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CURVES IN GRASSMANN VARIETIES

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§ 0. Introduction

The following question was the main motivation of this paper: which are the necessary and sufficient conditions for a non-singular subvariety of a Grassmann variety to have an ample normal bundle. Knowing that a non-singular subvariety of a Grassmann variety has an ample normal bundle we can apply on it several well-known theorems.

a) A vanishing theorem on formal schemes (Hartshorne [6], Theorem 4.1.).

b) A theorem on meromorphic functions (Hartshorne [7], Chapter 6.).

c) Results on the cohomological dimension of a projective variety minus a subvariety (Hartshorne [7], Chapter 7.).

Let E be the universal subbundle on the Grassmannian $G(r, n)$, Q its universal quotient bundle, Y a closed subscheme of $G(r, n)$, and C a non-singular curve in $G(r, n)$. In this paper we prove the following results.

1) Necessary and sufficient conditions for $\check{E}|_Y$ to be ample.

2) Necessary and sufficient conditions for $Q|_Y$ to be ample.

3) Necessary and sufficient conditions for $T_G|_Y$ to be ample, where T_G is the tangent bundle of $G = G(r, n)$.

4) Necessary and sufficient conditions for $N_{C/G}$ to be ample, where $N_{C/G}$ is the normal bundle of a non-singular curve C in $G = G(r, n)$.

5) Sufficient conditions for $N_{Y/G}$ to be ample, where in this case Y is a non-singular subvariety of $G = G(r, n)$. We also give examples of non-singular subvarieties of a $G(r, n)$ with ample normal bundle which do not satisfy conditions 5) to show that these are not necessary conditions.

The paper is organized as follows: § 1 contains the basic definitions

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and facts about Grassmann varieties, Schubert cycles and ample vector bundles which will be needed. § 2 contains the proofs of the main results of this paper. In § 3 several examples are given to illustrate our results. Also the different applications are explained. In the appendix we outline a proof of the fact that Ω^1_G, the sheaf of 1-differentials on $G = G(r, n)$ can be expressed as $E \otimes \check{Q}$.

I would like to thank my advisor Robin Hartshorne for introducing me in these questions and for his invaluable help.

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§ 1. Definitions and elementary properties

Let $G = G(r, n)$ be the Grassmannian of $\boldsymbol{P}^r$ spaces in a fixed $\boldsymbol{P}^n$ space. The fixed $\boldsymbol{P}^n$ induces a trivial bundle O_G^{n+1} on G of rank $n + 1$, and the $\boldsymbol{P}^r$ subspaces induce a subbundle E of O_G^{n+1} of rank $r + 1$. So on G we have the following canonical short exact sequence

$$0 \longrightarrow E \longrightarrow O_G^{n+1} \longrightarrow Q \longrightarrow 0 .$$

Q is a vector bundle on G of rank $n - r$, called the canonical quotient bundle on G, and E is called the canonical subbundle on G.

$G = G(r, n)$ is uniquely determined by the following universal property of the canonical quotient bundle Q. For all schemes X, for all locally free sheaves Q_0 of rank $n - r$ on X and for all $n + 1$ global sections $s_0, \cdots, s_n \in \Gamma(X, Q_0)$ which generate the stalks of Q_0 at every point $p \in X$, there exists a unique morphism $f: X \to G$ such that:

a) $f^*Q = Q_0$ and

b) $s_i = f^*(x_i)$ for $i = 0, \cdots, n$ where $x_i \in \Gamma(G, Q)$ is the image of 1 in the i^{th} place of O_G^{n+1} under the canonical map

$$O_G^{n+1} \longrightarrow Q \longrightarrow 0 .$$

DEFINITION 1.1. Let $0 \leq a_0 < \cdots < a_r \leq n$ be $r + 1$ integers and let

$A_0 \subset \cdots \subset A_r \subset \boldsymbol{P}^n$ be $r+1$ linear subspaces such that $\dim A_i = a_i$. Let $\Omega(A_0, \cdots, A_r)$ be the subspace of $G(r, n)$ parametrizing all $\boldsymbol{P}^r \subset \boldsymbol{P}^n$ such that $\boldsymbol{P}^r \cap A_i \neq \emptyset$ and $\dim \boldsymbol{P}^r \cap A_i \geq i$ for $i = 0, \cdots, r$. We know from [9] that these subspaces are actually subvarieties of $G(r, n)$ and that they depend, up to rational equivalence, only on the integers $a_0, \cdots, a_r$ and not on the choice of the linear subspaces A_i. We call them Schubert subvarieties or Schubert cycles of $G(r, n)$.

We are interested in the following two Schubert cycles.

1) Let Z_1 be the Schubert cycle of $G(r, n)$ parametrizing all $\boldsymbol{P}^r \subset \boldsymbol{P}^n$ which are contained in a fixed $\boldsymbol{P}^{n-1}$ in $\boldsymbol{P}^n$. This is the Schubert cycle which satisfies the condition $(n-r-1, \cdots, n-2, n-1)$ and it is isomorphic to $G(r, n-1)$. The fixed $\boldsymbol{P}^{n-1}$ space induces a trivial subbundle of $O_G^{n+1}|_{Z_1}$ of rank n, namely $O_G^n|_{Z_1}$, and $E|_{Z_1}$ is a subbundle of $O_G^n|_{Z_1}$. In other words, the canonical short exact sequence,

$$0 \longrightarrow E|_{Z_1} \longrightarrow O_G^{n+1}|_{Z_1} \longrightarrow Q|_{Z_1} \longrightarrow 0$$

factors through

$$0 \longrightarrow E|_{Z_1} \longrightarrow O_G^n|_{Z_1} \longrightarrow Q' \longrightarrow 0$$

where Q' is the canonical quotient bundle of Z_1, when Z_1 is viewed as a $G(r, n-1)$ and $Q|_{Z_1} = Q' \oplus O_{Z_1}$ is the vector bundle on Z_1 of rank $n-r$, which induces the embedding of $Z_1 \hookrightarrow G(r, n)$.

2) Let Z_2 be the Schubert cycle of $G(r, n)$ parametrizing all $\boldsymbol{P}^r \subset \boldsymbol{P}^n$ such that $p \in \boldsymbol{P}^r \subset \boldsymbol{P}^n$, for some fixed point p in $\boldsymbol{P}^n$. This is a Schubert cycle satisfying the condition $(O, n-r+1, \cdots, n-1, n)$. The fixed point p induces a trivial subbundle of $E|_{Z_2}$ of rank 1. In other words, we have on Z_2 the following short exact sequence,

$$0 \longrightarrow O_{Z_2} \longrightarrow E|_{Z_2} \longrightarrow E' \longrightarrow 0$$

where E' is a vector bundle on Z_2 of rank r. Thus on Z_2 we have the short exact sequence,

$$0 \longrightarrow E' \longrightarrow O_G^n|_{Z_2} \longrightarrow Q|_{Z_2} \longrightarrow 0 .$$

By the Principle of Duality we can consider $G(r, n)$ equivalent to $G(n-r-1, n)$. On $G(n-r-1, n)$ we will have the dual short exact sequence,

$$0 \longrightarrow \check{Q} \longrightarrow O_G^{n+1} \longrightarrow \check{E} \longrightarrow 0 .$$

If we let $d = n - r - 1$, we can see from [9] that Z_2 as a Schubert cycle now of $G(d, n)$ it satisfies the condition $(n - d - 1, \cdots, n - 2, n - 1)$.

DEFINITION 1.2. A vector bundle E on a scheme X is ample if for every coherent sheaf F on X there exists an integer $n_0 > 0$ such that for all integers $n \geq n_0$ the sheaf $F \otimes S^n(E)$, (where $S^n(E)$ is the n^{th} symmetric product of E), is generated as an O_X-module by its global sections.

Throughout this paper we will be using basic properties of ample vector bundles which are proved in (Hartshorne [5], 2 and 4).

DEFINITION 1.3. Let L be a line bundle on a non-singular curve C. We define the degree of L to be the degree of D, where D is the corresponding divisor. If C is singular we define $\deg L = \deg L \otimes O_{\tilde{C}}$, where $\tilde{C}$ is the normalization of C. If E is a vector bundle of rank r then we define $\deg E = \deg \bigwedge^r E$.

The following results of Hartshorne and Gieseker will be very useful.

THEOREM 1.4 (Hartshorne [8], Theorem 2.4.). *Let X be a complete non-singular curve over k,* $\operatorname{char} k = 0$, *and E a vector bundle on X. Then E is ample if and only if every quotient bundle of E has positive degree.*

THEOREM 1.5 (Gieseker [4], Prop. 2.1.). *Let X be a complete scheme over a field k. Let E be a vector bundle on X which is generated by its global sections. Then E is ample if and only if for every curve $C \subseteq X$, and for every quotient line bundle L of $E|_C$,* $\deg L > 0$.

Throughout the paper we are working over a field k of characteristic 0.

§2. Curves with ample normal bundle

Our main theorem can be stated as follows.

THEOREM 2.1. *Let $C \subset G(r, n)$ be a non-singular curve. Let $N_{C/G}$ be the normal bundle of C in $G(r, n)$. Then $N_{C/G}$ is not ample if and only if C lies in some Z_3, where Z_3 is the Schubert cycle parametrizing $\{\boldsymbol{P}^r \subset \boldsymbol{P}^n |$ some fixed point $p \in \boldsymbol{P}^r \subset$ some fixed $\boldsymbol{P}^{n-1} \subset \boldsymbol{P}^n\}$.*

The proof of the theorem will follow immediately from the following four propositions. By Z_1 and Z_2 we denote the two Schubert cycles as defined in §1.

PROPOSITION 2.2. *Let Y be a subvariety of $G(r, n)$. Then $\check{E}|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_2.*

PROPOSITION 2.3. *Let Y be a subvariety of $G(r, n)$. Then $Q|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_1.*

PROPOSITION 2.4. *Let Y be a subvariety of $G(r, n)$. Then $T_G|_Y$ is not ample if and only if there is a curve C in Y which lies in some Z_3.*

PROPOSITION 2.5. *Let C be a non-singular curve in $G(r, n)$. Then $N_{C/G}$ is ample if and only if $T_G|_C$ is ample.*

We will need the following lemmas.

LEMMA 2.6. *Let Y be a subvariety of $G(r, n)$. Then there exists a splitting $Q|_Y = Q_0 \oplus O_Y$ if and only if Y lies in some $Z_1 = G(r, n-1) \subset G(r, n)$. Then $Q|_Y$ embeds $Y \hookrightarrow G(r, n)$ and Q_0 embeds $Y \hookrightarrow Z_1$.*

Proof of lemma. a) Assume Y lies in some $Z_1 \subset G(r, n)$. Then $Q|_Y = (Q|_{Z_1})|_Y = Q'|_Y \oplus O_Y$, where Q' is the canonical quotient bundle of $Z_1 = G(r, n-1)$.

b) Suppose there exists a splitting of $Q|_Y$, and $Q|_Y = Q_0 \oplus O_Y$. Q_0 is a vector bundle on Y of rank $n - r - 1$, generated by n global sections. Therefore, Q_0 induces a unique morphism,

$$f : Y \longrightarrow Z_1 = G(r, n-1)$$

such that $f^*(Q') = Q_0$, where Q' is the canonical quotient of Z_1, and $f^*(x_i) = s_i$ for all n global sections $s_i \in \Gamma(Y, Q_0)$ generating the stalks of Q_0 at every point p of Y. Let

$$g : Z_1 \hookrightarrow G(r, n)$$

be the embedding induced by $Q' \oplus O_{Z_1}$. By the universal property of the canonical quotient bundle Q on $G(r, n)$ we know that $g^*(Q) = Q' \oplus O_{Z_1}$ and $g^*(y_i) = x_i$, where $y_0, \cdots, y_n \in \Gamma(G, Q)$ and $x_0, \cdots, x_{n-1}, 1 \in \Gamma(Z_1, Q' \oplus O_{Z_1})$. Thus

$$f^*(g^*(Q)) = Q_0 \oplus O_Y$$

and

$$f^*(g^*(y_i)) = s_i ,$$

where $s_0, \cdots, s_{n-1}, 1$ are in $\Gamma(Y, Q_0 \oplus O_Y)$, so by uniqueness $g \circ f$ is the embedding induced by $Q_0 \oplus O_Y$, $Y \hookrightarrow G(r, n)$. Therefore,

$$f: Y \hookrightarrow Z_1$$

is an embedding. q.e.d.

By the Principle of Duality and its application on Z_2 that we explained in §1, the following lemma is equivalent to Lemma 2.6.

LEMMA 2.7. *Let Y be a subvariety of $G(r, n)$. Then there exists a splitting of $E|_Y = E_0 \oplus O_Y$ if and only if Y lies in some $Z_2 \subset G(r, n)$.*

We can now prove our four propositions.

Proof of Proposition 2.2. Suppose that there is a curve C in Y which lies in some Z_2. By Lemma 2.7. there is a splitting $\check{E}|_C = \check{E}_0 \oplus O_C$. Hence $\check{E}|_C$ is not ample and that implies that $\check{E}|_Y$ is not ample.

Now suppose that $\check{E}|_Y$ is not ample. Then by Theorem 1.5. there exists a curve C in Y and a quotient line bundle E' of $\check{E}|_C$ on C

$$\check{E}|_C \longrightarrow E' \longrightarrow 0$$

such that $\deg E' \leq 0$.

CLAIM. $E' = O_C$. We break the proof of the claim into two cases.

Case 1. Assume C is a non-singular curve. E' is a line bundle on C generated by its global sections. Therefore, E' corresponds to an effective divisor D on C such that $D = \sum_i n_i p_i$, where $n_i \geq 0$ and $\sum_i n_i = \deg E' \leq 0$. Therefore $\sum_i n_i = 0$ and $D = 0$, the zero divisor. Hence $E' = O_C$.

Case 2. Let C be any reduced, irreducible curve not necessarily non-singular. E' is generated by its global sections; hence there exists $0 \neq s \in \Gamma(C, E')$ which induces an injective map $O_C \to E'$, and E' has no torsion. Thus we have the short exact sequence,

$$0 \longrightarrow O_C \longrightarrow E' \longrightarrow F \longrightarrow 0\,, \tag{1)}$$

where F is the quotient sheaf. Let $\tilde{C}$ be the normalization of C. Tensoring the short exact sequence 1) by $O_{\tilde{C}}$ we get

$$O_{\tilde{C}} \longrightarrow E' \otimes O_{\tilde{C}} \longrightarrow F \otimes O_{\tilde{C}} \longrightarrow 0\,.$$

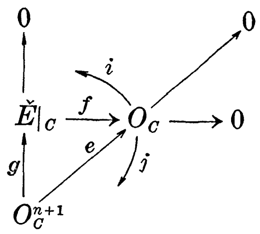
$\tilde{C}$ is non-singular and by definition $\deg E' \otimes O_{\tilde{C}} = \deg E' \leq 0$. Therefore by the non-singular case $E' \otimes O_{\tilde{C}} = O_{\tilde{C}}$. This implies that $F \otimes O_{\tilde{C}} = 0$.

The morphism $\tilde{C} \to C$ is finite and surjective, therefore $F = 0$ also. Hence $E' = O_C$.

Going now back to the proof of Prop. 2.2., we have shown that on C we have a surjective map

$$f: \check{E}|_C \longrightarrow O_C \longrightarrow 0 .$$

We actually have the following diagram on C



$e = f \circ g$ but e splits, i.e. there exists a map j such that $e \circ j = \mathrm{id}_{O_C}$. Now if we let $i = g \circ j$ we get

$$f \circ i = f \circ g \circ j = e \circ j = \mathrm{id}_{O_C} .$$

Therefore the map f splits and $\check{E}|_C = E_0 \oplus O_C$, where E_0 is some vector bundle on C of rank r. Thus by Lemma 2.7. C lies in some Z_2. q.e.d.

Proof of Proposition 2.3. Suppose that there is a curve C in Y which lies in some Z_1. Then by Lemma 2.6. there exists a splitting of $Q|_C$, $Q|_C = Q_0 \oplus O_C$. Thus $Q|_C$ is not ample. Hence $Q|_Y$ is not ample.

Now suppose that $Q|_Y$ is not ample. Then by Theorem 1.5. there exists a curve C in Y and a quotient bundle Q' of $Q|_C$ on C

$$Q|_C \longrightarrow Q' \longrightarrow 0$$

such that $\deg Q' \leq 0$. $Q|_C$ is generated by its global sections and hence Q' is also, so we can go through the same steps as in the proof of Proposition 2.2. and show that $Q|_C = Q_0 \oplus O_C$, where Q_0 is some vector bundle on C. Thus by Lemma 2.6. C lies in some Z_1.

With the two propositions we just proved and with the fact that $T_G = Q \otimes \check{E}$ (see Appendix), where T_G is the tangent bundle on $G = G(r, n)$, we can prove immediately the next proposition.

Proof of Proposition 2.4. Assume that $T_G|_Y$ is not ample. Hence by Theorems 1.4. and 1.5. there exists a curve C in Y such that $T_G|_C = Q|_C \otimes \check{E}|_C$ is not ample. Since both $Q|_C$ and $\check{E}|_C$ are generated by global sections and their tensor product $Q|_C \otimes \check{E}|_C$ is not ample, we get from (Hartshorne [5], 2) that neither $Q|_C$ nor $\check{E}|_C$ is ample. Hence by Propositions 2.2. and 2.3. C lies in some $Z_3 = Z_1 \cap Z_2$.

Now assume that C lies in some $Z_3 = Z_1 \cap Z_2$, where C is a curve in Y. By Lemmas 2.6. and 2.7. we have the following splittings

$$Q|_C = Q_0 \oplus O_C \quad \text{and}$$
$$\check{E}|_C = \check{E}_0 \oplus O_C \,.$$

Thus

$$T_G|_C = Q|_C \otimes \check{E}|_C = (Q_0 \otimes \check{E}_0) \oplus Q_0 \oplus \check{E}_0 \oplus O_C$$

and hence $T_G|_C$ is not ample. Therefore $T_G|_Y$ is not ample. q.e.d.

Proof of Proposition 2.5. Assume that $N_{C/G}$ is not ample. Then from the short exact sequence

$$0 \longrightarrow T_C \longrightarrow T_G|_C \longrightarrow N_{C/G} \longrightarrow 0$$

we have that $T_G|_C$ is not ample.

Now assume that $T_G|_C$ is not ample. Hence by Proposition 2.4. C lies in some Z_3. The cycle Z_3 is isomorphic to the Grassmannian $G(r-1, n-2)$. So let Q_Z and E_Z be its canonical quotient bundle and its canonical subbundle, respectively. Let T_Z be its tangent bundle. Hence $T_Z = Q_Z \otimes \check{E}_Z$. We have on Z_3 the short exact sequence

$$0 \longrightarrow T_Z \longrightarrow T_G|_Z \longrightarrow N_{Z/G} \longrightarrow 0$$

and furthermore

$$T_G|_Z = Q|_Z \otimes \check{E}|_Z = (Q_Z \oplus O_Z) \otimes (\check{E}_Z \oplus O_Z)$$
$$= T_Z \oplus Q_Z \oplus \check{E}_Z \oplus O_Z \,.$$

Therefore

$$N_{Z/G} = Q_Z \oplus \check{E}_Z \oplus O_Z \quad \text{and}$$
$$N_{Z/G}|_C = (Q_Z \oplus \check{E}_Z)|_C \oplus O_C \,.$$

Hence $N_{Z/G}|_C$ is not ample. Finally from the short exact sequence

$$0 \longrightarrow N_{C/Z} \longrightarrow N_{C/G} \longrightarrow N_{Z/G}|_C \longrightarrow 0$$

we get that $N_{C/G}$ is not ample. q.e.d.

Remark. One can easily show that any Schubert cycle in $G(r, n)$ which is isomorphic to a lower Grassmannian does not have an ample normal bundle. In particular if Z is a cycle of type Z_1, then its normal bundle is not ample and it is actually isomorphic to $\check{E}|_Z$.

For non-singular subvarieties of $G(r, n)$ of dimension higher than 1, we have the following sufficient conditions.

PROPOSITION 2.8. *Let Y be a non-singular subvariety of $G(r, n)$. Assume that Y does not contain any curve C which lies in a Z_3. Then the normal bundle $N_{Y/G}$ is ample.*

Proof. $N_{Y/G}$ is generated by its global sections. Hence if $N_{Y/G}$ is not ample, then there is a curve C in Y such that $N_{Y/G}|_C$ is not ample. Thus from the short exact sequence

$$0 \longrightarrow T_Y|_C \longrightarrow T_G|_C \longrightarrow N_{Y/G}|_C \longrightarrow 0$$

we get that $T_G|_C$ is not ample. Therefore by Proposition 2.4. C lies in some Z_3, and this contradicts our hypothesis. q.e.d.

For non-singular subvarieties of codimension 1 in $G(r, n)$ we can show that their normal bundle is always ample.

PROPOSITION 2.9. *Let Y be a non-singular subvariety of $G(r, n)$ with $\mathrm{cod}_G\, Y = 1$. Then $N_{Y/G}$ is ample.*

Proof. A subvariety Y of codimension 1 can be seen as an effective divisor on $G = G(r, n)$. Therefore it corresponds to a line bundle L. Since $\mathrm{Pic}\,(G) \cong Z$, $L = O_G(v)$. Y is an effective divisor. This implies that $v > 0$. We have the short exact sequence

$$0 \longrightarrow O_G(-v) \longrightarrow O_G \longrightarrow O_Y \longrightarrow 0\,,$$

where $O_G(-v)$ is the sheaf of ideals defining Y in G. Therefore

$$(\check{N}_{Y/G}) = O_G(-v) \otimes O_Y = O_Y(-v) \qquad \text{and}$$
$$N_{Y/G} = O_Y(v) \qquad \text{where } v > 0\,.$$

Hence $N_{Y/G}$ is ample. q.e.d.

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§ 3. Applications and examples

Let $G = G(1,3)$, the Grassmannian parametrizing $\boldsymbol{P}^1$ spaces in a fixed $\boldsymbol{P}^3$. Then $\dim G = 4$ and G is a quadric in $\boldsymbol{P}^5$.

The Schubert cycles of G can be described as follows.

a) $Z_0 = G \cap H$, H a hyperplane in $\boldsymbol{P}^5$, $\dim Z_0 = 3$,
$Z_0 = \{\boldsymbol{P}^1 \subset \boldsymbol{P}^3 | \boldsymbol{P}^1 \cap \text{some fixed } \boldsymbol{P}^1 \neq \emptyset\}$.

b) $Z_1 = \{\boldsymbol{P}^1 \subset \boldsymbol{P}^3 | \boldsymbol{P}^1 \subset \text{some fixed } \boldsymbol{P}^2 \subset \boldsymbol{P}^3\} \cong \boldsymbol{P}^2$.
$Z_2 = \{\boldsymbol{P}^1 \subset \boldsymbol{P}^3 | \text{ some fixed point } p \in \boldsymbol{P}^1 \subset \boldsymbol{P}^3\} \cong \boldsymbol{P}^2$.

c) $Z_3 = \{\boldsymbol{P}^1 \subset \boldsymbol{P}^3 | \text{ some fixed point } p \in \boldsymbol{P}^1 \subset \text{some fixed } \boldsymbol{P}^2 \subset \boldsymbol{P}^3\} \cong \boldsymbol{P}^1$.

d) $p =$ any point.

So according to our Theorem 2.1., the only curves in $G(1,3)$ that do not have an ample normal bundle are the Schubert cycles of the type $Z_3 \cong \boldsymbol{P}^1$. Using Grothendieck's theorem on the decomposition of vector bundles on $\boldsymbol{P}^1$, one can actually calculate the normal bundle of a curve C in $G(1,3)$ of the type Z_3 and show that $N_{C/G} = O_C(1) \oplus O_C(1) \oplus O_C$.

By § 2, the non-singular subvarieties of G of dimension 3 always have an ample normal bundle, the non-singular surfaces in G which do not contain a cycle of type Z_3 have an ample normal bundle, while the surfaces which are cycles of type Z_1 or Z_2 do not. The case left unanswered is the case of a non-singular surface in G which is not a Schubert cycle but which contains a curve of type Z_3. With the following example we show that such a surface in G could have an ample normal bundle. Hence we show that the conditions in Proposition 2.8. for a non-singular subvariety Y in a $G(r,n)$ to have an ample normal bundle are sufficient but not necessary conditions.

EXAMPLE 3.1. Let $Y = G(1,3) \cap$ some general linear $\boldsymbol{P}^3 \subset \boldsymbol{P}^5$. Y is a proper intersection of $G(1,3)$ and $\boldsymbol{P}^3$ in $\boldsymbol{P}^5$. Therefore

$$N_{Y/G} \cong N_{\boldsymbol{P}^3/\boldsymbol{P}^5}|_Y$$

which is ample. It is easy to see that Y contains a cycle of type Z_3. $G(1,3) \subset \boldsymbol{P}^5$ is a quadric defined by the equation $x_0x_1 + x_2x_3 + x_4x_5 = 0$, $(x_0, x_1, x_2, x_3, x_4, x_5)$ are the homogeneous coordinates of $\boldsymbol{P}^5$. Let $\boldsymbol{P}^3$ be given by $(x_0 = x_2 = 0)$. Let C be given by $(x_0 = x_2 = x_4 = x_5 = 0)$. Then C is a cycle of type Z_3 and lies in $G(1,3) \cap \boldsymbol{P}^3 = Y$.

Therefore Y is an example of a subvariety of a Grassmannian such that $T_G|_Y$ is not ample though $N_{Y/G}$ is ample. Hence Proposition

2.5. can not be extended to subvarieties of a $G(r, n)$ of dimension greater than 1.

Another application of our main theorem is the following corollary.

COROLLARY 3.2. *Let* $G = G(r, n)$. *Then* T_G *is ample if and only if* G *is a trivial Grassmannian, i.e.* $G \cong \boldsymbol{P}^n$.

Proof. Assume $G = G(r, n)$ is a non-trivial Grassmannian, i.e. $r \neq 0$, $r \neq n - 1$. Therefore if $Z \subset G(r, n)$ is a cycle of type Z_3, then $\dim Z \geq 1$. Hence it contains at least a curve. Hence $T_G|_Z$ is not ample, which implies that T_G is not ample.

Now assume that $G = \boldsymbol{P}^n$. On $\boldsymbol{P}^n$ we have the following short exact sequence

$$0 \longrightarrow E \longrightarrow O_{\boldsymbol{P}^n}^{n+1} \longrightarrow O_{\boldsymbol{P}^n}(1) \longrightarrow 0 .$$

where rank $E = n$ and $O_{\boldsymbol{P}^n}(1) = Q$. Tensoring the short exact sequence by $\check{Q}$ and then dualizing we get

$$0 \longrightarrow O_{\boldsymbol{P}^n} \longrightarrow O_{\boldsymbol{P}^n}^{n+1}(1) \longrightarrow \check{E} \otimes Q \longrightarrow 0 .$$

Hence $T_{\boldsymbol{P}^n} = \check{E} \otimes Q$ is ample.

To subvarieties of $G(r, n)$ with ample normal bundle we can apply the following well-known results.

THEOREM 3.3 (Hartshorne [6], Theorem 4.1.). *Let* X *be a non-singular projective variety over a field* k, char $k = 0$, *and let* Y *be a non-singular subvariety of* X *of dimension* d. *Let* $\hat{X}$ *be the formal completion of* X *along* Y. *Assume that* $N_{Y/X}$ *is ample. Then* $H^i(\hat{X}, \hat{F})$ *is a finite-dimensional* k*-vector space for all locally free sheaves* F *on* X *and all* $i < d$.

DEFINITION 3.4. Let S be a noetherian scheme of finite Krull dimension. We define the cohomological dimension of S, written $cd(S)$, to be the smallest integer $n \geq 0$ such that $H^i(S, F) = 0$ for all $i > n$ and for all coherent sheaves F on S.

DEFINITION 3.5. Let S be a scheme of finite type over k. We define the integer $q(S)$ to be the smallest integer $n \geq -1$ such that $H^i(S, F)$ is a finite-dimensional k-vector space for all $i > n$, and for all coherent sheaves F on S.

DEFINITION 3.6. Let S be as above. We define the integer $p(S)$ to be the largest integer n (or ∞) such that $H^i(S, F)$ is a finite-dimensional k-vector space for all $i < n$, and for all locally free sheaves F on S.

THEOREM 3.7 (Hartshorne [7], Corollary 5.5.). *Let X, Y and d be as in Theorem* 3.3. *Then*

a) $p(U) = q(U) = n - d - 1$, *where* $U = X - Y$, $n = \dim X$,

b) *If $O_X(1)$ is an ample line bundle on X, then $H^i(U, F(m)) = 0$ for all $i \geq n - d$, m 0, and for all coherent sheaves F on $U = X - Y$.*

THEOREM 3.8 (Hartshorne [7], Corollary 5.4.). *Let X and Y be as above and assume further that X is connected and that $d \geq 1$. Then*

$$H^0(\hat{X}, O_{\hat{X}}) = k ,$$

i.e., there are no non-constant holomorphic functions on X.

Also from (Hartshorne [7], Corollary 6.8.) we get that any subvariety of a Grassmannian with ample normal bundle is $G2$, i.e. the field of meromorphic functions $K(\hat{G})$ on the formal completion of G along the subvariety Y is a finite algebraic extension of $K(G)$.

Appendix

In this paper we have been using the following well-known fact.

THEOREM A.1. *Let $G = G(r, n)$, E the canonical subbundle on G and Q the canonical quotient bundle on G. Then the sheaf of* 1*-differentials on G can be expressed as follows,*

$$\Omega^1_G = E \otimes \check{Q} .$$

We will outline here a proof of this fact, since we were not able to find a reference for it.

We will recall first some basic properties of the associated projective r-space bundle over a projective variety X and state the theorems that are needed for the proof.

DEFINITION A.2. Let X be a projective variety, E a vector bundle on X of rank r. We let $\boldsymbol{P}(E)$ be the associated projective $(r-1)$-space bundle: the fibre over a point $x \in X$ is a projective space of hyperplanes of the vector space E_x.

Let $p : \boldsymbol{P}(E) \to X$ be the projection map. Then the bundle E lifts to a bundle $p^*(E)$ on $\boldsymbol{P}(E)$. There is a canonical quotient line bundle of

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$p^*(E)$, L_E, which to each point $y \in \boldsymbol{P}(E)$ it associates the 1-dimensional vector space $E/E_{p(y)}$, where $E_{p(y)}$ is the hyperplane corresponding to the point y. Then $p_* L_E = E$ and $p_* \check{L}_E = 0$.

THEOREM A.3 ([3], SGA 1, II, Thm. 4.3.). *Let X and Y be non-singular projective varieties over a field k. Let $p: X \to Y$ be a smooth morphism. Then there is a short exact sequence of differentials*

$$0 \longrightarrow p^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 ,$$

where $\Omega^1_{X/Y}$ is the sheaf of relative differentials.

THEOREM A.4 (Manin [11], Prop. 17.12). *Let $X = \boldsymbol{P}(E)$ be the associated projective $(r-1)$-space bundle over Y, where E is a vector bundle of rank r on Y. Let $p: \boldsymbol{P}(E) \to Y$ be the projection map. Then there is a short exact sequence*

$$0 \longrightarrow \Omega^1_{X/Y} \longrightarrow p^*(E) \otimes \check{L}_E \longrightarrow O_X \longrightarrow 0 ,$$

where L_E is the canonical quotient line bundle of $p^*(E)$.

THEOREM A.5 (Manin [11], Prop. 7.7.). *Let X and Y be as above. Then*

a) $R^i p_*(p^*(H) \otimes F) = H \otimes R^i p_* F$ *for all $i \geq 0$, for all vector bundles H on Y and for all vector bundles F on X.*

b) $R^0 p_*(O_X) = O_Y$ *and*

$R^i p_*(O_X(n)) = 0$ *for all $i > 0$ and for all $n \geq 0$.*

Proof of Theorem A.1. We want to show that

$$\Omega^1_G = E \otimes \check{Q} ,$$

where $G = G(r, n)$ is any Grassmannian and E, Q are the canonical subbundle and quotient bundle on G, respectively. We have seen that the theorem is true for $r = n - 1$. So we are going to use descending induction on r and the incidence correspondence. Let $G_{r,r-1}$ be the subspace of

$$G(r, n) \times G(r-1, n)$$

parametrizing all chains $(\boldsymbol{P}^{r-1} \subset \boldsymbol{P}^r)$ of subspaces of a fixed $\boldsymbol{P}^n$. $G_{r,r-1}$ is called the incidence correspondence.

Let p, q be the restrictions to $G_{r,r-1}$ of the projections of $G(r, n) \times G(r-1, n)$ onto the first and second factor, respectively. Let $G = G_{r,r-1}$,

$G' = G(r, n)$ and $G'' = G(r-1, n)$. So we have the following diagram

$$\begin{array}{ccc} & G = G_{r,r-1} & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} \\ G' = G(r,n) & & G'' = G(r-1,n) \end{array}$$

We also have the following canonical short exact sequences

$$0 \longrightarrow E' \longrightarrow O_{G'}^{n+1} \longrightarrow Q' \longrightarrow 0 \qquad \text{on } G' ,$$

and

$$0 \longrightarrow E'' \longrightarrow O_{G''}^{n+1} \longrightarrow Q'' \longrightarrow 0 \qquad \text{on } G'' .$$

CLAIM. a) $G = \boldsymbol{P}(E')$,
b) $G = \boldsymbol{P}(\check{Q}'')$.

Proof.

a) $$\begin{array}{c} G = G_{r,r-1} \\ {\scriptstyle p}\big\downarrow \\ G' = G(r,n) \end{array}$$

Each point $y \in G$ corresponds to a chain $(\boldsymbol{P}^{r-1} \subset \boldsymbol{P}^r)$, so it corresponds to a hyperplane of the vector space $E'_{p(y)}$. Thus $G = \boldsymbol{P}(E')$ over G. Furthermore, on $\boldsymbol{P}(E')$ we have the canonical quotient line bundle, $L_{E'}$. The kernel of the map

$$p^*(E') \longrightarrow L_{E'} \longrightarrow 0$$

corresponds to hyperplanes and each hyperplane corresponds to a $\boldsymbol{P}^{r-1}$. Therefore, the kernel is exactly $q^*(E'')$. So on $\boldsymbol{P}(E')$ we have the short exact sequence

$$0 \longrightarrow q^*(E'') \longrightarrow p^*(E') \longrightarrow L_{E'} \longrightarrow 0 .$$

b) $$\begin{array}{c} G = G_{r,r-1} \\ {\scriptstyle q}\big\downarrow \\ G'' = G(r-1,n) \end{array}$$

Each point $y \in G$ corresponds to a chain $(\boldsymbol{P}^{r-1} \subset \boldsymbol{P}^r)$, hence to a line in the vector space $Q''_{q(y)}$. So each point $y \in G$ corresponds to a hyperplane of the vector space $\check{Q}''_{q(y)}$. Hence $G = \boldsymbol{P}(\check{Q}'')$. Furthermore,

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on $\boldsymbol{P}(\check{Q}'')$ we have the canonical line bundle, $M_{Q''}$. The kernel of the map

$$q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0$$

associates to each point $y \in G$ a hyperplane of the vector space $\check{Q}''_{q(y)}$. Hence the kernel is $p^*(\check{Q}')$. So on $\boldsymbol{P}(\check{Q}'')$ we have the short exact sequence

$$0 \longrightarrow p^*(\check{Q}') \longrightarrow q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0 \ .$$

From the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & L_{E'} & & & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \\
0 & \longrightarrow & p^*(E') & \longrightarrow & p^*(O_{G'}^{n+1}) & \longrightarrow & p^*(Q') & \longrightarrow & 0 \\
 & & \uparrow & & \wr\Vert & & \uparrow & & \\
0 & \longrightarrow & q^*(E'') & \longrightarrow & q^*(O_{G''}^{n+1}) & \longrightarrow & q^*(Q'') & \longrightarrow & 0 \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & L_{E'} & & \\
 & & & & & & \uparrow & & \\
 & & & & & & 0 & &
\end{array}$$

we can see that $M_{Q''} = \check{L}_{E'}$.

So we have shown that on G we have the following short exact sequences and relations

1) $$0 \longrightarrow q^*(E'') \longrightarrow p^*(E') \longrightarrow L_{E'} \longrightarrow 0$$

2) $$0 \longrightarrow p^*(\check{Q}') \longrightarrow q^*(\check{Q}'') \longrightarrow M_{Q''} \longrightarrow 0$$

and
$$\begin{aligned}
p_* L_{E'} &= p_* \check{M}_{Q''} = E' \\
p_* \check{L}_{E'} &= p_* M_{Q''} = 0 \\
q_* M_{Q''} &= q_* \check{L}_{E'} = \check{Q}'' \\
q_* \check{M}_{Q''} &= q_* L_{E'} = 0 \ .
\end{aligned}$$

Applying Theorem A.3. on G'' we get

3) $$0 \longrightarrow q^* \Omega^1_{G''} \longrightarrow \Omega^1_G \longrightarrow \Omega^1_{G/G''} \longrightarrow 0 \ .$$

Applying Theorem A.4.. on G'' we get

$$0 \longrightarrow \Omega^1_{G/G''} \longrightarrow q^*(\check{Q}'') \otimes \check{M}_{Q''} \longrightarrow O_G \longrightarrow 0 \ .$$

From the short exact sequence 2) we get

4) $$0 \longrightarrow p^*(\check{Q}') \otimes \check{M}_{Q''} \longrightarrow q^*(\check{Q}'') \otimes \check{M}_{Q''} \longrightarrow O_G \longrightarrow 0 .$$

Therefore $\Omega^1_{G/G''} = p^*(\check{Q}') \otimes \check{M}_{Q''}$ and 3) becomes

5) $$0 \longrightarrow q^*\Omega^1_{G''} \longrightarrow \Omega^1_G \longrightarrow p^*(\check{Q}') \otimes \check{M}_{Q''} \longrightarrow 0 .$$

Applying q_* to 5) we get

$$0 \longrightarrow \Omega^1_{G''} \longrightarrow q_*\Omega^1_G \longrightarrow q_*(p^*(\check{Q}') \otimes \check{M}_{Q''}) \longrightarrow \cdots$$

The third term though is zero for the following reason

$$\begin{array}{c} 0 \longrightarrow q_*(p^*(\check{Q}') \otimes \check{M}_{Q''}) \longrightarrow \check{Q}'' \otimes q_*(M_{Q''}) \longrightarrow \cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \end{array}$$

Here we applied q_* to the short exact sequence 4) and used the projection formula. Therefore

6) $$\Omega^1_{G''} \cong q_*\Omega^1_G .$$

We will use now our induction hypothesis. We assume that for $G' = G(r, n)$ $\Omega^1_{G'} = E' \otimes \check{Q}'$ and we will show that for $G'' = G(r-1, n)$ $\Omega^1_{G''} = E'' \otimes \check{Q}''$. By Theorem A.3. we have on G'

$$0 \longrightarrow p^*\Omega^1_{G'} \longrightarrow \Omega^1_G \longrightarrow \Omega^1_{G/G'} \longrightarrow 0 ,$$

and by Theorem A.4. we have on G'

$$0 \longrightarrow \Omega^1_{G/G'} \longrightarrow p^*(E') \otimes \check{L}_{E'} \longrightarrow O_G \longrightarrow 0 .$$

From the short exact sequence 1) we get

$$0 \longrightarrow q^*(E'') \otimes \check{L}_{E'} \longrightarrow p^*(E') \otimes \check{L}_{E'} \longrightarrow O_G \longrightarrow 0 .$$

Therefore

$$\Omega^1_{G/G'} = q^*(E'') \otimes \check{L}_{E'}$$

and on G' we have

7) $$0 \longrightarrow p^*\Omega^1_{G'} \longrightarrow \Omega^1_G \longrightarrow q^*(E'') \otimes \check{L}_{E'} \longrightarrow 0 .$$

Applying q_* to 7) we get the following long exact sequence

8) $$\begin{array}{l} 0 \longrightarrow q_*p^*\Omega^1_{G'} \longrightarrow q_*\Omega^1_G \xrightarrow{\tau} E'' \otimes \check{Q}'' \longrightarrow \\ \longrightarrow R^1q_*p^*\Omega^1_{G'} \xrightarrow{\chi} R^1q_*\Omega^1_G \longrightarrow E'' \otimes R^1q_*\check{L}_{E'} \end{array}$$

Using the two short exact sequences 1) and 2) one can easily show that $q_*p^*\Omega^1_{G'} = 0$. Hence the map τ in the long exact sequence 8) is injective. Furthermore, using Theorem A.5. one can show that the map χ in 8) is also injective. Thus τ is an isomorphism and this completes the proof by 6). q.e.d.

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