i_{J(q(\delta))} = i_{J(\delta)} \sqcup \{i_\ell\}.$$

Hence,

$$|D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta))| = |J_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta)) \backslash \{i_\ell\}| = |i_{J(\delta)}| = |J(\delta)|.$$

By (5.4) and Proposition 3.1, the inclusion relation is also preserved:

$$\begin{aligned} J(\delta) \subset J(\tau) &\Leftrightarrow J_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta)) \subset J_\alpha(v^\alpha_\gamma \circ b \circ \iota(\tau)) \\ &\Leftrightarrow D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta)) \subset D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\tau)). \end{aligned}$$

Now we can replace (5.12) with

$$\sum_{\delta \in \Gamma^2_{\ell-1},\, D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta)) \subset D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\tau))} (-1)^{|D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\delta))|} f \circ v^\alpha_\gamma \circ b \circ \iota(\delta) \equiv 0 \bmod \alpha^{|D_\alpha(v^\alpha_\gamma \circ b \circ \iota(\tau))|}.$$

Replacing $v^\alpha_\gamma \circ b \circ \iota(\delta)$ and $v^\alpha_\gamma \circ b \circ \iota(\tau)$ with δ and γ respectively, we get (5.10).

Case 3. $x = s_\alpha \pi(\gamma_{\min})$. For $\ell > 0$, this case can be obtained from Case 2 by interchanging e and s_α. We get the following version of (5.12):

$$\sum_{\delta \in \Gamma^2_{\ell-1},\, J(\delta) \subset J(\tau)} (-1)^{|J(\delta)|} f \circ v^\alpha_\gamma \circ b' \circ \iota(\delta) \equiv 0 \bmod \alpha^{|J(\tau)|}, \tag{5.13}$$

where $b' : Z \to Z \times \{e\}$ adds the point e to the last position. We have a similar commutative diagram

$$\begin{array}{ccc} \Gamma^2_{\ell-1} & \xrightarrow{\iota} & Z \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle b'} \\ \Gamma^2_{\ell} & \xrightarrow[\iota]{} & Z \times \{e\} \end{array}$$

where $q'(\delta_1, \ldots, \delta_{\ell-1}) = (\delta_1, \ldots, \delta_{\ell-1}, \delta_{\ell-1} \cdots \delta_2\delta_1)$. Here again, $v^\alpha_\gamma \circ b \circ \iota(\delta)$ runs over the set $\{\delta \in \Gamma_{s_\alpha x} \mid \delta \sim_\alpha \gamma\}$ as δ runs over $\Gamma^2_{\ell-1}$. By (5.4),

$$J_\alpha(v^\alpha_\gamma \circ b' \circ \iota(\delta)) = J_\alpha(v^\alpha_\gamma \circ \iota \circ q'(\delta)) = i_{J(q'(\delta))} = i_{J(\delta)}.$$

Hence,

$$|D_\alpha(v^\alpha_\gamma \circ b' \circ \iota(\delta))| = |J_\alpha(v^\alpha_\gamma \circ b' \circ \iota(\delta)) \backslash \{i_\ell\}| = |i_{J(\delta)}| = |J(\delta)|$$

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and the same is true for the inclusion

$$J(\delta) \subset J(\tau) \Leftrightarrow J_\alpha(\nu_\gamma^\alpha \circ b' \circ \iota(\delta)) \subset J_\alpha(\nu_\gamma^\alpha \circ b' \circ \iota(\tau)) \\ \Leftrightarrow D_\alpha(\nu_\gamma^\alpha \circ b' \circ \iota(\delta)) \subset D_\alpha(\nu_\gamma^\alpha \circ b' \circ \iota(\tau)).$$

Therefore, we again get (5.10).

Finally, assume that $\ell = 0$. In this case, $\gamma = \gamma_{\min}$ and $\operatorname{im}\nu_\gamma^\alpha \backslash \pi^{-1}(x) = \{\gamma\}$. Hence the restriction $H_T^\bullet(\operatorname{im}\nu_\gamma^\alpha \backslash \pi^{-1}(x)) \to H_T^\bullet(\{\delta \in \bar{\Gamma}_x \mid \delta \sim_\alpha \gamma\})$ is the identity map. Thus f^γ is always in its image. On the other hand, condition (5.10) is also satisfied, as $D_\alpha(\gamma) = \varnothing$.

All the cases being considered, it suffices to apply Corollary 2.5 to conclude the proof. □

5.4. Versions of Lemmas 4.6, 4.7, and 4.8 for $\bar{\Sigma}_x$. First, we get the following results similar to the first two lemmas.

Lemma 5.3 (Cf. Lemma 4.6). $\dot{\bar{\mathcal{X}}}_x = \bar{\mathcal{X}}_{xs_r}$.

Proof. Take any $f \in \bar{\mathcal{X}}_x$. By Proposition 5.2, in order to prove that $\dot{f} \in \bar{\mathcal{X}}_{xs_r}$, we must check the following equivalences:

$$\sum_{\delta \in \Gamma, \delta \sim_\alpha \gamma, J_\alpha(\delta) \subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|J_\alpha(\gamma)|}$$

for $\gamma \in \bar{\Gamma}_{\{xs_r, s_\alpha xs_r\}}$ and

$$\sum_{\delta \in \Gamma_{s_\alpha xs_r}, \delta \sim_\alpha \gamma, D_\alpha(\delta) \subset D_\alpha(\gamma)} (-1)^{|D_\alpha(\delta)|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|D_\alpha(\gamma)|}$$

for $\gamma \in \Gamma_{s_\alpha xs_r}$. The first equivalence can be proved exactly as in Lemma 4.6. Note that $\gamma, \dot{\gamma} \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ if $r \in M_\alpha(\gamma)$ (for Case 2). In view of the properties listed in Section 4.3, the second equivalence can be rewritten in the form

$$\sum_{\dot{\delta} \in \Gamma_{s_\alpha x}, \dot{\delta} \sim_\alpha \dot{\gamma}, D_\alpha(\dot{\delta}) \subset D_\alpha(\dot{\gamma})} (-1)^{|D_\alpha(\dot{\delta})|} f(\dot{\delta}) \equiv 0 \mod \alpha^{|D_\alpha(\dot{\gamma})|}.$$

It holds by Proposition 5.2. □

Lemma 5.4 (Cf. Lemma 4.7). *Let $f \in \bar{\mathcal{X}}_x$, $r > 0$ and $t \in \{e, s_r\}$. We define $f' \in H_T^\bullet(\bar{\Gamma}'_{xt})$ by $f'(\gamma') = f(\gamma' \cdot t)$. Then $f' \in \bar{\mathcal{X}}'_{xt}$.*

Proof. The same embedding ι as in Section 4.4 induces the following commutative diagram:

$$\begin{array}{ccc} H_T^\bullet(\bar{\Sigma}_x) & \longrightarrow & H_T^\bullet(\bar{\Sigma}'_x) \\ \downarrow & & \downarrow \\ H_T^\bullet(\bar{\Gamma}_x) & \longrightarrow & H_T^\bullet(\bar{\Gamma}'_x) \end{array}$$

Therefore the lemma holds for $t = e$. In order to prove it for $t = s_r$, consider $\dot{f}$ and apply Lemma 5.3. □

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The version of Lemma 4.8 requires however a more accurate choice of t.

Lemma 5.5 (Cf. Lemma 4.8). *Let $x \in W$, $f \in \bar{X}_x$ and $r > 0$. We can choose a unique $t \in \{e, s_r\}$ such that $xt < xts_r$. Suppose that $f(\gamma) = 0$ for $\gamma_r \neq t$. Then $f(\gamma)$ is divisible by $\boldsymbol{\beta}_r(\gamma)$ for any $\gamma \in \bar{\Gamma}_x$. Moreover, the function $\gamma' \mapsto f(\gamma' \cdot t)/\boldsymbol{\beta}_r(\gamma' \cdot t)$, where $\gamma' \in \bar{\Gamma}'_{xt}$, belongs to $\bar{X}'_{xt}$.*

Proof. First, we show how to choose t. Let us choose $t \in \{e, s_r\}$ arbitrarily. The elements xt and xts_r are comparable with respect to the Bruhat order, as they differ by a reflection. The element t is already chosen if $xt < xts_r$. Suppose that $xt > xts_r$. Then we set $t' = ts_r$ and get $xt' = xts_r < xt = xt's_r$. The uniqueness is clear from $xt < xts_r \Leftrightarrow xt' > xt's_r$ with t' as before.

As in the proof of Lemma 4.8, we shall prove the divisibility claim by induction with respect to $\trianglelefteq$. Suppose that $f(\delta)$ is divisible by $\boldsymbol{\beta}_r(\delta)$ for any $\delta \in \bar{\Gamma}_x$ such that $\delta \vartriangleleft \gamma$ for some $\gamma \in \bar{\Gamma}_x$. We must prove that $f(\gamma)$ is divisible by $\boldsymbol{\beta}_r(\gamma)$. Clearly, we need only to consider the case $\gamma_r = t$.

We take for α the positive of the two roots $\boldsymbol{\beta}_r(\gamma)$ and $-\boldsymbol{\beta}_r(\gamma)$. Thus $r \in M_\alpha(\gamma)$. Note the following chain of equivalences:

$$\gamma \in \Gamma_{s_\alpha x} \Leftrightarrow \gamma^r = s_\alpha x = \gamma^r s_r (\gamma^r)^{-1} x \Leftrightarrow \gamma^r s_r = x \Leftrightarrow \gamma^r = xs_r \Leftrightarrow \gamma \in \Gamma_{xs_r}. \quad (5.14)$$

Similarly, we get $\dot{\gamma} \in \Gamma_{s_\alpha x} \Leftrightarrow \dot{\gamma} \in \Gamma_{xs_r}$.

The case $\gamma \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ is identical to Cases 1 and 2 of Lemma 4.8, where one applies Proposition 5.2 instead of Proposition 4.1. Note that in this case $\dot{\gamma} \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ by (5.14), Case 1 corresponds to $r \in J_\alpha(\gamma)$ and Case 2 corresponds to $r \notin J_\alpha(\gamma)$.

Consider the case $\gamma \in \Gamma_{s_\alpha x}$. By Proposition 5.2,

$$\sum_{\delta \in \Gamma_{s_\alpha x}, \delta \sim_\alpha \gamma, D_\alpha(\delta) \subset D_\alpha(\gamma)} (-1)^{|D_\alpha(\delta)|} f(\delta) \equiv 0 \mod \alpha^{|D_\alpha(\gamma)|}.$$

We have $\pi(\delta) = \pi(\gamma)$ in the summation. It follows from this fact and Proposition 3.1 that $J_\alpha(\delta) \subset J_\alpha(\gamma)$. Hence $\delta \trianglelefteq \gamma$.

It remains to check that $D_\alpha(\gamma) \neq \varnothing$. By (5.14), we get $\gamma \in \Gamma_{xs_r}$. Thus $\gamma^{r-1} = \gamma^r t = xs_r t$, whence our condition $xt < xts_r$ implies $\gamma^{r-1} s_r < \gamma^{r-1}$ and $r \in D_\alpha(\gamma)$.

Let us prove the last claim. We denote by f' the function under consideration: $f'(\gamma') = f(\gamma' \cdot t)/\boldsymbol{\beta}_r(\gamma' \cdot t)$. By Proposition 5.2, we must check the equivalence

$$\sum_{\delta' \in \Gamma', \delta' \sim_\alpha \gamma', J_\alpha(\delta') \subset J_\alpha(\gamma')} (-1)^{|J_\alpha(\delta')|} f'(\delta') \equiv 0 \mod \alpha^{|J_\alpha(\gamma')|}$$

for any $\gamma' \in \bar{\Gamma}'_{\{xt, s_\alpha xt\}}$ and the equivalence

$$\sum_{\delta' \in \Gamma'_{s_\alpha xt}, \delta' \sim_\alpha \gamma', D_\alpha(\delta') \subset D_\alpha(\gamma')} (-1)^{|D_\alpha(\delta')|} f'(\delta') \equiv 0 \mod \alpha^{|D_\alpha(\gamma')|} \quad (5.15)$$

for any $\gamma' \in \bar{\Gamma}'_{s_\alpha xt}$. The first one can be proved exactly as in Lemma 4.8, as $\gamma \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ and $\dot{\gamma} \in \bar{\Gamma}_{\{x, s_\alpha x\}}$ if $r \in M_\alpha(\gamma)$ (Case 2), where $\gamma = \gamma' \cdot t$.

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It remains to prove (5.15). In this case, $\gamma = \gamma' \cdot t \in \Gamma_{s_\alpha x}$. We consider the following cases.

Case a. $r \notin M_\alpha(\gamma)$. In this case $\boldsymbol{\beta}_t(\gamma) \neq \pm\alpha$. Hence $s_\alpha x(-\alpha_r) \neq \pm\alpha$. By Proposition 5.2,

$$\sum_{\delta\in\Gamma_{s_\alpha x},\delta\sim_\alpha\gamma,\delta_r=t,D_\alpha(\delta)\subset D_\alpha(\gamma)} (-1)^{|D_\alpha(\delta)|} f(\delta) \equiv 0 \mod \alpha^{|D_\alpha(\gamma)|}. \tag{5.16}$$

As $\delta_r = \gamma_r = t$ in this summation, we can rewrite the above equivalence as

$$s_\alpha x(-\alpha_r) \sum_{\delta'\in\Gamma_{s_\alpha xt},\delta'\sim_\alpha\gamma',D_\alpha(\delta')\subset D_\alpha(\gamma')} (-1)^{|D_\alpha(\delta')|} f'(\delta') \equiv 0 \mod \alpha^{|D_\alpha(\gamma')|}.$$

Cancelling out $s_\alpha x(-\alpha_r)$, we get (5.15).

Case b. $r \in M_\alpha(\gamma)$. In this case, $xs_r = s_\alpha x$. For any $\delta \in \Gamma_{s_\alpha x}$ such that $\delta_r = t$ and $\delta \sim_\alpha \gamma$, we have $\delta^{r-1}s_r = s_\alpha xts_r = xt < xts_r = s_\alpha xt = \delta^{r-1}$. Hence $r \in D_\alpha(\delta)$. Therefore we can rewrite (5.16) as

$$\pm\alpha \sum_{\delta'\in\Gamma_{s_\alpha xt},\delta'\sim_\alpha\gamma',D_\alpha(\delta')\subset D_\alpha(\gamma')} (-1)^{|D_\alpha(\delta')|} f'(\delta') \equiv 0 \mod \alpha^{|D_\alpha(\gamma')|+1}.$$

Cancelling out $\pm\alpha$, we get (5.15). □

5.5. Basis for $\bar{\mathcal{X}}_x$. For any gallery $\gamma \in \Gamma$ and $x \in W$, we define

$$\mathbf{c}^x_\varnothing = 1, \quad \mathbf{c}^x_\gamma = \begin{cases} \Delta(\mathbf{c}^{x\gamma_r}_{\gamma'}) & \text{if } x\gamma_r > x\gamma_r s_r, \\ \nabla_{\gamma_r}(\mathbf{c}^{x\gamma_r}_{\gamma'}) & \text{if } x\gamma_r < x\gamma_r s_r. \end{cases}$$

By Lemmas 4.3 and 4.4, we get $\mathbf{c}^x_\gamma \in \mathcal{X}$.

Theorem 5.6. *The set $\{\mathbf{c}^x_\gamma|_{\bar{\Gamma}_x} \mid \gamma \in \bar{\Gamma}_x\}$ is an S-basis of $\bar{\mathcal{X}}_x$. In particular, the restrictions $\mathcal{X} \to \bar{\mathcal{X}}_x$ and $H^\bullet_T(\Sigma) \to H^\bullet_T(\bar{\Sigma}_x)$ are surjective.*

Proof. We apply induction on r, the result being obvious for $r = 0$. Now let $r > 0$ and f be an element of $\bar{\mathcal{X}}_x$. Choose $q \in \{e, s_r\}$ so that $xq > xqs_r$ and define $f'(\gamma') = f(\gamma' \cdot q)$ for $\gamma' \in \bar{\Gamma}'_{xq}$. By Lemma 5.4, we get $f' \in \bar{\mathcal{X}}'_{xq}$. By the inductive hypothesis, $f' = \sum_{\gamma'\in\bar{\Gamma}'_{xq}} a_{\gamma'} \mathbf{c}^{xq}_{\gamma'}|_{\bar{\Gamma}'_{xq}}$ for some $a_{\gamma'} \in S$. Consider the difference

$$h = f - \sum_{\gamma\in\bar{\Gamma}_x,\gamma_r=q} a_{\gamma'} \mathbf{c}^x_\gamma|_{\bar{\Gamma}_x}. \tag{5.17}$$

By the above definitions, we get $h(\delta) = 0$ for any $\delta \in \bar{\Gamma}_x$ such that $\delta_r = q$:

$$\begin{aligned} h(\delta) = f(\delta) - \sum_{\gamma\in\bar{\Gamma}_x,\gamma_r=q} a_{\gamma'} \mathbf{c}^x_\gamma(\delta) &= f'(\delta') - \sum_{\gamma\in\bar{\Gamma}_x,\gamma_r=q} a_{\gamma'} \Delta(\mathbf{c}^{xq}_{\gamma'})(\delta) \\ &= \sum_{\gamma'\in\bar{\Gamma}'_{xq}} a_{\gamma'} \mathbf{c}^{xq}_{\gamma'}(\delta') - \sum_{\gamma\in\bar{\Gamma}_x,\gamma_r=q} a_{\gamma'} \mathbf{c}^{xq}_{\gamma'}(\delta') = 0. \end{aligned}$$

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Let t be the element of $\{e, s_r\}$ distinct from q. We clearly have $xt < xts_r$. Thus by Lemma 5.5, we get that the function h' defined by $h'(\gamma') = h(\gamma' \cdot t)/\boldsymbol{\beta}_r(\gamma' \cdot t)$ for $\gamma' \in \bar{\Gamma}'_{xt}$ is a well-defined element of $\bar{\mathcal{X}}'_{xt}$. By induction, $h' = \sum_{\gamma' \in \bar{\Gamma}'_{xt}} b_{\gamma'} \mathbf{c}^{xt}_{\gamma'}|_{\bar{\Gamma}'_{xt}}$ for some $b_{\gamma'} \in S$. We get $h = \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} b_{\gamma'} \mathbf{c}^x_{\gamma}|_{\bar{\Gamma}_x}$. Indeed, both sides evaluate to 0 at $\delta \in \bar{\Gamma}_x$ such that $\delta_r = q$, and for $\delta \in \bar{\Gamma}_x$ such that $\delta_r = t$,

$$\begin{aligned} h(\delta) - \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} b_{\gamma'} \mathbf{c}^x_{\gamma}(\delta) &= h(\delta' \cdot t) - \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} b_{\gamma'} \nabla_t(\mathbf{c}^{xt}_{\gamma'})(\delta' \cdot t) \\ &= \boldsymbol{\beta}_r(\delta) h'(\delta') - \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} b_{\gamma'} \boldsymbol{\beta}_r(\delta) \mathbf{c}^{xt}_{\gamma'}(\delta') \\ &= \boldsymbol{\beta}_r(\delta) \sum_{\gamma' \in \bar{\Gamma}'_{xt}} b_{\gamma'} \mathbf{c}^{xt}_{\gamma'}(\delta') - \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} b_{\gamma'} \boldsymbol{\beta}_r(\delta) \mathbf{c}^{xt}_{\gamma'}(\delta') = 0. \end{aligned}$$

Hence and from (5.17), we get that f is an S-linear combination of elements of our set.

Finally, let us prove the linear independence. Suppose that we have $\sum_{\gamma \in \bar{\Gamma}_x} a_\gamma \mathbf{c}^x_\gamma = 0$ for some $a_\gamma \in S$. We can write this sum as

$$\sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = q} a_\gamma \Delta(\mathbf{c}^{xq}_{\gamma'}) + \sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} a_\gamma \nabla_t(\mathbf{c}^{xt}_{\gamma'}) = 0. \tag{5.18}$$

Evaluation at $\delta \in \bar{\Gamma}_x$ with $\delta_r = q$ yields $\sum_{\gamma' \in \bar{\Gamma}'_{xq}} a_\gamma \mathbf{c}^{xq}_{\gamma'}(\delta') = 0$. Hence by the inductive hypothesis, $a_\gamma = 0$ for any $\gamma \in \bar{\Gamma}_x$ such that $\gamma_r = q$. Therefore (5.18) takes the form $\sum_{\gamma \in \bar{\Gamma}_x, \gamma_r = t} a_\gamma \nabla_t(\mathbf{c}^{xt}_{\gamma'}) = 0$. Evaluation at $\delta \in \bar{\Gamma}_x$ with $\delta_r = t$ yields $\sum_{\gamma' \in \bar{\Gamma}'_{xt}} a_\gamma \boldsymbol{\beta}_r(\delta) \mathbf{c}^{xt}_{\gamma'}(\delta') = 0$, whence $\sum_{\gamma' \in \bar{\Gamma}'_{xt}} a_\gamma \mathbf{c}^{xt}_{\gamma'}(\delta') = 0$. By the inductive hypothesis, $a_\gamma = 0$ for any $\gamma \in \bar{\Gamma}_x$ such that $\gamma_r = t$. □

Remark 5.7. For each $x \in W$, we can define the tree $\xi_r(x) \in \Upsilon$ just as we defined the tree $\rho_r(x)$ in Remark 4.13 but with the opposite choice of the element corresponding to the empty sequence:

$$x\xi_r(x)_\varnothing < x\xi_r(x)_\varnothing s_r, \quad \xi_r(x)'_0 = \xi_{r-1}(x\xi_r(x)_\varnothing s_r), \quad \xi_r(x)'_1 = \xi_{r-1}(x\xi_r(x)_\varnothing).$$

Similarly to Remark 4.13, one can easily prove that $\{\mathbf{c}^x_\gamma \mid \gamma \in \Gamma\} = B_{\xi_r(x)}$.

6. The costalk-to-stalk embedding and the decomposition of the direct image

6.1. Change of coefficients. Let k be a principal ideal domain with invertible 2 if the root system contains a component of type C_n. Consider the canonical ring homomorphism $\mathbb{Z}' \to k$. It extends to the ring homomorphism $S \to S_k$. We get the following commutative diagram:

$$\begin{array}{ccc} H^i(\Sigma) \otimes_{\mathbb{Z}'} k & \longrightarrow & H^i(\Sigma, k) \\ \downarrow & & \downarrow \\ H^i(\Gamma) \otimes_{\mathbb{Z}'} k & \longrightarrow & H^i(\Gamma, k) \end{array} \tag{6.1}$$

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As $H^i(\Sigma)$ and $H^i(\Sigma, k)$ vanish in odd degrees, Σ is compact and Γ is finite, Proposition 5.1 implies that the horizontal arrows are isomorphisms. Hence we get the following chain of isomorphisms:

$$H_T^\bullet(\Sigma) \otimes_S S_k \simeq (H^\bullet(\Sigma) \otimes_{\mathbb{Z}'} S) \otimes_S S_k \simeq (H^\bullet(\Sigma) \otimes_{\mathbb{Z}'} k) \otimes_k S_k \\ \simeq H^\bullet(\Sigma, k) \otimes_k S_k \simeq H_T^\bullet(\Sigma, k).$$

The similar chain yields an isomorphism $H_T^\bullet(\Gamma) \otimes_S S_k \xrightarrow{\sim} H_T^\bullet(\Gamma, k)$. Diagram (6.1) proves that these isomorphisms are compatible with the restriction from Σ to Γ. This means that we get the following commutative diagram:

$$\begin{array}{ccccc} & \overset{\sim}{\nearrow} & H_T^\bullet(\Sigma) \otimes_S S_k & \xrightarrow{\sim} & H_T^\bullet(\Sigma, k) \\ & & \big\downarrow & & \big\downarrow \\ \mathcal{X} \otimes_S S_k & \hookrightarrow & H_T^\bullet(\Gamma) \otimes_S S_k & \xrightarrow{\sim} & H_T^\bullet(\Gamma, k) \end{array}$$

If we go along the upper path, then we get an isomorphism of S_k-modules $\mathcal{X} \otimes_S S_k \xrightarrow{\sim} \mathcal{X}(k)$. However this map is the same as the map of the lower path. Hence we get the following result.

Lemma 6.1. *There exists an isomorphism of S_k-modules (dashed arrow) such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{X} \otimes_S S_k & \overset{\exists}{\underset{\sim}{\dashrightarrow}} & \mathcal{X}(k) \\ \big\downarrow & & \big\downarrow \\ H_T^\bullet(\Gamma) \otimes_S S_k & \xrightarrow[\sim]{} & H_T^\bullet(\Gamma, k) \end{array}$$

Arguing similarly with Σ_x and Γ_x, we get the following result.

Lemma 6.2. *There exists an isomorphism of S_k-modules (dashed arrow) such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{X}_x \otimes_S S_k & \overset{\exists}{\underset{\sim}{\dashrightarrow}} & \mathcal{X}_x(k) \\ \big\downarrow & & \big\downarrow \\ H_T^\bullet(\Gamma_x) \otimes_S S_k & \xrightarrow[\sim]{} & H_T^\bullet(\Gamma_x, k) \end{array}$$

The case of $\bar{\mathcal{X}}_x(k)$ is more difficult, as $\bar{\Sigma}_x$ is in general not compact. However, we can use the Poincaré duality

$$H^i(\bar{\Sigma}_x, k) \simeq \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma - i}(\bar{\Sigma}_x, k), k)$$

established in Section 5.1. We get the following sequence of canonical maps:

$$H^i(\bar{\Sigma}_x) \otimes_{\mathbb{Z}'} k \simeq \mathrm{Hom}_{\mathbb{Z}'\text{-mod}}(H_c^{2\dim\Sigma - i}(\bar{\Sigma}_x), \mathbb{Z}') \otimes_{\mathbb{Z}'} k \xrightarrow{\varphi} \\ \to \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma - i}(\bar{\Sigma}_x) \otimes_{\mathbb{Z}'} k, k) \simeq \mathrm{Hom}_{k\text{-mod}}(H_c^{2\dim\Sigma - i}(\bar{\Sigma}_x, k), k) \\ \simeq H^i(\bar{\Sigma}_x, k).$$

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From Section 5.1, we know that $H_c^{2\dim\Sigma-i}(\bar{\Sigma}_x)$ is a finitely generated free $\mathbb{Z}'$-module. Hence we conclude that the morphism φ in the sequence above is an isomorphism. If we replace $\bar{\Sigma}_x$ by $\bar{\Gamma}_x$ in this argument, then we get an isomorphism $H^i(\bar{\Gamma}_x)\otimes_{\mathbb{Z}'} k \simeq H^i(\bar{\Gamma}_x, k)$. It is rather easy to see that these isomorphisms are compatible with the restriction from $\bar{\Sigma}_x$ to $\bar{\Gamma}_x$. An argument similar to the one preceding Lemma 6.1 proves the following result.

Lemma 6.3. *There exists an isomorphism of S_k-modules (dashed arrow) such that the following diagram is commutative:*

$$\begin{array}{ccc} \bar{\mathcal{X}}_x \otimes_S S_k & \overset{\exists}{\underset{\sim}{\dashrightarrow}} & \bar{\mathcal{X}}_x(k) \\ \big\downarrow & & \big\downarrow \\ H_T^\bullet(\bar{\Gamma}_x)\otimes_S S_k & \xrightarrow[\sim]{} & H_T^\bullet(\bar{\Gamma}_x, k) \end{array}$$

These three lemmas allow us to construct bases of $\mathcal{X}(k)$, $\mathcal{X}_x(k)$, $\bar{\mathcal{X}}_x(k)$ from the bases of $\mathcal{X}$, $\mathcal{X}_x$, $\bar{\mathcal{X}}_x$ given by Theorems 4.9, 4.11, 5.6 respectively. Moreover, we can construct operators Δ and ∇_t on $H_T^\bullet(\Gamma', k)$ similarly to Section 4.2 and obtain analogs of Lemmas 4.3 and 4.4.

6.2. Description of the costalk-to-stalk embedding. From the T-equivariant distinguished triangle

$$i_* i^! \underline{k}_\Sigma \to \underline{k}_\Sigma \to j_* j^* \underline{k}_\Sigma \xrightarrow{+1},$$

where i and j are as in Section 5.1, we get the following exact sequence:

$$\mathbb{H}_T^n(\Sigma_x, i^! \underline{k}_\Sigma) \to H_T^n(\Sigma, k) \to H_T^n(\bar{\Sigma}_x, k) \to \mathbb{H}_T^{n+1}(\Sigma_x, i^! \underline{k}_\Sigma) \to \tag{6.2}$$

We are actually interested in the left map. It would be very convenient if we could prove that its source $\mathbb{H}_T^n(\Sigma_x, i^! \underline{k}_\Sigma)$ vanishes in odd degrees. This is fortunately true, as the sequence

$$H_T^{2m}(\Sigma, k) \to H_T^{2m}(\bar{\Sigma}_x, k) \to \mathbb{H}_T^{2m+1}(\Sigma_x, i^! \underline{k}_\Sigma) \to H_T^{2m+1}(\Sigma, k) = 0$$

is exact by (6.2) and the left map is surjective by Theorem 5.6 and Lemmas 6.1 and 6.3.

Consider the following commutative diagram with the exact first row:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{H}_T^{2m}(\Sigma_x, i^! \underline{k}_\Sigma) & \longrightarrow & H_T^{2m}(\Sigma, k) & \longrightarrow & H_T^{2m}(\bar{\Sigma}_x, k) & \longrightarrow & 0 \\ & & \exists \big\downarrow \wr & & \wr \big\downarrow & & \wr \big\downarrow & & \\ 0 & \longrightarrow & \ker \varphi^{2m} & \longrightarrow & \mathcal{X}(k)^{2m} & \xrightarrow{\varphi^{2m}} & \bar{\mathcal{X}}_x(k)^{2m} & \longrightarrow & 0 \end{array}$$

Here, φ^{2m} is the restriction map $f \mapsto f|_{\bar{\Gamma}_x}$ and the solid vertical arrows are induced by embeddings $\Gamma \hookrightarrow \Sigma$ and $\bar{\Gamma}_x \hookrightarrow \bar{\Sigma}_x$ respectively. We have thus proved the following result.

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Lemma 6.4. *There exists an isomorphism of S_k-modules (dashed arrow) such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathbb{H}_T^\bullet(\Sigma_x, i^! \underline{k}_\Sigma) & \longrightarrow & H_T^\bullet(\Sigma, k) \\ {\scriptstyle \exists}\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\ \mathcal{X}^x(k) & \hookrightarrow & \mathcal{X}(k) \end{array}$$

where $\mathcal{X}^x(k) = \{f \in \mathcal{X}(k) \mid f|_{\bar{\Gamma}_x} = 0\}$ and the bottom arrow is the natural embedding.

Corollary 6.5. *The functor $\mathbb{H}_T^\bullet(\Sigma_x, \_)$ applied to the natural morphism $i^! \underline{k}_\Sigma \to i^* \underline{k}_\Sigma$, where $i : \Sigma_x \hookrightarrow \Sigma$ is the embedding, yields a map isomorphic to the embedding $\mathcal{X}^x(k) \hookrightarrow \mathcal{X}_x(k)$.*

It remains to discuss the behaviour of $\mathcal{X}^x(k)$ with respect to the change of the ring of coefficients k. By the remark at the end of Section 5.5, we have $\{\mathbf{c}^x_\gamma \mid \gamma \in \Gamma\} = B_{\xi_r(x)}$. Thus we can write $B_{\xi_r(x)} = \{b_1, \dots, b_m, b_{m+1}, \dots, b_n\}$ so that $\{b_1|_{\bar{\Gamma}_x}, \dots, b_m|_{\bar{\Gamma}_x}\}$ is a basis of $\bar{\mathcal{X}}_x$. We have the following decompositions $b_j|_{\bar{\Gamma}_x} = \sum_{i=1}^m c_{i,j} b_i|_{\bar{\Gamma}_x}$ for some (homogeneous) $c_{i,j} \in S$. Let $u = \sum_{i=1}^n x_i b_i$, where $x_i \in S$, be an arbitrary element of $\mathcal{X}$. We get

$$\begin{aligned} u|_{\bar{\Gamma}_x} &= \sum_{i=1}^n x_i b_i|_{\bar{\Gamma}_x} \\ &= \sum_{i=1}^m x_i b_i|_{\bar{\Gamma}_x} + \sum_{j=m+1}^n x_j \sum_{i=1}^m c_{i,j} b_i|_{\bar{\Gamma}_x} = \sum_{i=1}^m \Big(x_i + \sum_{j=m+1}^n c_{i,j} x_j\Big) b_i|_{\bar{\Gamma}_x}. \end{aligned}$$

Hence $\mathcal{X}^x = \mathcal{X}^x(\mathbb{Z}')$ is a free S-module with basis $\{-\sum_{i=1}^m c_{i,j} b_i + b_j\}_{j=m+1}^n$. Arguing similarly, we get that $\mathcal{X}^x(k)$ is a free S_k-module with basis $\{-\sum_{i=1}^m (c_{i,j} \otimes 1_k)(b_i \otimes 1_k) + b_j \otimes 1_k\}_{j=m+1}^n$.

Lemma 6.6. *There exists an isomorphism of S_k-modules (dashed arrow) such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{X}^x \otimes_S S_k & \overset{\exists}{\underset{\sim}{\dashrightarrow}} & \mathcal{X}^x(k) \\ \downarrow & & \downarrow \\ H_T^\bullet(\Gamma) \otimes_S S_k & \underset{\sim}{\longrightarrow} & H_T^\bullet(\Gamma, k) \end{array}$$

6.3. Description of $\mathcal{X}^x(k)$. We are going to describe this module via the dual of $\mathcal{X}_x(k)$. This is a well-known description due to Fiebig [8, Lemmas 6.8, 6.9 and 6.13]. We present here an alternative proof that does not require invertibility of 2 in k. Let

$$D\mathcal{X}_x(k) = \{g \in \mathrm{Map}(\Gamma_x, Q_k) \mid (g, f) \in S_k \text{ for any } f \in \mathcal{X}_x(k)\},$$

where Q_k is the ring of quotients of S_k and $(g, f) = \sum_{\gamma \in \Gamma_x} g_\gamma f_\gamma$ (the standard scalar product). It will be convenient, for example in Lemma 6.7, to identify elements of $D\mathcal{X}_x(k)$ with their extensions by zero to Γ.

Consider $P_k \in H_T^\bullet(\Gamma, k)$ defined by

$$P_k(\gamma) = \prod_{i=1}^{r} \boldsymbol{\beta}_i(\gamma) \otimes 1_k = \pm \prod_{\alpha\in\Phi^+} (\alpha \otimes 1_k)^{|M_\alpha(\gamma)|}.$$

Note that any $P_k(\gamma)$ is divisible in S_k by the Euler class

$$e_x(k) = \prod_{\alpha\in\Phi^+, s_\alpha x<x} \alpha \otimes 1_k$$

(see for example [17, Lemma 4.9.7]).

Lemma 6.7. *It holds that* $\mathcal{X}^x(k) = P_k D\mathcal{X}_x(k)$.

Proof. First we prove by induction on r that $P_k g \in H_T^\bullet(\Gamma_x, k)$ for any $g \in D\mathcal{X}_x(k)$. This is clear for $r = 0$, so we assume that $r > 0$. Let $t \in \{e, s_r\}$. By Lemma 4.4,

$$S_k \ni (g, \nabla_t f'|_{\Gamma_x}) = \sum_{\gamma'\in\Gamma'_{xt}} g(\gamma' \cdot t)\boldsymbol{\beta}_r(\gamma' \cdot t) f'(\gamma')$$

for any $f' \in \mathcal{X}'(k)$. Hence the function

$$g'(\gamma') = \boldsymbol{\beta}_r(\gamma' \cdot t) g(\gamma' \cdot t),$$

where $\gamma' \in \Gamma'_{xt}$, belongs to $D\mathcal{X}'_{xt}(k)$. By the inductive hypothesis, the product $P'_k g'$ has values in S_k:

$$S_k \ni P'(\gamma')g'(\gamma') = \left(\prod_{i=1}^{r-1} \boldsymbol{\beta}_i(\gamma')\right) \boldsymbol{\beta}_r(\gamma' \cdot t) g(\gamma' \cdot t) = P(\gamma' \cdot t) g(\gamma' \cdot t).$$

As t is arbitrary, the function $P_k g$ has values in S_k.

Now we are going to prove the lemma for $k = \mathbb{Z}'$. In this case, we write $P = P_{\mathbb{Z}'}$ and $D\mathcal{X}_x = D\mathcal{X}_x(\mathbb{Z}')$. We apply induction on r, the result being obvious for $r = 0$. Assume that $r > 0$.

Let us prove that $Pg \in \mathcal{X}^x$ for $g \in D\mathcal{X}_x$. We actually must prove that the extension by zero of Pg to Γ belongs to $\mathcal{X}$, which by Proposition 4.1 is equivalent to checking that

$$\sum_{\delta\in\Gamma_x, \delta\sim_\alpha\gamma, J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} P(\delta) g(\delta) \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}$$

for any $\alpha \in \Phi^+$ and $\gamma \in \Gamma$. In this summation, $P(\delta)$ is clearly divisible by $\alpha^{|M_\alpha(\delta)|} = \alpha^{|M_\alpha(\gamma)|}$. Hence it also divisible by $\alpha^{|J_\alpha(\gamma)|}$. Therefore it remains to prove that the function

$$p_\gamma^\alpha(\delta) = \begin{cases} (-1)^{|J_\alpha(\delta)|} P(\delta)/\alpha^{|J_\alpha(\gamma)|} & \text{if } \delta \sim_\alpha \gamma \text{ and } J_\alpha(\delta) \subset J_\alpha(\gamma), \\ 0 & \text{otherwise,} \end{cases}$$

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where $\delta \in \Gamma$, belongs to $\mathcal{X}$. It is rather difficult to prove it directly, applying Proposition 4.1. Therefore, we define the following function q^{α}_{γ} by induction: $q^{\alpha}_{\varnothing} = 1$ and

$$q^{\alpha}_{\gamma} = \begin{cases} \nabla_{\gamma_r} q^{\alpha}_{\gamma'} & \text{if } r \notin M_\alpha(\gamma), \\ \nabla_{\gamma_r} q^{\alpha}_{\gamma'} + \nabla_{\gamma_r s_r} q^{\alpha}_{\gamma'} - \alpha\Delta q^{\alpha}_{\gamma'} & \text{if } r \in M_\alpha(\gamma)\backslash J_\alpha(\gamma), \\ -\Delta q^{\alpha}_{\gamma'} & \text{if } r \notin J_\alpha(\gamma), \end{cases}$$

if $r > 0$. By Lemmas 4.3 and 4.4, we get $q^{\alpha}_{\gamma} \in \mathcal{X}$. Therefore, it suffices to prove that

$$2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} p^{\alpha}_{\gamma} = q^{\alpha}_{\gamma}.$$

This formula is obvious for $r = 0$. Therefore, we consider the case $r > 0$ and apply induction.

Case 1. $r \notin M_\alpha(\gamma)$. If $\delta \nsim_\alpha \gamma$, then either $\delta_r \neq \gamma_r$ or $\delta' \nsim_\alpha \gamma'$. In both cases, $q^{\alpha}_{\gamma}(\delta) = \nabla_{\gamma_r} q^{\alpha}_{\gamma'}(\delta) = 0$. Now assume that $\delta \sim_\alpha \gamma$. Then $\delta_r = \gamma_r$, $\delta' \sim_\alpha \gamma'$, $J_\alpha(\delta) = J_\alpha(\delta')$, $J_\alpha(\gamma) = J_\alpha(\gamma')$, $M_\alpha(\gamma) = M_\alpha(\gamma')$. If $J_\alpha(\delta) \not\subset J_\alpha(\gamma)$, then $J_\alpha(\delta') \not\subset J_\alpha(\gamma')$ and

$$q^{\alpha}_{\gamma}(\delta) = \nabla_{\gamma_r} q^{\alpha}_{\gamma'}(\delta) = \boldsymbol{\beta}_r(\delta) q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} \boldsymbol{\beta}_r(\delta) p^{\alpha}_{\gamma'}(\delta') = 0.$$

If $J_\alpha(\delta) \subset J_\alpha(\gamma)$, then $J_\alpha(\delta') \subset J_\alpha(\gamma')$ and

$$\begin{aligned} q^{\alpha}_{\gamma}(\delta) &= \nabla_{\gamma_r} q^{\alpha}_{\gamma'}(\delta) = \boldsymbol{\beta}_r(\delta) q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} \boldsymbol{\beta}_r(\delta) p^{\alpha}_{\gamma'}(\delta') \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} (-1)^{|J_\alpha(\delta)|} \boldsymbol{\beta}_r(\delta) P(\delta')/\alpha^{|J_\alpha(\gamma)|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} (-1)^{|J_\alpha(\delta)|} P(\delta)/\alpha^{|J_\alpha(\gamma)|} = 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} p^{\alpha}_{\gamma}(\delta). \end{aligned}$$

Case 2. $r \in M_\alpha(\gamma)\backslash J_\alpha(\gamma)$. If $\delta \nsim_\alpha \gamma$, then $\delta' \nsim_\alpha \gamma'$. In this case, $q^{\alpha}_{\gamma}(\delta) = 0$, as $q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^{\alpha}_{\gamma'}(\delta') = 0$. If $J_\alpha(\delta) \not\subset J_\alpha(\gamma)$, then either $r \in J_\alpha(\delta)$ or $J_\alpha(\delta') \not\subset J_\alpha(\gamma')$. In the former case, we get $\boldsymbol{\beta}_r(\delta) = \alpha$ and

$$q^{\alpha}_{\gamma}(\delta) = \nabla_{\gamma_r} q^{\alpha}_{\gamma'}(\delta) + \nabla_{\gamma_r s_r} q^{\alpha}_{\gamma'}(\delta) - \alpha\Delta q^{\alpha}_{\gamma'}(\delta) = \boldsymbol{\beta}_r(\delta) q^{\alpha}_{\gamma'}(\delta') - \alpha q^{\alpha}_{\gamma'}(\delta') = 0.$$

In the latter case, we get $q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^{\alpha}_{\gamma'}(\delta') = 0$, whence $q^{\alpha}_{\gamma}(\delta) = 0$.

Now suppose that $\delta \sim_\alpha \gamma$ and $J_\alpha(\delta) \subset J_\alpha(\gamma)$. Then $\delta' \sim_\alpha \gamma'$, $J_\alpha(\delta') \subset J_\alpha(\gamma')$ and $r \notin J_\alpha(\delta)$. It follows from the last formula that $\boldsymbol{\beta}_r(\delta) = -\alpha$. Hence,

$$\begin{aligned} q^{\alpha}_{\gamma}(\delta) &= \nabla_{\gamma_r} q^{\alpha}_{\gamma'}(\delta) + \nabla_{\gamma_r s_r} q^{\alpha}_{\gamma'}(\delta) - \alpha\Delta q^{\alpha}_{\gamma'}(\delta) = \boldsymbol{\beta}_r(\delta) q^{\alpha}_{\gamma'}(\delta') - \alpha q^{\alpha}_{\gamma'}(\delta') \\ &= -2\alpha q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|+1} (-1)^{|J_\alpha(\delta')|} (-\alpha) P(\delta')/\alpha^{|J_\alpha(\gamma')|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} (-1)^{|J_\alpha(\delta)|} P(\delta)/\alpha^{|J_\alpha(\gamma)|} = 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} p^{\alpha}_{\gamma}(\delta). \end{aligned}$$

Case 3. $r \in J_\alpha(\gamma)$. If $\delta \nsim_\alpha \gamma$, then $\delta' \nsim_\alpha \gamma'$. In this case, $q^{\alpha}_{\gamma}(\delta) = 0$, as $q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^{\alpha}_{\gamma'}(\delta') = 0$. If $J_\alpha(\delta) \not\subset J_\alpha(\gamma)$, then $J_\alpha(\delta') \not\subset J_\alpha(\gamma')$ and we again get $q^{\alpha}_{\gamma}(\delta) = 0$, as $q^{\alpha}_{\gamma'}(\delta') = 2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^{\alpha}_{\gamma'}(\delta') = 0$.

Now suppose that $\delta \sim_\alpha \gamma$ and $J_\alpha(\delta) \subset J_\alpha(\gamma)$. Then $\delta' \sim_\alpha \gamma'$ and $J_\alpha(\delta') \subset J_\alpha(\gamma')$.

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If $r \notin J_\alpha(\delta)$, then $\boldsymbol{\beta}_r(\delta) = -\alpha$ and

$$\begin{aligned} q^\alpha_\gamma(\delta) = -\Delta q^\alpha_{\gamma'}(\delta) = -q^\alpha_{\gamma'}(\delta') &= -2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^\alpha_{\gamma'}(\delta') \\ &= -2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta')|} P(\delta')/\alpha^{|J_\alpha(\gamma')|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta')|}(-\alpha) P(\delta')/\alpha^{|J_\alpha(\gamma)|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta)|} P(\delta)/\alpha^{|J_\alpha(\gamma)|} = 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} p^\alpha_\gamma(\delta). \end{aligned}$$

If $r \in J_\alpha(\delta)$, then $\boldsymbol{\beta}_r(\delta) = \alpha$ and

$$\begin{aligned} q^\alpha_\gamma(\delta) = -\Delta q^\alpha_{\gamma'}(\delta) = -q^\alpha_{\gamma'}(\delta') &= -2^{|M_\alpha(\gamma')|-|J_\alpha(\gamma')|} p^\alpha_{\gamma'}(\delta') \\ &= -2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta')|} P(\delta')/\alpha^{|J_\alpha(\gamma')|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta)|}\alpha P(\delta')/\alpha^{|J_\alpha(\gamma)|} \\ &= 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}(-1)^{|J_\alpha(\delta)|} P(\delta)/\alpha^{|J_\alpha(\gamma)|} = 2^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} p^\alpha_\gamma(\delta). \end{aligned}$$

Finally, we prove the inverse inclusion. Let $f \in \mathcal{X}^x$. We apply induction on the cardinality of the following set (lower closure of the support of f):

$$\widehat{C}(f) = \{\delta \in \Gamma_x \mid \text{ there exists } \gamma \in \Gamma_x \text{ such that } \delta \leqslant \gamma \text{ and } f(\gamma) \neq 0\}.$$

If $\widehat{C}(f) = \varnothing$, then $f = 0$ and the result follows. Suppose now that $\widehat{C}(f) \neq \varnothing$ and let γ be its maximal element with respect to $<$. Let α be a positive root.

First suppose that $s_\alpha x > x$. In this case, $|J_\alpha(\gamma)| = |D_\alpha(\gamma)|$. From [11, Theorem 6.2(3)],

$$\sum_{\delta \in \Gamma_x, \delta \sim_\alpha \gamma, J_\alpha(\gamma) \subset J_\alpha(\delta)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|}.$$

Proposition 3.1 and (4.17) imply that $\delta \geqslant \gamma$ for any δ in the above summation. Thus $f(\gamma)$ is divisible by $\alpha^{|M_\alpha(\gamma)|-|J_\alpha(\gamma)|} = \alpha^{|M_\alpha(\gamma)|-|D_\alpha(\gamma)|}$.

Now suppose that $s_\alpha x < x$. In this case, $|J_\alpha(\gamma)| = |D_\alpha(\gamma)| + 1$. Let j be the greatest element of $M_\alpha(\gamma)$. Note that j is also the greatest element of $J_\alpha(\gamma)$. Let $\widetilde{\gamma}$ be obtained from γ by replacing γ_j with $\gamma_j s_j$. We clearly have $\pi(\widetilde{\gamma}) = s_\alpha x$, $\widetilde{\gamma} \sim_\alpha \gamma$ and $J_\alpha(\gamma) = J_\alpha(\widetilde{\gamma}) \sqcup \{j\}$, whence $|J_\alpha(\widetilde{\gamma})| = |J_\alpha(\gamma)| - 1 = |D_\alpha(\gamma)|$. From [11, Theorem 6.2(3)],

$$\sum_{\delta \in \Gamma_x, \delta \sim_\alpha \gamma, J_\alpha(\widetilde{\gamma}) \subset J_\alpha(\delta)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|M_\alpha(\gamma)|-|D_\alpha(\gamma)|}.$$

As $j \in J_\alpha(\delta)$ for any δ in the above summation, we can replace there the condition $J_\alpha(\widetilde{\gamma}) \subset J_\alpha(\delta)$ with $J_\alpha(\gamma) \subset J_\alpha(\delta)$. Hence again $\delta \geqslant \gamma$ in the above summation and $f(\gamma)$ is divisible by $\alpha^{|M_\alpha(\gamma)|-|D_\alpha(\gamma)|}$.

As a result, we get that $f(\gamma)$ is divisible by

$$\prod_{\alpha \in \Phi^+} \alpha^{|M_\alpha(\gamma)|-|D_\alpha(\gamma)|} = \pm \frac{P(\gamma)}{\mathbf{a}(\gamma)}.$$

It follows from Theorem 4.11 that $D\mathcal{X}_x$ has an S-basis $\{\hat{\mathbf{b}}_\gamma\}_{\gamma \in \Gamma_x}$ such that $\hat{\mathbf{b}}_\gamma(\gamma) = 1/\mathbf{a}(\gamma)$ and $\hat{\mathbf{b}}_\gamma(\delta) = 0$ for $\delta \in \Gamma_x$ with $\delta > \gamma$. To get this basis, one should invert and transpose the matrix given by (4.20).

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Consider the difference $h = f - f(\gamma)/(P(\gamma)/\mathbf{a}(\gamma))P\hat{\mathbf{b}}_\gamma$. We get $C(h) \subset \{\delta \in \Gamma_x \mid \delta < \gamma\} \subsetneq C(f)$. By induction, h belongs to the $P D\mathcal{X}_x$. Thus, so does f.

Finally, let us return to the general case. By Lemma 6.2, the basis $\{\hat{\mathbf{b}}_\gamma\}_{\gamma\in\Gamma_x}$ of $D\mathcal{X}_x$ mentioned above and the similar basis for $D\mathcal{X}_x(k)$ yield an isomorphism (dashed arrow) making the following diagram commutative:

$$\begin{array}{ccc} PD\mathcal{X}_x \otimes_S S_k & \overset{\sim}{\dashrightarrow} & P_k D\mathcal{X}_x(k) \\ \downarrow & & \downarrow \\ H^\bullet_T(\Gamma) \otimes_S S_k & \xrightarrow{\ \sim\ } & H^\bullet_T(\Gamma, k) \end{array}$$

Multiplying it by P_k, we get by Lemma 6.6 an isomorphism (dashed arrow) making the following diagram commutative:

$$\begin{array}{ccccc} & \mathcal{X}^x \otimes_S S_k & & \xrightarrow{\ \sim\ } & \mathcal{X}^x(k) \\ (PD\mathcal{X}_x) \otimes_S S_k & = & \xrightarrow{\ \sim\ } & P_k D\mathcal{X}_x(k) & \overset{\sim}{\dashrightarrow} \\ \downarrow & \swarrow & & \downarrow & \swarrow \\ H^\bullet_T(\Gamma) \otimes_S S_k & & \xrightarrow{\ \sim\ } & H^\bullet_T(\Gamma, k) & \end{array}$$

The commutativity of the right triangle means that the isomorphism represented by the dashed arrow is over $H^\bullet_T(\Gamma, k)$, which proves that it is the equality of subsets. □

Proposition 6.8. *Let H_x be the matrix defined by (4.20) and P_x be the diagonal matrix with γth entry $P_{\mathbb{Z}'}(\gamma)$. We set $H_{x,k} = H_x \otimes_S S_k$ and $P_{x,k} = P_x \otimes_S S_k$. The costalk-to-stalk embedding $\mathcal{X}^x(k) \hookrightarrow \mathcal{X}_x(k)$ is described by the transition matrix $(H_{x,k}^{-1})^T P_{x,k} H_{x,k}^{-1}$. All entries of this matrix are divisible in S_k by the Euler class $e_x(k) = \prod_{\alpha\in\Phi^+, s_\alpha x<x} \alpha \otimes 1_k$.*

Proof. We need only to prove the divisibility. It suffices to consider the case $k = \mathbb{Z}'$. We write $e_x = e_x(\mathbb{Z}')$. Let $f \in \mathcal{X}^x$. By Proposition 4.1,

$$\sum_{\delta\in\Gamma_x, \delta\sim_\alpha\gamma, J_\alpha(\delta)\subset J_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta) \equiv 0 \bmod \alpha^{|J_\alpha(\gamma)|}$$

for any $\alpha \in \Phi^+$. If $s_\alpha x < x$, then $|J_\alpha(\gamma)| = |D_\alpha(\gamma)| + 1$. By Proposition 3.1 and (4.17), we get $\delta \leqslant \gamma$ in the above summation. Hence we get by induction that $f(\gamma)$ is divisible in S by $e_x = e_x(\mathbb{Z}')$. Dividing the above equivalence by α if $s_\alpha x < x$ and taking into account that different roots are not proportional,

$$\sum_{\delta\in\Gamma_x, \delta\sim_\alpha\gamma, D_\alpha(\delta)\subset D_\alpha(\gamma)} (-1)^{|J_\alpha(\delta)|} f(\delta)/e_x \equiv 0 \bmod \alpha^{|D_\alpha(\gamma)|}.$$

Thus we have proved that $f/e_x \in \mathcal{X}_x$. □

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6.4. Euler classes. Consider the closed T-equivariant embedding

$$f : \{0\} \hookrightarrow V = \mathbb{C}_{\lambda_1} \oplus \cdots \oplus \mathbb{C}_{\lambda_d},$$

where λ_i are characters of T and $\mathbb{C}_{\lambda_i}$ is the corresponding one-dimensional representation of T (see the proof of Theorem 2.4). We assume that k is a field such that $\lambda_i \otimes_{\mathbb{Z}} k \neq 0$ for any i. We get the following exact sequence:

$$\mathbb{H}^\bullet_T(\{0\}, f^! \underline{k}_V) \to H^\bullet_T(V, k) \to H^\bullet_T(V \backslash \{0\}, k). \tag{6.3}$$

We want to look more closely at the right map. By [3, (2.15)], we have the following commutative diagram with exact top row:

$$\begin{array}{ccccc} H^\bullet_T(V, V \setminus \{0\}, k) & \longrightarrow & H^\bullet_T(V, k) & \longrightarrow & H^\bullet_T(V \setminus \{0\}, k) \\ \wr\uparrow\Phi & \overset{f_*}{\nearrow} & \wr\downarrow f^* & & \\ H^{\bullet-2d}_T(\{0\}, k) & \xrightarrow{e\cup ?} & H^\bullet_T(\{0\}, k) & & \end{array}$$

where Φ is the Thom isomorphism, f_* is the push-forward and the corresponding Euler class is $e = \prod_{i=1}^d \lambda_i \otimes_{\mathbb{Z}} k$. By our assumption, $e \neq 0$. Hence $H^\bullet_T(V \backslash \{0\}, k)$ vanishes in odd degrees and $H^\bullet_T(V, k) \to H^\bullet_T(V \backslash \{0\}, k)$ is epimorphic in any degree. Coming back to (6.3), we obtain the isomorphisms $\mathbb{H}^\bullet_T(\{0\}, f^! \underline{k}_V) \simeq H^{\bullet-2d}_T(\{0\}, k) \simeq S_k[-2d]$ and $H^\bullet_T(V, k) \simeq S_k$ under which the map $\mathbb{H}^\bullet_T(\{0\}, f^! \underline{k}_V) \to H^\bullet_T(V, k)$ becomes the multiplication by e.

6.5. Defect of a homomorphism. Recall that S_k has the maximal ideal $\mathfrak{m} = \bigoplus_{i>0} S^i_k$. We clearly have $S_k/m \simeq k$. Let $\varphi : U \to V$ be a homomorphism of graded S_k-modules. Then we can consider the quotient $\operatorname{im} \varphi/\mathfrak{m}V$, which is a graded $S_k/\mathfrak{m}$-module and thus also a graded k-vector space. If U is a finitely generated S_k-module, then we can define the *defect* of φ as the graded dimension of this quotient:

$$\mathrm{d}(\varphi) = \sum_{n \in \mathbb{Z}} \dim_k(\operatorname{im} \varphi/\mathfrak{m}V)^n v^{-n}.$$

This is an element of the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$. Clearly $\mathrm{d}(\varphi_1 \oplus \varphi_2) = \mathrm{d}(\varphi_1) + \mathrm{d}(\varphi_2)$ and $\mathrm{d}(\varphi[n]) = v^n \mathrm{d}(\varphi)$. If φ is an embedding and U and V are finitely generated free S_k-modules, then $\mathrm{d}(\varphi)$ can be calculated as follows.

Proposition 6.9 [17, Corollary 3.3.3]. *Let $\varphi : U \hookrightarrow V$ be an embedding of graded S_k-modules. Let $\{u^{(n)}_i\}_{n\in\mathbb{Z},1\leqslant i\leqslant l_n}$ and $\{v^{(n)}_j\}_{n\in\mathbb{Z},1\leqslant j\leqslant k_n}$ be bases of U and V, respectively, labelled in such a way that $u^{(n)}_i$ and $v^{(n)}_j$ have degree n. Let*

$$\varphi(u^{(n)}_i) = \sum\nolimits_{m\in\mathbb{Z},1\leqslant j\leqslant k_m} a^{(m,n)}_{j,i} v^{(m)}_j$$

for corresponding homogeneous $a^{(m,n)}_{j,i} \in S_k$. For each $n \in \mathbb{Z}$, we denote by $A^{(n)}$ the $k_n \times l_n$-matrix whose jith entry is $a^{(n,n)}_{j,i} \in k$. Then $\mathrm{d}(\varphi) = \sum_{n\in\mathbb{Z}} \mathrm{rk}_k A^{(n)} v^{-n}$.

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Finally, we describe a homomorphism of graded modules $\varphi : U \to V$ that can be divided by a homogeneous element $e \in S_k^d$ that is no zero divisor for V. Suppose that for any $u \in U$ there exists $(\varphi/e)(u) \in V$ such that $\varphi(u) = e(\varphi/e)(u)$. Then we get a uniquely defined homomorphism $\varphi/e : U \to V$ of S_k-modules such that $(\varphi/e)(U_{i+d}) \subset V_i$. It is a homomorphism of graded modules $\varphi/e : U[d] \to V$.

6.6. Application to parity sheaves. Our calculations have not yet involved any stratifications. In this section, we are going to apply our results to the stratification $G/B = \bigsqcup_{x \in W} BxB/B$. We write $X = G/B$ and $X_x = BxB/B$ for brevity and assume that k is a field. By [15] there exists the decomposition

$$\pi_* \underline{k}_\Sigma [r] = \bigoplus_{x \in W} \bigoplus_{d \in \mathbb{Z}} \mathscr{E}(x,k)[-d]^{\oplus m(x,d)},$$

where $\mathscr{E}(x,k) \in D_T(X,k)$ is the T-equivariant parity sheaf such that $\operatorname{supp} \mathscr{E}(x,k) \subset \overline{X_x}$ and $\mathscr{E}(x,k)|_{X_x} = \underline{k}_{X_x}[d_x]$, where $d_x = \dim X_x$. Our aim is to calculate the multiplicities $m(x,d)$.

We rewrite the above decomposition as

$$\pi_* \underline{k}_\Sigma = \bigoplus_{x \in W} \bigoplus_{d \in \mathbb{Z}} \mathscr{E}(x,k)[-d-r]^{\oplus m(x,d)} \tag{6.4}$$

and consider the natural embedding $i_x : \{x\} \hookrightarrow X$. We are going to take the following steps:

- apply to both sides of (6.4) the morphism of functors $\mathbb{H}_T^\bullet(\{x\}, i_x^! -) \to \mathbb{H}_T^\bullet(\{x\}, i_x^* -)$;
- divide it by the Euler class $e_x = \prod_{\alpha \in \Phi^+, s_\alpha x < x} \alpha \otimes 1_k$;
- take the defect of the resulting map.

First consider the left-hand side of (6.4). We have the following Cartesian diagram:

$$\begin{array}{ccc} \Sigma_x & \xrightarrow{\ i\ } & \Sigma \\ \big\downarrow{\scriptstyle \pi_x} & & \big\downarrow{\scriptstyle \pi} \\ \{x\} & \xrightarrow{\ i_x\ } & X \end{array}$$

As π is proper, the base change yields $i_x^! \pi_* \underline{k}_\Sigma \simeq (\pi_x)_* i^! \underline{k}_\Sigma$ and $i_x^* \pi_* \underline{k}_\Sigma \simeq (\pi_x)_* i^* \underline{k}_\Sigma$. Hence the map $\mathbb{H}_T^\bullet(\{x\}, i_x^! \pi_* \underline{k}_\Sigma) \to \mathbb{H}_T^\bullet(\{x\}, i_x^* \pi_* \underline{k}_\Sigma)$ is isomorphic to $\mathbb{H}_T^\bullet(\Sigma_x, i^! \underline{k}_\Sigma) \to \mathbb{H}_T^\bullet(\Sigma_x, i^* \underline{k}_\Sigma)$, which in its turn is isomorphic to $\mathcal{X}^x(k) \hookrightarrow \mathcal{X}_x(k)$ by Corollary 6.5.

In order to tackle the right-hand side of (6.4), let us compute the map

$$\mathbb{H}_T^\bullet(\{x\}, i_x^! \mathscr{E}(y,k)) \to \mathbb{H}_T^\bullet(\{x\}, i_x^* \mathscr{E}(y,k)). \tag{6.5}$$

If $x \notin \overline{X_y}$, then $i_x^! \mathscr{E}(y,k) = i_x^* \mathscr{E}(y,k) = 0$ as $\mathscr{E}(y,k)|_{X \setminus \overline{X_y}} = 0$. Therefore, we must only consider the case $x \in \overline{X_y}$ that is $x \leqslant y$.

Let $U = \bigsqcup_{z \geqslant x} X_z$. This is an open subset of X that contains X_x as a closed subset. Moreover, the restriction $\mathscr{F} = \mathscr{E}(y,k)|_U$ is an indecomposable parity sheaf on U.

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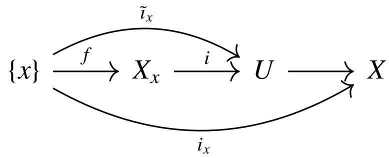
Let $i : X_x \hookrightarrow U$, $f : \{x\} \hookrightarrow X_x$ and $\tilde{\imath}_x : \{x\} \hookrightarrow U$ denote the natural embeddings:



Let $\varphi : i^! \mathscr{F} \to i^* \mathscr{F}$ denote the natural morphism.

Suppose first that $x < y$. Then $\mathscr{F}$ has no direct summands supported on X_x. Let us write $i^! \mathscr{F} = \bigoplus_{m \in \mathbb{Z}} Q^m$ and $i^* \mathscr{F} = \bigoplus_{n \in \mathbb{Z}} P^n$, where $Q^m = H^m(i^! \mathscr{F})[-m]$ and $P^n = H^n(i^* \mathscr{F})[-n]$. We denote by $\varphi_{n,m} : Q^m \to P^n$ the corresponding morphism of the direct summands. By [15, Corollary 2.22], we get that $\varphi_{n,m} = 0$ for $m \leqslant n$.

We denote by $A : f^! \to f^*$ the natural morphism of functors. Then the composition $f^* \varphi \circ A(i^! \mathscr{F})$ is the natural morphism $\tilde{\imath}_x^! \mathscr{F} \to \tilde{\imath}_x^* \mathscr{F}$:

$$\tilde{\imath}_x^! \mathscr{F} = f^! \circ i^! \mathscr{F} \xrightarrow{A(i^! \mathscr{F})} f^* \circ i^! \mathscr{F} \xrightarrow{f^* \varphi} f^* \circ i^* \mathscr{F} = \tilde{\imath}_x^* \mathscr{F}.$$

Recalling our decompositions of $i^! \mathscr{F}$ and $i^* \mathscr{F}$, we represent this morphism as the direct sum of the following morphisms:

$$f^! Q^m \xrightarrow{A(Q^m)} f^* Q^m \xrightarrow{f^* \varphi_{n,m}} f^* P^n \tag{6.6}$$

for $m > n$. We have the decompositions $Q^m = \underline{k}_{X_x}[-m]^{\oplus a(x,m)}$ and $P^n = \underline{k}_{X_x}[-n]^{\oplus b(x,n)}$ for some nonnegative integers $a(x,m)$ and $b(x,n)$. Applying $\mathbb{H}_T^\bullet(\{x\}, \_)$ to (6.6), we get maps from

$$\begin{aligned} \mathbb{H}_T^\bullet(\{x\}, f^! Q^m) &= \mathbb{H}_T^\bullet(\{x\}, f^! \underline{k}_{X_x}[-m])^{\oplus a(x,m)} \\ &= \mathbb{H}_T^{\bullet - m}(\{x\}, f^! \underline{k}_{X_x})^{\oplus a(x,m)} \\ &= H_T^{\bullet - m - 2d_x}(\{x\}, k)^{\oplus a(x,m)} = S_k[-m - 2d_x]^{\oplus a(x,m)} \end{aligned}$$

to

$$\begin{aligned} \mathbb{H}_T^\bullet(\{x\}, f^* Q^m) &= \mathbb{H}_T^\bullet(\{x\}, f^* \underline{k}_{X_x}[-m])^{\oplus a(x,m)} \\ &= H_T^{\bullet - m}(\{x\}, k)^{\oplus a(x,m)} = S_k[-m]^{\oplus a(x,m)} \end{aligned}$$

and finally to

$$\begin{aligned} \mathbb{H}_T^\bullet(\{x\}, f^* P^n) &= \mathbb{H}_T^\bullet(\{x\}, f^* \underline{k}_{X_x}[-n])^{\oplus b(x,n)} \\ &= H_T^{\bullet - n}(\{x\}, k)^{\oplus b(x,n)} = S_k[-n]^{\oplus b(x,n)}. \end{aligned}$$

So we get the following sequence of maps (see Section 6.4):

$$S_k[-m - 2d_x]^{\oplus a(x,m)} \xrightarrow{e_x \cup ?} S_k[-m]^{\oplus a(x,m)} \longrightarrow S_k[-n]^{\oplus b(x,n)}.$$

The division by e_x yields a map $S_k[-m]^{\oplus a(x,m)} \to S_k[-n]^{\oplus b(x,n)}$. As $m > n$, the defect of this map is zero. Thus we have proved that the defect of the map $\mathbb{H}^\bullet_T(\{x\}, \tilde{\imath}^!_x \mathscr{F}) \to \mathbb{H}^\bullet_T(\{x\}, \tilde{\imath}^*_x \mathscr{F})$ divided by e_x is zero. Recalling that $\mathscr{F} = \mathscr{E}(y,k)|_U$, we get that the defect of (6.5) divided by e_x is also zero.

It remains to calculate the defect of (6.5) divided by e_x in the case $x = y$. Consider the natural embedding $j : U \backslash \overline{X_x} \hookrightarrow U$. We have $j^* \mathscr{F} = \mathscr{E}(x,k)|_{U \backslash \overline{X_x}} = 0$ as $U \backslash \overline{X_x} \subset X \backslash \overline{X_x}$. The distinguished triangle

$$0 = j_! j^* \mathscr{F} \to \mathscr{F} \to i_* i^* \mathscr{F} \xrightarrow{+1}$$

yields $\mathscr{F} = i_* i^* \mathscr{F} = i_* \underline{k}_{X_x}[d_x]$. For any $? \in \{!, *\}$,

$$i^?_x \mathscr{E}(x,k) = \tilde{\imath}^?_x \mathscr{F} = \tilde{\imath}^?_x i_* \underline{k}_{X_x}[d_x] = f^? i^? i_* \underline{k}_{X_x}[d_x] = f^? \underline{k}_{X_x}[d_x].$$

Hence (6.5) becomes the natural map

$$\mathbb{H}^{\bullet + d_x}_T(\{x\}, f^! \underline{k}_{X_x}) \to \mathbb{H}^{\bullet + d_x}_T(\{x\}, f^* \underline{k}_{X_x}).$$

Under the identifications of Section 6.4, this map becomes

$$S[-d_x] \xrightarrow{e_x \cup ?} S[d_x] \; .$$

Division by e_x leaves us with the identity map $S_k[d_x] \to S_k[d_x]$, whose defect is obviously v^{d_x}. Hence we get the following result.

THEOREM 6.10. *The defect of the inclusion $\mathcal{X}^x(k) \hookrightarrow \mathcal{X}_x(k)$ divided by e_x is*

$$\sum_{d \in \mathbb{Z}} m(x,d) v^{d_x - d - r}.$$

Once we compute the above inclusion, for example by Proposition 6.8, we can recover the coefficients $m(x,d)$.

6.7. Example of torsion. We use here the notation of Proposition 6.8. Let $G = \mathrm{SL}_8(\mathbb{C})$, $\Pi = \{\alpha_1, \ldots, \alpha_7\}$ and

$$s = (s_3, s_2, s_1, s_5, s_4, s_3, s_2, s_6, s_5, s_4, s_3, s_7, s_6, s_5), \qquad x = s_2 s_3 s_2 s_5 s_6 s_5,$$

where $s_i = s_{\alpha_i}$. We arrange elements of Γ_x in ascending order with respect to $<$. The matrix $H_x = \{h_{i,j}\}_{i,j=1}^{29}$ computed by (4.20) has the following nonzero entries: $h_{1,j} = 1$

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for $1 \leqslant j \leqslant 29$,

$$
\begin{aligned}
&h_{13,13} = h_{13,14} = h_{13,15} = h_{13,16} = h_{13,17} = h_{13,18} = h_{13,19} = \alpha_5 + \alpha_6,\\
&h_{27,27} = \alpha_6\alpha_5(\alpha_2 + \alpha_3),\ h_{5,14} = h_{6,19} = -\alpha_3\alpha_6,\ h_{5,6} = h_{14,27} = -\alpha_2\alpha_5,\\
&h_{3,3} = h_{3,6} = h_{3,8} = h_{3,15} = h_{3,17} = h_{3,22} = h_{3,27} = \alpha_2 + \alpha_3,\\
&h_{7,7} = h_{7,8} = h_{7,16} = h_{7,17} = \alpha_3 + \alpha_4 + \alpha_5,\ h_{26,27} = -\alpha_6\alpha_2\alpha_5,\\
&h_{20,20} = h_{20,21} = h_{20,22} = h_{20,23} = h_{20,24} = h_{20,25}\\
&\quad = h_{20,26} = h_{20,27} = h_{20,28} = h_{20,29} = \alpha_6,\\
&h_{4,4} = h_{4,5} = h_{4,6} = h_{4,11} = h_{4,12} = h_{13,25} = h_{13,26}\\
&\quad = h_{13,27} = h_{13,28} = h_{13,29} = \alpha_5,\\
&h_{9,9} = h_{9,10} = h_{9,11} = h_{9,12} = h_{9,18} = h_{9,19} = h_{9,23}\\
&\quad = h_{9,24} = h_{9,28} = h_{9,29} = \alpha_2,\\
&h_{2,2} = h_{2,5} = h_{2,14} = h_{2,21} = h_{2,26} = h_{3,10}\\
&\quad = h_{3,12} = h_{3,19} = h_{3,24} = h_{3,29} = \alpha_3,\\
&h_{4,16} = h_{4,17} = -\alpha_3 - \alpha_4 - \alpha_5 - \alpha_6,\ h_{2,7} = h_{2,16} = -\alpha_4 - \alpha_5,\\
&h_{2,8} = h_{2,17} = -\alpha_2 - \alpha_3 - \alpha_4 - \alpha_5,\ h_{28,28} = h_{28,29} = \alpha_6\alpha_2\alpha_5,\\
&h_{26,26} = h_{27,29} = \alpha_6\alpha_3\alpha_5,\ h_{12,12} = h_{19,29} = \alpha_2\alpha_3\alpha_5,\\
&h_{6,17} = -(\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_3),\ h_{24,24} = h_{24,29} = \alpha_2\alpha_3\alpha_6,\\
&h_{10,10} = h_{10,12} = h_{10,19} = h_{10,24} = h_{10,29} = \alpha_2\alpha_3,\\
&h_{5,5} = h_{6,12} = h_{14,26} = h_{15,29} = \alpha_3\alpha_5,\ h_{22,22} = h_{22,27} = \alpha_6(\alpha_2 + \alpha_3),\\
&h_{11,18} = h_{11,19} = h_{21,22} = h_{21,27} = -\alpha_2\alpha_6,\ h_{6,15} = -\alpha_6(\alpha_2 + \alpha_3),\\
&h_{14,14} = h_{15,19} = (\alpha_5 + \alpha_6)\alpha_3,\ h_{2,3} = h_{2,6} = h_{2,15} = h_{2,22} = h_{2,27} = -\alpha_2,\\
&h_{19,19} = (\alpha_5 + \alpha_6)\alpha_2\alpha_3,\ h_{15,15} = h_{15,17} = (\alpha_5 + \alpha_6)(\alpha_2 + \alpha_3),\\
&h_{14,16} = -(\alpha_5 + \alpha_6)(\alpha_4 + \alpha_5),\ h_{5,7} = (\alpha_3 + \alpha_4)(\alpha_4 + \alpha_5),\\
&h_{14,17} = -(\alpha_5 + \alpha_6)(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5),\ h_{14,15} = -(\alpha_5 + \alpha_6)\alpha_2,\\
&h_{18,18} = h_{18,19} = (\alpha_5 + \alpha_6)\alpha_2,\ h_{5,8} = (\alpha_3 + \alpha_4)(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5),\\
&h_{11,11} = h_{11,12} = h_{18,28} = h_{18,29} = \alpha_2\alpha_5,\ h_{12,19} = -\alpha_2\alpha_3\alpha_6,\\
&h_{16,16} = h_{16,17} = (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4 + \alpha_5),\ h_{29,29} = \alpha_2\alpha_3\alpha_5\alpha_6,\\
&h_{4,7} = h_{4,8} = -\alpha_3 - \alpha_4,\ h_{4,13} = h_{4,14} = h_{4,15} = h_{4,18} = h_{4,19} = -\alpha_6,\\
&h_{5,15} = h_{23,23} = h_{23,24} = h_{23,28} = h_{23,29} = \alpha_2\alpha_6,\\
&h_{6,6} = h_{15,27} = \alpha_5(\alpha_2 + \alpha_3),\ h_{8,8} = h_{8,17} = (\alpha_3 + \alpha_4 + \alpha_5)(\alpha_2 + \alpha_3),\\
&h_{17,17} = (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4 + \alpha_5)(\alpha_2 + \alpha_3),\\
&h_{6,8} = -(\alpha_3 + \alpha_4)(\alpha_2 + \alpha_3),\ h_{5,16} = (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\alpha_4 + \alpha_5),\\
&h_{5,17} = (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5),\\
&h_{21,21} = h_{21,26} = h_{22,24} = h_{22,29} = \alpha_3\alpha_6,\\
&h_{25,25} = h_{25,26} = h_{25,27} = h_{25,28} = h_{25,29} = \alpha_5\alpha_6.
\end{aligned}
$$

The 29th row of the matrix $(H_x^{-1})^T$ has minimal degree. Its precise value is

$$r_{29} = \Big(\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, -\frac{1}{(\alpha_2+\alpha_3)\alpha_6\alpha_5\alpha_3}, -\frac{1}{\alpha_2\alpha_6\alpha_5(\alpha_2+\alpha_3)}, -\frac{1}{(\alpha_5+\alpha_6)\alpha_2\alpha_5\alpha_3},$$
$$\frac{1}{(\alpha_5+\alpha_6)(\alpha_2+\alpha_3)\alpha_5\alpha_3}, \frac{1}{\alpha_2(\alpha_5+\alpha_6)\alpha_5(\alpha_2+\alpha_3)}, 0, 0, -\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3},$$
$$\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, \frac{1}{(\alpha_5+\alpha_6)\alpha_2\alpha_5\alpha_3}, -\frac{1}{(\alpha_5+\alpha_6)\alpha_2\alpha_5\alpha_3}, -\frac{1}{\alpha_6(\alpha_5+\alpha_6)\alpha_2\alpha_3},$$
$$\frac{1}{\alpha_6(\alpha_2+\alpha_3)(\alpha_5+\alpha_6)\alpha_3}, \frac{1}{\alpha_2\alpha_6(\alpha_5+\alpha_6)(\alpha_2+\alpha_3)}, 0, 0, \frac{1}{\alpha_6(\alpha_5+\alpha_6)\alpha_2\alpha_3},$$
$$-\frac{1}{\alpha_6(\alpha_5+\alpha_6)\alpha_2\alpha_3}, -\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, \frac{1}{(\alpha_2+\alpha_3)\alpha_6\alpha_5\alpha_3}, \frac{1}{\alpha_2\alpha_6\alpha_5(\alpha_2+\alpha_3)},$$
$$\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, -\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, \frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, -\frac{1}{(\alpha_2+\alpha_3)\alpha_6\alpha_5\alpha_3}, -\frac{1}{\alpha_2\alpha_6\alpha_5(\alpha_2+\alpha_3)},$$
$$-\frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}, \frac{1}{\alpha_6\alpha_2\alpha_5\alpha_3}\Big).$$

We have the Euler class $e_x = \alpha_3(\alpha_2+\alpha_3)\alpha_6\alpha_2(\alpha_5+\alpha_6)\alpha_5$. Hence the matrix P_x/e_x has the diagonal

$$p = (\alpha_3\alpha_6\alpha_2\alpha_5\alpha_1\alpha_4(\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$-\alpha_3(\alpha_2+\alpha_3)\alpha_1\alpha_5(\alpha_3+\alpha_4)\alpha_6(\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$-(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4)\alpha_2\alpha_6(\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$-\alpha_3\alpha_2\alpha_1\alpha_5(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)(\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$\alpha_3(\alpha_2+\alpha_3)\alpha_1\alpha_5(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_5+\alpha_6)\alpha_7,$$
$$(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4+\alpha_5)^2\alpha_2(\alpha_5+\alpha_6)\alpha_7,$$
$$-(\alpha_2+\alpha_3)\alpha_1(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)(\alpha_3+\alpha_4)\alpha_7,$$
$$(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_2+\alpha_3+\alpha_4+\alpha_5)(\alpha_5+\alpha_6)(\alpha_3+\alpha_4)\alpha_7,$$
$$-\alpha_3\alpha_2(\alpha_1+\alpha_2)\alpha_5\alpha_4\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$\alpha_3(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4)\alpha_2\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$\alpha_3\alpha_2(\alpha_1+\alpha_2)\alpha_5(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$-\alpha_3(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4+\alpha_5)\alpha_2(\alpha_5+\alpha_6)(\alpha_2+\alpha_3+\alpha_4+\alpha_5)\alpha_7,$$
$$-\alpha_3\alpha_2\alpha_1(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)\alpha_6(\alpha_3+\alpha_4+\alpha_5)(\alpha_5+\alpha_6+\alpha_7),$$
$$\alpha_3(\alpha_2+\alpha_3)\alpha_1(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_5+\alpha_6)\alpha_6(\alpha_5+\alpha_6+\alpha_7),$$
$$(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_3+\alpha_4+\alpha_5)^2\alpha_2(\alpha_5+\alpha_6)\alpha_6(\alpha_5+\alpha_6+\alpha_7),$$
$$(\alpha_2+\alpha_3)\alpha_1(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7),$$
$$-(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_3+\alpha_4+\alpha_5)^2(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$$
$$\times(\alpha_5+\alpha_6)(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7),$$

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$\alpha_3\alpha_2(\alpha_1+\alpha_2)(\alpha_4+\alpha_5)(\alpha_5+\alpha_6)\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5)(\alpha_5+\alpha_6+\alpha_7)$,
$-\alpha_3(\alpha_1+\alpha_2+\alpha_3)(\alpha_3+\alpha_4+\alpha_5)\alpha_2(\alpha_5+\alpha_6)\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5)(\alpha_5+\alpha_6+\alpha_7)$,
$-\alpha_3\alpha_2\alpha_1\alpha_5\alpha_4\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_6+\alpha_7)$,
$\alpha_3(\alpha_2+\alpha_3)\alpha_1\alpha_5(\alpha_3+\alpha_4)\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_6+\alpha_7)$,
$(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4)\alpha_2\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_6+\alpha_7)$,
$\alpha_3\alpha_2(\alpha_1+\alpha_2)\alpha_5\alpha_4\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_6+\alpha_7)$,
$-\alpha_3(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4)\alpha_2\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_6+\alpha_7)$,
$\alpha_3\alpha_2\alpha_1\alpha_5(\alpha_4+\alpha_5)\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7)$,
$-\alpha_3(\alpha_2+\alpha_3)\alpha_1\alpha_5(\alpha_3+\alpha_4+\alpha_5)\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7)$,
$-(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4+\alpha_5)\alpha_2\alpha_6(\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7)$,
$-\alpha_3\alpha_2(\alpha_1+\alpha_2)\alpha_5(\alpha_4+\alpha_5)\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7)$,
$\alpha_3(\alpha_1+\alpha_2+\alpha_3)\alpha_5(\alpha_3+\alpha_4+\alpha_5)\alpha_2\alpha_6(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)(\alpha_5+\alpha_6+\alpha_7))$

and zeros elsewhere.

Consider the triple scalar product of rows: $(a,b,c)=\sum_{i=1}^{29} a_ib_ic_i$. A (computer) calculation shows that $(r_{29},r_{29},p)=2$. Thus we have proved that the defect of the inclusion $\mathcal{X}^x(k)\hookrightarrow\mathcal{X}_x(k)$ divided by $e_x(k)$ is 0 if $\operatorname{char} k=2$ and v^{-8} otherwise. By Theorem 6.10,

$$\sum_{d\in\mathbb{Z}} m(x,d)v^{-d-8}=\begin{cases}0 & \text{if } \operatorname{char} k=2,\\ v^{-8} & \text{otherwise.}\end{cases}$$

Hence, we get the following result.

THEOREM 6.11. *Let $G=\mathrm{SL}_8(\mathbb{C})$ and $\Pi=\{\alpha_1,\dots,\alpha_7\}$ be the set of simple roots. Consider the Bott–Samelson variety Σ for the sequence $s=(s_3,s_2,s_1,s_5,s_4,s_3,s_2,s_6,s_5,s_4,s_3,s_7,s_6,s_5)$ and take $x=s_2s_3s_2s_5s_6s_5$. Let $\Sigma\to G/B$ be the canonical resolution and k be a field. If $\operatorname{char} k=2$, then $\pi_*\underline{k}_\Sigma[14]$ has no direct summand of the form $\mathcal{E}(x,k)[d]$. If $\operatorname{char} k\neq 2$, then $\mathcal{E}(x,k)$ is its only direct summand of this form. Moreover, it occurs with multiplicity* 1.

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