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Examples of dHYM connections in a variable background

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Abstract. We study deformed Hermitian Yang–Mills (dHYM) connections on ruled surfaces explicitly, using the momentum construction. As a main application, we provide many new examples of dHYM connections coupled to a variable background Kähler metric. These are solutions of the moment map partial differential equations given by the Hamiltonian action of the extended gauge group, coupling the dHYM equation to the scalar curvature of the background. The large radius limit of these coupled equations is the Kähler–Yang–Mills system of Álvarez-Cónsul, Garcia-Fernandez, and García-Prada, and in this limit, our solutions converge smoothly to those constructed by Keller and Tønnesen-Friedman. We also discuss other aspects of our examples including conical singularities, realization as B-branes, the small radius limit, and canonical representatives of complexified Kähler classes.

1 Background and main results

1.1 dHYM connections

Let $L \to X$ denote a holomorphic line bundle over a compact n-dimensional Kähler manifold, with a fixed background Kähler form ω. A Hermitian metric h on the fibers of L determines the two notions of *Lagrangian phase* and *radius* of the line bundle. Namely, writing $F(h) = \sqrt{-1}F$, $F \in \mathcal{A}^{1,1}(X, \mathbb{R})$ for the curvature of the Chern connection, one introduces the endomorphism of the tangent bundle given by $\omega^{-1}F$, with eigenvalues λ_i. Then we have

$$\frac{(\omega - \sqrt{-1}F)^n}{\omega^n} = \prod_{i=1}^{n}(1 - \sqrt{-1}\lambda_i) = r_\omega(F)e^{\sqrt{-1}\Theta_\omega(F)},$$

where the Lagrangian phase and radius are defined (using the background metric ω), respectively, as

$$\Theta_\omega(F) = -\sum_{i=1}^{n}\arctan(\lambda_i),\ r_\omega(F) = \prod_{i=1}^{n}(1+\lambda_i^2)^{1/2}.$$

The *deformed Hermitian Yang–Mills (dHYM) equation* (introduced in [17, 18] and surveyed in [4, 5]) is the condition of having constant Lagrangian phase,

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$$\Theta_\omega(F) = \hat{\theta} \mod 2\pi. \tag{1.1}$$

The work of Leung, Yau, and Zaslow [17] shows that, at least under suitable assumptions, the dHYM equation (1.1) is mapped to the special Lagrangian equation under mirror symmetry. Therefore, this equation has attracted considerable interest in mathematical physics and complex differential geometry (see, e.g., the foundational works [3, 6, 15] and the recent contributions [11, 12, 20, 21]).

In the present paper, we study dHYM connections in the very special case when X is a complex ruled surface. While this is a classical test bed for equations in complex differential geometry, here we allow a rather general setup, as we now discuss.

1.2 Variable background dHYM

Most importantly, we couple the dHYM equation (1.1) to a variable background Kähler metric ω, through the equations

$$\begin{cases} \Theta_\omega(F) = \hat{\theta} \mod 2\pi, \\ s(\omega) - \alpha r_\omega(F) = \hat{s} - \alpha \hat{r}, \end{cases} \tag{1.2}$$

where $s(\omega)$, $\hat{s}$, $\hat{r}$ denote the scalar curvature and its average (resp. the average radius), and $\alpha \in \mathbb{R}$ is an arbitrary coupling constant. The quantities $\hat{s}$, $\hat{r}$, and $\hat{\theta}$ are fixed by cohomology, and in particular we have

$$\hat{r} = \frac{1}{n! \operatorname{Vol}(X, \omega)} \left| \int_X (\omega - F(h))^n \right|$$

and

$$e^{-\sqrt{-1}\hat{\theta}} = \frac{1}{n! \operatorname{Vol}(X, \omega) \hat{r}} \int_X (\omega - F(h))^n.$$

Equations (1.2) are obtained by combining the moment map pictures for dHYM connections (due to Thomas and Collins-Yau, see [6, 22]) and for constant scalar curvature Kähler (cscK) metrics (due to Donaldson and Fujiki, see [8, 10]) in a very natural way, through the action of the extended gauge group (a canonical extension of the group of unitary gauge transformations by Hamiltonian symplectomorphisms): this is explained in [19], building on the results of [1]. The coupling constant α is a scale parameter for the relevant symplectic form on the space of integrable connections. Thus, only the case when $\alpha > 0$ corresponds to a genuine Kähler reduction (rather than just a symplectic reduction).

Let Σ be a compact Riemann surface of genus h, with Kähler metric g_Σ of constant scalar curvature $2s_\Sigma$, and let $\mathcal{L} \xrightarrow{p} \Sigma$ denote a holomorphic line bundle of degree $k \in \mathbb{Z}_{>0}$, with $2\pi c_1(\mathcal{L}) = [\omega_\Sigma]$. Since $\operatorname{Vol}(\Sigma) = 2\pi k$, by the Gauss–Bonnet formula, we have

$$s_\Sigma = \frac{1}{\operatorname{Vol}(\Sigma)} \int_\Sigma s_\Sigma \omega_\Sigma = \frac{1}{\operatorname{Vol}(\Sigma)} \int_\Sigma \rho_\Sigma = \frac{2(1-h)}{k},$$

where ρ_Σ denotes the Ricci 2-form of g_Σ.

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We will construct solutions of the coupled equations (1.2) on ruled surfaces of Hirzebruch type, obtained by the projectivization

$$X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma,$$

where $\mathcal{O}$ denotes the trivial holomorphic line bundle. (It is well known that such X does not admit cscK metrics.) Our solutions are obtained by extending the classical *momentum construction* (also known as the Calabi ansatz, see [14]) to equations (1.2) (see (2.2) and (3.1) for our ansatz).

Let $E_0 = \mathbb{P}(0 \oplus \mathcal{O})$ and $E_\infty = \mathbb{P}(\mathcal{L} \oplus 0)$ denote, respectively, the zero section and the infinity section of the $\mathbb{CP}^1$-bundle X over Σ, with general fiber C. We introduce the real parameters k_1, k_2, and $k' > 0$, and consider the cohomology classes

$$\begin{aligned} [\omega] &= 2\pi[2E_0 + k'C], \\ [F] &= 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C], \end{aligned} \tag{1.3}$$

where we slightly abuse the notation and denote the Poincaré duals of E_0 and C, respectively, by $[E_0]$ and $[C]$. Then $[\omega]$ is a Kähler class and $[F/(2\pi)]$ is integral, provided k_1, k_2, k' are integers and $k' > 0$, so it is possible to find a holomorphic line bundle $L \to X$ such that $-2\pi c_1(L) = [F]$.

Remark 1 Equation (1.1) is equivalent to

$$\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega - \sqrt{-1}F\right)^n\right) = 0,$$

and the latter condition is preserved when $F \to -F$ and $\hat{\theta} \to -\hat{\theta}$, which should be interpreted geometrically as considering the dHYM equation on L^{-1} instead of L. With our choice of parametrization, this implies that the set of parameters corresponding to solutions of the system (1.2) is invariant under $k_i \to -k_i$, for $i = 1, 2$. When $k_2 = 0$, it follows from (1.3) that, for any choice of Kähler class, the unique solution of the dHYM equation (1.1) is given by $F = k_1\omega$ (note that uniqueness for dHYM solutions is known in general by the results of [15]). In this case, the Lagrangian radius is also constant $r_\omega(k_1\omega) = (1 + k_1^2)^2$ and, since X does not admit cscK metrics, the system (1.2) has no solution. In Section 2, it will be clear that also for $k_1 = 0$ the dHYM equation has a trivial solution; in this case, $\hat{\theta} = 0$ and we can solve also (1.2). In the following, we will focus on the less trivial choices of parameters, assuming

$$k_1 < 0, \qquad k_2 \neq 0.$$

It is also convenient to introduce the quantity

$$x = \frac{k}{k + k'} \in (0, 1).$$

Theorem 2 *Suppose the "stability condition"*

$$(1 + (k_1 + k_2)^2) > x(1 + (k_1 - k_2)^2) \tag{1.4}$$

holds. Then, there exist a unique Kähler form ω and a curvature form F, with cohomology classes given by (1.3), such that they are obtained by the momentum construction (see

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(2.2) and (3.1)) and solve the coupled equations (1.2) on the ruled surface X, for the unique value of the coupling constant

$$\alpha = \frac{\sqrt{4k_1^2 + (1 - k_1^2 + k_2^2)^2}}{2(1 + (k_1 - k_2)^2)k_2^2} (-2 + s_\Sigma x) .$$

If equality holds instead in (1.4), then there is a smooth solution on $X \backslash E_\infty$, with underlying metric $\omega \in C^{1,1/2}(X) \cap C^\infty(X \backslash E_\infty)$.

Theorem 2 is proved in Section 4.

1.3 Solutions with conical singularities

The main limitation of Theorem 2 concerns the sign of the coupling constant: it is straightforward to check that in the situation of that result, we always have $\alpha < 0$, since $s_\Sigma \leq 2$ and $x < 1$. In order to gain more flexibility, we allow the background metric ω to develop conical singularities along the divisors E_0, E_∞. Fix $0 < \beta_0 \leq 1$, and let

$$\beta_\infty = \frac{-2 + \beta_0(1 + x)}{-1 + x} \geq 1.$$

Theorem 3 *Suppose the "stability condition" (1.4) holds. Then, there exist a unique Kähler form ω and a curvature form F, such that they are obtained by the momentum construction (see (2.2) and (3.1)), ω has conical singularities with cone angles $2\pi\beta_0$ along E_0 and $2\pi\beta_\infty$ along E_∞, the corresponding cohomology classes (in the sense of currents) are given by (1.3), and they solve the coupled equations (1.2), for the unique value of the coupling constant*

$$\alpha = \frac{\sqrt{4k_1^2 + (1 - k_1^2 + k_2^2)^2}}{2(1 + (k_1 - k_2)^2)k_2^2} \frac{3 + x + s_\Sigma x^2 - 3(1 + x)\beta_0}{x}.$$

Theorem 3 is proved in Section 5. Note that this gives a generalization of Theorem 2: when $\beta_0 = 1$, we recover precisely the smooth solutions provided by that result.

Corollary 4 *For sufficiently small cone angle $2\pi\beta_0$ and sufficiently large $k' > 0$, the coupling constant α is positive.*

1.4 Relation to twisted KE metrics

As usual, under a suitable cohomological condition, the equation in (1.2) involving the scalar curvature may be reduced to a condition involving the Ricci curvature. In our case, this condition is given by

$$[\mathrm{Ric}(\omega)] + \frac{\alpha}{2 \sin \hat{\theta}} [F] = \frac{\hat{s} - \alpha \hat{r}}{4} [\omega].$$

Then, the equation

$$s(\omega) - \alpha r_\omega(F) = \hat{s} - \alpha \hat{r}$$

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reduces to the twisted Kähler–Einstein equation

$$\mathrm{Ric}(\omega) + \frac{\alpha}{2\sin\hat{\theta}}F = \frac{\hat{s} - \alpha\hat{r}}{4}\omega. \tag{1.5}$$

We provide an explicit criterion for when this reduction occurs for the class of examples provided by Theorem 3 (in which case $\mathrm{Ric}(\omega)$, F, and ω extend to closed currents on X).

Proposition 5 *The condition $s(\omega) - \alpha r_\omega(F) = \hat{s} - \alpha\hat{r}$ reduces to the twisted Kähler–Einstein equation (1.5) iff we have*

$$\begin{aligned}&(1 + k_1^2 + k_2^2)(x-1)\left(s_\Sigma x^2 - 3\beta_0(x+1) + x + 3\right)\\ &= 2k_1k_2\left(-3\beta_0 + s_\Sigma x^3 - x^2(\beta_0 + s_\Sigma - 1) + 3\right).\end{aligned} \tag{1.6}$$

Moreover, there are infinitely many admissible values of k_1, k_2, k' which satisfy this equality for some β_0 and for which the "stability condition" (1.4) holds (so that the corresponding coupled equations are solvable).

This result is proved in Section 6. Writing the dHYM equation on the surface X in Monge–Ampère form (as in [3]), we see that in the twisted Kähler–Einstein case, the coupled equations (1.2) become

$$\begin{cases}\left(-\sin(\hat{\theta})F + \cos(\hat{\theta})\omega\right)^2 = \omega^2,\\ \mathrm{Ric}(\omega) + \frac{\alpha}{2\sin\hat{\theta}}F = \frac{\hat{s}-\alpha\hat{r}}{4}\omega,\end{cases}$$

and so they are closely related to the systems of coupled Monge–Ampère equations studied by Hultgren and Wytt-Nyström [13].

1.5 Realization as B-branes

Given the origin of the dHYM equation in mirror symmetry, it seems interesting to ask whether the special dHYM connections appearing in Theorem 2, i.e., solutions of the coupled equations (1.2), can in fact be realized as B-branes (i.e., for our purposes, holomorphic submanifolds endowed with a dHYM connection) in some ambient Calabi–Yau manifold (this is how the dHYM equation appears in mathematical physics; see, e.g., [5]). Thus, we are asking for a Calabi–Yau manifold $\check{M}$ with a Ricci flat Kähler metric $\omega_{\check{M}}$, and a holomorphic embedding $\iota\colon X \hookrightarrow \check{M}$, such that the Kähler form ω constructed in Theorem 1.2 is given by the restriction $\omega = \iota^*\omega_{\check{M}}$. We show that this can be achieved at least locally around X, relying on the classical results on Feix [9] on the hyperkähler extension of real analytic Kähler metrics.

Proposition 6 *The Kähler form ω and curvature form F provided by Theorem 2 are real analytic. Thus, ω extends to a hyperkähler metric defined on an open neighborhood of the zero section in the holomorphic cotangent bundle T^*X, and F extends to the curvature form of a hyperholomorphic line bundle defined on the same open neighborhood.*

This result is proved in Section 4.

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1.6 Large and small radius limits

In the mathematical physics literature (see, e.g., [2, Chapter 1]), the dHYM equation involves a "slope" parameter $\alpha' > 0$ (related to the "string length" by $\alpha' = l_s^2$), which appears simply as a scale parameter for the curvature form, $F \mapsto \alpha' F$. The corresponding coupled equations (1.2) are given by

$$\begin{cases} \Theta_\omega(\alpha' F) = \hat{\theta} \mod 2\pi\,, \\ s(\omega) - \alpha r_\omega(\alpha' F) = \hat{s} - \alpha \hat{r}. \end{cases} \tag{1.7}$$

The expressions "large radius limit" (or "zero slope limit") refer to the behavior of the dHYM equations and their solutions as $\alpha' \to 0$. As explained in [19], the large radius limit of our coupled equations is the (rank 1 case of) the Kähler–Yang–Mills system introduced by Álvarez-Cónsul, Garcia-Fernandez, and García-Prada [1]. We can prove a much stronger result, at the level of solutions, on the ruled surface X.

Theorem 7 *For all sufficiently small α', depending only on the fixed parameters k_1, k_2, k', (i.e., on the fixed cohomology classes $[\omega]$ and $[F]$), the coupled equations (1.7) are uniquely solvable on X with the momentum construction. Moreover, as $\alpha' \to 0$, the corresponding solutions $\omega_{\alpha'}$, $F_{\alpha'}$ converge smoothly to a solution of the Kähler–Yang–Mills system*

$$\begin{cases} \Lambda_\omega F = \mu, \\ s(\omega) + \tilde{\alpha}\, \Lambda_\omega^2 (F \wedge F) = c, \end{cases} \tag{1.8}$$

for some (explicit) coupling constant $\tilde{\alpha}$.

The particular solutions of the Kähler–Yang–Mills system obtained in this limit are due to Keller and Tønnesen-Friedman [16].

Similarly, the "small radius limit" (or "infinite slope limit") concerns the behavior of the coupled equations (1.7) as $\alpha' \to \infty$.

Theorem 8 *Fix parameters k_1, k_2, k' (i.e., cohomology classes $[\omega]$ and $[F]$) such that the "stability condition"*

$$(k_1 + k_2)^2 > x(k_1 - k_2)^2$$

holds. Then the coupled equations (1.7) are uniquely solvable on X with the momentum construction, for all $\alpha' > 0$. Moreover, as $\alpha' \to \infty$, the corresponding solutions $\omega_{\alpha'}$ and $F_{\alpha'}$ converge smoothly to a solution of the system

$$\begin{cases} F \wedge \omega = c_1 F^2, \\ s(\omega) - \hat{\alpha}\, \Lambda_\omega F = c_2, \end{cases}$$

for some (explicit) coupling constant $\hat{\alpha}$.

At least in the case when F is Kähler, this system couples the J-equation $\Lambda_F \omega = c_1'$ for F to a twisted cscK equation for ω. In general, these limiting equations belong to a class of coupled PDEs studied by Datar and Pingali [7].

Theorems 7 and 8 are proved in Section 7.

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1.7 Complexified Kähler classes

Complexified Kähler classes are expressions of the form $[\omega + \sqrt{-1}B]$, where ω is a Kähler form and $[B] \in H^2(M,\mathbb{R})/H^2(M,\mathbb{Z})$ is known as the B-field. They play an important role in mirror symmetry (see, e.g., [23, Section 2]). Let M be a compact Kähler manifold with no holomorphic 2-forms. Collins and Yau [6, Section 8] consider a dHYM equation on M of the form

$$\Theta_\omega(F+B) = \hat{\theta} \mod 2\pi,$$

where $\sqrt{-1}F$ is the unknown curvature form of a Hermitian holomorphic line bundle $L \to M$ and B is a fixed representative of a (lift of a) B-field. Arguing from mirror symmetry, they propose that the existence of a solution F should be related, conjecturally, to a suitable notion of stability of the object L with respect to the complexified Kähler class $[\omega + \sqrt{-1}B]$.

In the special case when L is the trivial bundle $\mathcal{O}_M$, the equation becomes

$$\Theta_\omega(B + \sqrt{-1}\partial\bar{\partial}u) = \hat{\theta} \mod 2\pi,$$

so we are effectively trying to find a canonical representative of the B-field $[B]$ with respect to a background Kähler form ω; the existence of such a representative should be related to the stability of the object $\mathcal{O}_M$ with respect to $[\omega + \sqrt{-1}B]$.

Our coupled equations

$$\begin{cases} \Theta_\omega(B) = \hat{\theta} \mod 2\pi, \\ s(\omega) - \alpha r_\omega(B) = \hat{s} - \alpha\hat{r}, \end{cases} \tag{1.9}$$

with $[B]$ a (lift of a) class in $H^2(M,\mathbb{R})/H^2(M,\mathbb{Z})$, can then be thought of as trying to prescribe a canonical representative of the complexified Kähler class $[\omega + \sqrt{-1}B]$. Note that in the Calabi–Yau case, at zero coupling $\alpha = 0$ and in the large radius limit, these equations for the complex form $\omega + \sqrt{-1}B$ reduce to the conditions

$$\begin{cases} \Delta_\omega B = 0, \\ \mathrm{Ric}(\omega) = 0, \end{cases}$$

which are standard in the physics literature (see, e.g., [2, Section 1.1]).

As an example, we shall discuss the existence of such a canonical representative for the complexified Kähler class

$$[\omega + \sqrt{-1}B] = 2\pi(2E_0 + (k' + \sqrt{-1}k'')C)$$

on our ruled surfaces X, where the Kähler condition is equivalent to $k' > 0$. The key observation is that this can be expressed in the form

$$[\omega] = 2\pi[2E_0 + k'C],$$
$$[B] = 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C]$$

with the special choices

$$k_1 = k_2 = \frac{k''}{2(k+k')},$$

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provided we have $k'' < 0$. Thus, we may apply Theorem 2 (and, more generally, Theorem 3 in the case of conical singularities) to show that the coupled equations (1.9) are solvable, uniquely under the momentum construction, iff the "stability condition"

$$(1 + (k_1 + k_2)^2) = 1 + \left(\frac{k''}{k + k'}\right)^2 > x(1 + (k_1 - k_2)^2) = \frac{k}{k + k'}$$

holds. But, clearly, this is automatically satisfied. By Remark 1, the same argument works for the case $k'' > 0$.

Corollary 9 *The complexified Kähler class*

$$[\omega + \sqrt{-1}B] = 2\pi(2E_0 + (k' + \sqrt{-1}k'')C),$$

where $k' > 0$, $k'' \neq 0$, admits a canonical representative. This also holds allowing conical singularities; the corresponding coupling constant is given by

$$\alpha = \frac{2\sqrt{(k + k')^2 + (k'')^2}\left(k^2(-6\beta_0 + s_\Sigma + 4) + (7 - 9\beta_0)kk' - 3(\beta_0 - 1)(k')^2\right)}{k(k'')^2}.$$

Note that a canonical representative with vanishing B-field $B = 0$ would correspond to a cscK metric, which does not exist. The coupling constant α diverges as $k'' \to 0$. It seems interesting that a nontrivial B-field can stabilize the unstable ruled surface X.

Plan of the paper. In Section 2, we set up the momentum construction on our ruled surfaces. Section 3 solves the dHYM equation on our ruled surfaces explicitly using the momentum construction, under the necessary "stability condition" (1.4). This result is applied in Section 4 in order to solve the coupled equations (1.2). All of this is extended to allow conical singularities in Section 5; the main advantage is that in this case there exist solutions with positive coupling constants. Finally, Section 7 contains our results on the large and small radius limits.

2 Momentum construction

Let $X = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \to \Sigma$ be a ruled surface as in the Introduction. Let

$$E_0 = \mathbb{P}(0 \oplus \mathcal{O}), \; E_\infty = \mathbb{P}(\mathcal{L} \oplus 0)$$

denote, respectively, the zero section and the infinity section of the $\mathbb{CP}^1$-bundle X over Σ, with general fiber C. We have the straightforward intersection formulae:

$$E_0 \cdot E_0 = -E_\infty \cdot E_\infty = k, \quad C \cdot C = 0, \quad C \cdot E_0 = C \cdot E_\infty = 1. \tag{2.1}$$

We will follow the standard *momentum construction* (sometimes called the Calabi ansatz; see, e.g., [14]) for metrics on the complement of the zero section $X_0 = \mathcal{L} \backslash E_0$, which extend across the zero and infinity sections of X under suitable conditions.

Thus, we consider metrics of the form

$$\omega = \frac{p^*\omega_\Sigma}{x} + \sqrt{-1}\partial\bar{\partial} f(s), \tag{2.2}$$

where x is a real parameter satisfying $0 < x < 1$, while f is a strictly convex function, such that $f' : X_0 \to (-1, 1)$. The real coordinate s is the log norm of the Hermitian metric $h(z)$ on $\mathcal{L}$ for which $-\partial_z\overline{\partial}_z \log(h) = F(h) = -\sqrt{-1}\omega_\Sigma$. Considering a trivialization $U \subset \mathcal{L}$ with adapted bundle coordinates (z, w), s is given by

$$s = \log|(z,w)|_h^2 = \log|w|^2 + \log h(z),$$

and it follows that

$$\sqrt{-1}\partial_w\overline{\partial}_w f(s) = \sqrt{-1}f''(s)\frac{dw \wedge d\bar{w}}{|w|^2}$$

and

$$\sqrt{-1}\partial_z\overline{\partial}_z f(s) = -f'(s)\omega_\Sigma + \sqrt{-1}f''(s)\frac{\partial_z h\overline{\partial}_z h}{h^2}.$$

If we choose U such that $d\log h(z_0) = 0$ in (z_0, w_0), at this point, all the mixed derivatives vanish and so we find

$$\omega = \frac{1 - xf'(s)}{x}\omega_\Sigma + \sqrt{-1}f''(s)\frac{dw \wedge d\bar{w}}{|w|^2};$$

moreover, we also have, globally,

$$\omega^2 = \frac{2}{|w|^2}\frac{1 - xf'(s)}{x}f''(s)\omega_\Sigma \wedge \sqrt{-1}dw \wedge d\bar{w}.$$

Since $f(s)$ is strictly convex, we may consider its Legendre transform $u(\tau)$, a function of the variable $\tau = f'(s)$, and define the *momentum profile*

$$\phi(\tau) = \frac{1}{u''(\tau)} = f''(s),$$

which must satisfy the condition

$$\phi(\tau) > 0, \quad -1 < \tau < 1, \tag{2.3}$$

required for ω to be positive. Moreover, the momentum construction shows that in order to extend ω across $w = 0$ and $w = \infty$, $\phi(\tau)$ must satisfy the boundary conditions

$$\lim_{\tau\to\pm1}\phi(\tau) = 0, \quad \lim_{\tau\to\pm1}\phi'(\tau) = \mp1. \tag{2.4}$$

The space $H^2(X, \mathbb{R})$ is generated by the Poincaré duals of E_0 and C. Following [16], we define the 2-form

$$\beta = \frac{x^2}{(1 - xf'(s))^2}\left(\frac{1 - xf'(s)}{x}\omega_\Sigma - \sqrt{-1}f''(s)\frac{dw \wedge d\bar{w}}{|w|^2}\right).$$

A direct computation shows that β is a closed $(1,1)$-form, traceless with respect to ω, and $\{\omega, \beta\}$ is a basis for the space $H^2(X, \mathbb{R})$. We consider now a real $(1,1)$ cohomology class and its representative

$$F_0 = c_1\omega + c_2\beta. \tag{2.5}$$

In order to identify $\sqrt{-1}F_0$ with the curvature form of a connection on some line bundle over X, $[F_0/(2\pi)]$ must be an integral class. For $[F_0] = a[E_0] + b[C]$, using

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the identities (2.1), we have

$$a = \int_C F, \qquad b = \int_{E_0} F_0 - k \int_C F_0. \tag{2.6}$$

Since $E_0 = (f')^{-1}(-1)$, we get

$$\int_{E_0} \omega = \frac{(1+x)}{x} \int_\Sigma \omega_\Sigma = 2\pi k \frac{(1+x)}{x}$$

and

$$\int_{E_0} \beta = \frac{x}{(1+x)} \int_\Sigma \omega_\Sigma = 2\pi k \frac{x}{(1+x)}.$$

For the general fiber C, let w denote the bundle adapted coordinate along the fiber and define $r = |w|$, such that $s = 2\log r$ and $d/ds = \frac{r}{2} d/dr$. Using the boundary conditions (2.4), we have

$$\begin{aligned}
\int_C \omega &= \int_{\mathbb{C}\setminus\{0\}} \sqrt{-1} f''(s) \frac{dw \wedge d\bar{w}}{|w|^2} \\
&= \int_{-\infty}^{+\infty} \int_0^{2\pi} \frac{d}{dr} f'(s) dr \wedge d\theta \\
&= 2\pi \left(\lim_{s\to\infty} f'(s) - \lim_{s\to-\infty} f'(s) \right) \\
&= 4\pi
\end{aligned}$$

and similarly

$$\int_C \beta = -4\pi \frac{x^2}{1-x^2}.$$

Using (2.6), we obtain

$$\left[\frac{F_0}{2\pi}\right] = \left(2c_1 - 2\frac{x^2}{1-x^2} c_2\right) E_0 + \left(\frac{1-x}{x} kc_1 + \frac{x}{1-x} kc_2\right) C.$$

If we introduce the new parametrization

$$x = \frac{k}{k+k'}, \; c_1 = k_1, \; c_2 = \frac{1-x^2}{x^2} k_2, \tag{2.7}$$

for real k_1, k_2, and $k' > 0$, then a direct calculation shows that the cohomology classes of $[\omega]$ and $[F_0]$ are given by our previous formulae

$$\begin{aligned}
[\omega] &= 2\pi[2E_0 + k'C], \\
[F_0] &= 2\pi[2(k_1 - k_2)E_0 + (2kk_2 + k'(k_1 + k_2)C].
\end{aligned}$$

In particular, we see that the choices $k' \in \mathbb{Z}_{>0}$ and $k_i \in \mathbb{Z}$, for $i = 1, 2$, correspond to integral classes.

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3 dHYM on ruled surfaces

In this section, we will solve the dHYM equation (1.1) on X explicitly, with respect to a fixed Kähler metric ω obtained by the momentum construction (2.2). Given a class $[F]$ satisfying the integrality conditions (2.7), we may fix a holomorphic line bundle $L \to X$ with the first Chern class $-2\pi[c_1(L)] = [F]$.

Recall that the parameter $\hat{\theta}$ is a topological constant determined by the condition

$$\int_X (\omega - \sqrt{-1}F)^2 \in \mathbb{R}_{>0} e^{\sqrt{-1}\hat{\theta}}.$$

Lemma 10 *We have*

$$e^{\sqrt{-1}\hat{\theta}} = \frac{\left(1 - k_1^2 + k_2^2 - 2\sqrt{-1}k_1\right)}{\sqrt{\left(1 - k_1^2 + k_2^2\right)^2 + \left(2k_1\right)^2}}.$$

Proof Since β is traceless with respect to ω, we only need to compute the quantities $\int_X \omega^2, \int_X \beta^2$. We have

$$\begin{aligned}\int_X \omega^2 &= 2\int_X f''(s)\frac{(1 - xf'(s))}{x}\omega_\Sigma \wedge \frac{dw \wedge d\bar{w}}{|w|^2}\\ &= 4\pi \int_\Sigma \omega_\Sigma \int_0^\infty \frac{d}{dr}\left(\frac{f'(s)}{x} + \frac{(f'(s))^2}{2}\right)dr\\ &= \frac{16\pi^2 k}{x}\end{aligned}$$

and similarly

$$\int_X \beta^2 = -\frac{16\pi^2 k}{x}\frac{x^4}{(1 - x^2)^2}.$$

Using (2.7), we find

$$\int_X \left(\omega - \sqrt{-1}F\right)^2 = \frac{16\pi^2 k}{x}\left(1 - k_1^2 + k_2^2 - 2\sqrt{-1}k_1\right),$$

from which the claim follows immediately. ∎

In order to solve the dHYM equation in the class $[F_0]$, we extend the momentum construction by making the ansatz

$$F = F_g = F_0 + \sqrt{-1}\partial\bar{\partial}g(s). \tag{3.1}$$

It will be convenient to introduce the function $v(\tau)$ given by the image of $g'(s)$ under the Legendre transform diffeomorphism relative to $f(s)$.

Lemma 11 *The form $\sqrt{-1}\partial\bar{\partial}g(s)$ extends smoothly to an exact form on X iff $v(\tau)$ extends smoothly to the interval $[-1, 1]$ and vanishes at the boundary points.*

Proof The component of $\sqrt{-1}\partial\bar{\partial}g(s)$ in the fiber direction is

$$\sqrt{-1}g''(s)\frac{dw \wedge d\bar{w}}{|w|^2} = \sqrt{-1}v'(\tau)\phi(\tau)\frac{dw \wedge d\bar{w}}{|w|^2}.$$

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So $\sqrt{-1}\partial\bar{\partial}g(s)$ extends smoothly to X iff $v(\tau)$ extends smoothly to $[-1,1]$. In order to derive the appropriate boundary behavior so that this extension is still exact, we compute

$$\int_{E_0} \partial\bar{\partial}g = -2\pi k\left(\lim_{s\to-\infty} g'(s)\right)$$

and

$$\int_{C} \partial\bar{\partial}g = 2\pi\left(\lim_{s\to\infty} g'(s) - \lim_{s\to-\infty} g'(s)\right).$$

Using (2.6), the only conditions we need to impose are

$$\lim_{\tau\to\pm 1} v(\tau) = 0. \tag{3.2}$$

∎

Our next result shows how to reduce the dHYM equation to an ODE. It is convenient to introduce the new variable

$$t = 1/x - \tau$$

as well as the auxiliary function

$$H(t) = k_1 t + \frac{k_2}{t}\frac{1-x^2}{x^2} - v(t). \tag{3.3}$$

Proposition 12 *Under the momentum construction (2.2) and (3.1), the dHYM equation is equivalent to the ODE*

$$H'(t) = \frac{t\sin\hat{\theta} + H(t)\cos\hat{\theta}}{H(t)\sin\hat{\theta} - t\cos\hat{\theta}}, \tag{3.4}$$

together with the boundary conditions

$$\begin{aligned} H\left(\frac{1+x}{x}\right) &= k_1\left(\frac{1+x}{x}\right) + k_2\left(\frac{1-x}{x}\right), \\ H\left(\frac{1-x}{x}\right) &= k_1\left(\frac{1-x}{x}\right) + k_2\left(\frac{1+x}{x}\right). \end{aligned} \tag{3.5}$$

Proof At a point (z_0, w_0) such that $d\log h(z_0) = 0$, we have

$$\begin{aligned} &\omega - iF_g = \\ &\left(\left(1-\sqrt{-1}k_1\right)\frac{1-xf'}{x} - \sqrt{-1}\frac{k_2}{x}\frac{1-x^2}{1-xf'} + \sqrt{-1}g'\right)\omega_\Sigma + \\ &\left(f''\left(1-\sqrt{-1}k_1 + \sqrt{-1}k_2\frac{1-x^2}{1-xf'}\right) - \sqrt{-1}g''\right)\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}, \end{aligned}$$

and we obtain the global identity

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$$\frac{1}{2}\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega-\sqrt{-1}F_g\right)^2\right)/\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_\Sigma =$$

$$-\sin\hat{\theta}\left(f''\frac{1-xf'}{x}+\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1f'\right)\left(g''+k_1f''-k_2f''\frac{1-x^2}{(1-xf')^2}\right)\right)$$

(3.6)

$$+\cos\hat{\theta}\left(\frac{1-xf'}{x}\left(k_2f''\frac{1-x^2}{(1-xf')^2}-g''-k_1f''\right)+f''\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1f'\right)\right).$$

This expression becomes much simpler under the Legendre transform diffeomorphism in terms of the variable $\tau=f'(s)$, for which $d\tau/ds=\phi(\tau)$, and the additional affine change of variable $t=1/x-\tau$. Setting

$$H(t)=k_1t+\frac{k_2}{t}\frac{1-x^2}{x^2}-v(t),$$

the dHYM equation is equivalent to

$$2\phi\left(\cos\hat{\theta}\,(H+tH')+\sin\hat{\theta}\,(t-HH')\right)=0$$

and, since $\phi>0$, also to

$$H'=\frac{t\sin\hat{\theta}+H\cos\hat{\theta}}{H\sin\hat{\theta}-t\cos\hat{\theta}}.$$

A direct computation shows that the boundary conditions (3.2) for $g(s)$, rephrased in term of $H(s)$, become the constraints (3.5). ∎

Corollary 13 *The ODE (3.4) is solvable with the boundary conditions (3.5) iff the "stability condition"*

$$\left(1+(k_1+k_2)^2\right)>x\left(1+(k_1-k_2)^2\right)$$

holds.

Proof Setting $tv=H$, equation (3.4) becomes

$$tv'=-2\frac{\xi(v)}{\xi'(v)},\tag{3.7}$$

with $\xi(v)=v^2\sin\hat{\theta}-2v\cos\hat{\theta}-\sin\hat{\theta}$. Solving (3.7) by separation of variables, we get

$$\xi(v)=\frac{C}{t^2},$$

which has two solutions given by

$$H_\pm(t)=t\cot\hat{\theta}\pm\sqrt{\left(\cot^2\hat{\theta}+1\right)(t^2+C')},\tag{3.8}$$

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with $C' = C \sin\hat{\theta}$. We need to impose the appropriate boundary conditions (3.5). The first condition at $1/x+1$ holds iff we choose the solution H_- in (3.8) and set

$$C = \frac{-2k_2\left(1+(k_1+k_2)^2 - x^2 - (k_1-k_2)^2 x^2\right)}{x^2\sqrt{(1-k_1^2+k_2^2)^2+(2k_1)^2}}.$$

In this case, at $1/x-1$, we have

$$\begin{aligned} H_-\left(\frac{1-x}{x}\right) &= \frac{1}{2xk_1}\left(-k_1^2(-1+x)+\left(1+k_2^2\right)(-1+x)\right) \\ &+\left|\frac{-1-(k_1+k_2)^2+x+(k_1-k_2)^2 x}{2k_1 x}\right| \\ &= \begin{cases} k_1\left(\frac{1-x}{x}\right)+k_2\left(\frac{1+x}{x}\right), & \text{if } \left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right), \\ \frac{(1+k_2^2)(x-1)-k_1k_2(1+x)}{k_1 x}, & \text{if } \left(1+(k_1+k_2)^2\right) < x\left(1+(k_1-k_2)^2\right), \end{cases} \end{aligned}$$

so the second condition in (3.5) holds iff we have

$$\left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right).$$

■

Remark 14 Jacob and Yau [15] showed that the solvability of the dHYM equation on compact Kähler surfaces is equivalent to a certain numerical "stability condition." Considering the closed, real $(1,1)$-form

$$\Omega = \cot\hat{\theta}\,\omega - F,$$

the relevant condition is $[\Omega] > 0$. In our setting, when we regard $H^2(X,\mathbb{R})$ as $\mathbb{R}^2$ with the basis provided by the Poincaré duals of E_0 and C and coordinates (a_1,a_2), the Kähler cone is identified with the subset $\{a_1>0, a_2>0\}$. A computation shows that the $[\Omega]$ is positive precisely when the condition (1.4) is satisfied.

Remark 15 Suppose equality holds instead in our "stability condition" (1.4),

$$\left(1+(k_1+k_2)^2\right) = x\left(1+(k_1-k_2)^2\right).$$

A direct computation then shows that the quantity $t^2 + C'$ vanishes at the endpoint $t = 1/x - 1$. By our explicit formula (3.8), we see that the function $H_-(t)$ is smooth on the interval $(1/x-1, 1/x+1]$ and extends to a $C^{1/2}$ function on its closure. Thus, for fixed background ω, we obtain a corresponding solution to the dHYM equation which is smooth on $X\backslash E_\infty$ and extends to a form with $C^{1/2}$ coefficients on X. This should be compared with a result of Takahashi [21] which holds for a general compact Kähler surface X, and states that under suitable assumptions, when the class $[\Omega]$ above is only semipositive, then there exists a solution to the dHYM equation which is smooth on the complement of finitely many holomorphic curves of negative self-intersection and which extends to a closed current on X.

4 Coupled equations

In the previous section, we solved the dHYM equation in suitable integral classes, determining explicitly the Legendre transform of the curvature form F in terms of the Kähler metric ω. More precisely, let us assume that the "stability condition"

$$\left(1+(k_1+k_2)^2\right) > x\left(1+(k_1-k_2)^2\right)$$

holds, and let us denote by $F = F(\omega)$ the unique curvature form constructed in the previous section.

In this section, we will complete the proof of Theorem 2 by solving the second equation in (1.2). We also establish the real analyticity of our solutions, Proposition 6.

Recall that we are concerned with the equation

$$s(\omega) - \alpha \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\frac{\left(\omega-\sqrt{-1}F\right)^2}{\omega^2}\right) = \hat{s} - \alpha\hat{r}, \tag{4.1}$$

where the constants $\hat{s}$ and $\hat{r}$ can be computed as

$$\hat{s} = 2xs_\Sigma + 2,\ \hat{r} = \sqrt{\left(1-k_1^2+k_2^2\right)^2+4k_1^2}.$$

Lemma 16 *In terms of the variable $t = 1/x - \tau$ and the function $H(t)$ appearing in (3.4), we have*

$$\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\frac{\left(\omega-\sqrt{-1}F\right)^2}{\omega^2}\right)$$
$$= \cos\hat{\theta}\left(1-\frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t)+\frac{H(t)}{t}\right).$$

Proof As in the proof of Proposition 12, at a point (z_0, w_0) such that $d\log h(z_0) = 0$, we have the global identities

$$x\frac{1}{2}\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}}\left(\omega-\sqrt{-1}F_g\right)^2\right)/\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_\Sigma =$$
$$\cos\hat{\theta}\left(f''\frac{1-xf'}{x}+\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1f'\right)\left(g''+k_1f''-k_2f''\frac{1-x^2}{(1-xf')^2}\right)\right)$$
$$+\sin\hat{\theta}\left(\frac{1-xf'}{x}\left(k_2f''\frac{1-x^2}{(1-xf')^2}-g''-k_1f''\right)+f''\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1f'\right)\right)$$

and

$$\omega^2 = 2f''\frac{1-xf'}{x}\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\wedge\omega_\Sigma.$$

In terms of the variable t and the auxiliary function $H(t)$, we have

$$\left(f''\frac{1-xf'}{x}+\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1f'\right)\left(g''+k_1f''-k_2f''\frac{1-x^2}{(1-xf')^2}\right)\right)$$
$$= 1-\frac{H(t)H'(t)}{t},$$

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respectively,

$$\left(\frac{1-xf'}{x}\left(k_2 f''\frac{1-x^2}{(1-xf')^2}-g''-k_1 f''\right)+f''\left(g'-\frac{k_2}{x}\frac{1-x^2}{1-xf'}-\frac{k_1}{x}+k_1 f'\right)\right)$$
$$=-H'(t)-\frac{H(t)}{t},$$

from which our claim follows immediately. ■

Lemma 17 *Equation (4.1) becomes the ODE for the momentum profile $\phi(t)$ given by*

$$\left(\frac{2s_\Sigma}{t}-\frac{1}{t}(2t\phi(t))''\right)+2\alpha\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}}-\frac{\alpha}{\sin^3\hat{\theta}}\frac{t}{\sqrt{(\cot^2\hat{\theta}+1)(t^2+C')}}$$
$$-\frac{\alpha}{t\sin\hat{\theta}}\sqrt{(\cot^2\hat{\theta}+1)(t^2+C')}=\hat{s}-\alpha\hat{r},$$

with the boundary conditions

$$\lim_{t\to\frac{1}{x}\pm1}\phi(t)=0,\qquad \lim_{t\to\frac{1}{x}\pm1}\phi'(t)=\mp1.$$

Proof By a standard computation, the scalar curvature of ω can be expressed in terms of the variable τ as

$$s(\omega)=\frac{2s_\Sigma x}{1-x\tau}-\frac{x}{1-x\tau}\left(2\phi(\tau)\frac{1-x\tau}{x}\right)'',$$

with $\phi(\tau)$ satisfying (2.4). After the affine change of variable $t=1/x-\tau$, our claim follows directly from Lemma 16 and the explicit formula (3.8) for $H(t)$. ■

Setting $\psi(t)=2t\phi(t)$, we obtain the ODE

$$\psi''(t)=\left(2\alpha\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}}-\hat{s}+\alpha\hat{r}\right)t-\frac{\alpha}{\sin\hat{\theta}}\sqrt{(\cot^2\hat{\theta}+1)(t^2+C')}$$
$$-\frac{\alpha}{\sin^3\hat{\theta}}\frac{t^2}{\sqrt{(\cot^2\hat{\theta}+1)(t^2+C')}}+2s_\Sigma \tag{4.2}$$

with the boundary conditions

$$\lim_{t\to\frac{1}{x}\pm1}\psi(t)=0,\qquad \lim_{t\to\frac{1}{x}\pm1}\psi'(t)=\mp2\left(\frac{1}{x}\pm1\right), \tag{4.3}$$

and the positivity condition

$$\psi(t)>0,\qquad \frac{1}{x}-1<t<\frac{1}{x}+1. \tag{4.4}$$

By integrating twice, we get the general solution of (4.2) with integration constants d_0, d_1

$$\psi(t)=s_\Sigma t^2+\left(\frac{\alpha}{3}\frac{\cos\hat{\theta}}{\sin^2\hat{\theta}}-\frac{\hat{s}-\alpha\hat{r}}{6}\right)t^3-\frac{\alpha}{3}\sin\hat{\theta}\left((\cot^2\hat{\theta}+1)(t^2+C')\right)^{\frac{3}{2}}$$
$$+d_0+d_1 t, \tag{4.5}$$

which satisfies (4.3) if and only if we set

$$d_0 = -\frac{(-2+s_\Sigma x)\left(-3-3k_1^2-2k_1k_2-3k_2^2+3\left(1+(k_1-k_2)^2\right)x^2\right)}{3\left(1+(k_1-k_2)^2\right)x^3},$$

$$d_1 = -\frac{\left(-2\left(1+k_1^2+k_2^2\right)+\left(1+(k_1-k_2)^2\right)s_\Sigma x\right)\left(-1+x^2\right)}{4k_1k_2x^2},$$

$$\alpha = \frac{\sqrt{4k_1^2+(1-k_1^2+k_2^2)^2}}{2\left(1+(k_1-k_2)^2\right)k_2^2}(-2+s_\Sigma x).$$

In order to check the positivity condition (4.4), we observe that

$$\frac{d^4\psi}{dt^4} = -\frac{3}{4}\alpha\left(\frac{C'}{k_1}\right)^2\left(t^2+C'\right)^{-\frac{5}{2}} > 0. \tag{4.6}$$

Moreover, setting $t_- = 1/x - 1$ and $t_+ = 1/x + 1$, we get

$$\begin{aligned}&\psi''(t_-)-\psi''(t_+)\\&\quad = 4\frac{-3\left(1+(k_1+k_2)^2\right)^2+\left(1+(k_1-k_2)^2\right)^2x^2(x+s_\Sigma)}{-\left(1+(k_1+k_2)^2\right)^2+\left(1+(k_1-k_2)^2\right)^2x^2} > 0,\end{aligned} \tag{4.7}$$

since $s_\Sigma + x < 3$. Thus, ψ'' is a convex function defined on the interval $[t_-, t_+]$, such that $\psi''(t_-) > \psi''(t_+)$, and this, together with (4.3), implies the positivity condition (4.4).

Finally, let us note that if equality holds in our "stability condition"

$$\left(1+(k_1+k_2)^2\right) = x\left(1+(k_1-k_2)^2\right),$$

then the quantity $t^2 + C'$ vanishes at the endpoint $t = 1/x - 1$, and by our explicit formulae (3.8) and (4.5), we obtain a solution ω, F which is smooth on $X\backslash E_\infty$, and such that F extends to a form with $C^{1/2}$ coefficients on X, while ω extends with $C^{1,1/2}$ coefficients. This completes the proof of Theorem 2.

Remark 18 As we will be interested in the small and large limits of the coupled equations, we point out that (4.6) and (4.7) hold uniformly as the scaling parameter $\alpha' \to 0$ and, provided the "stability condition"

$$(k_1+k_2)^2 > x(k_1-k_2)^2$$

is satisfied, also for $\alpha' \to \infty$.

We can now prove Proposition 6. We first claim that the Kähler form ω constructed above is real analytic. Recall that ω is obtained by the momentum construction (2.2),

$$\omega = \frac{p^*\omega_\Sigma}{x} + \sqrt{-1}\partial\bar{\partial}f(s),$$

for a suitable convex function $f:\mathbb{R}\to\mathbb{R}$, where we have $s=\log|w|^2+\log h(z)$ with respect to bundle adapted holomorphic coordinates (z,w). The hyperbolic metric ω_Σ is real analytic, so we can choose a local holomorphic coordinate z such that its coefficients are real analytic. On the other hand, the real function $h(z)$ satisfies $-\sqrt{-1}\partial_z\overline{\partial}_z\log h(z)=\omega_\Sigma$, with the same choice of local coordinate, and so it is also real analytic. So our claim follows if we can show that the function $f:\mathbb{R}\to\mathbb{R}$ is real analytic. But f is related to the momentum profile ϕ by the ODE

$$f''(s)=\phi(\tau)=\phi(f'(s)),$$

and the momentum profile $\phi(\tau)$ or our solution is clearly a real analytic function of the variable $\tau\in(-1,1)$ by (4.5). Thus, $f(s)$ is real analytic and our claim on ω follows. In order to see that the curvature form F is also real analytic, recall that it is given by our ansatz (3.1), $F=F_0+\sqrt{-1}\partial\overline{\partial}g(s)$, and that the dHYM equation satisfied by F can be expressed in terms of $g(s)$ as the vanishing of the right-hand side of the expression (3.6). Thus, the real analyticity of $g(s)$ follows from that of $f(s)$.

5 Conical singularities

In the present section, we prove Theorem 3. This extends our existence result, Theorem 2, to allow a Kähler form ω with conical singularities. Our main motivation for this extension is describing examples of solutions to the coupled equations (1.2) with positive coupling constant $\alpha>0$.

We consider again Kähler forms ω given by the momentum construction (2.2),

$$\omega=\frac{p^*\omega_\Sigma}{x}+\sqrt{-1}\partial\overline{\partial}f(s),$$

with momentum profile $\phi(\tau)>0$ defined on the interval $(-1,1)$.

Lemma 19 *The Kähler form ω extends to a form with conical singularities on X, with cone angle $2\pi\beta_0$ along E_0 (resp. $2\pi\beta_\infty$ along E_∞), iff the momentum profile satisfies the boundary conditions*

$$\lim_{\tau\to\pm1}\phi(\tau)=0,\ \lim_{\tau\to-1}\phi'(\tau)=\beta_0,\ \lim_{\tau\to1}\phi'(\tau)=-\beta_\infty.$$

Proof For any open neighborhood $U\subset X$, in terms of the bundle adapted coordinates (z,w), $E_0\cap U=\{w=0\}$. We assume that, near $r=|w|=0$, f'' has the form

$$f''(s)=c_0r^{2\beta_0}+A(r)$$

with $c_0\neq0$ and $A(r)=o(r^{2\beta_0})$. Then we have

$$\begin{aligned}
\omega_{z\bar z}&=\left(\frac{1-xf'}{x}\right)\omega_{\Sigma,z\bar z}+\sqrt{-1}f''\frac{\partial h\overline{\partial}h}{h^2}=O(1),\\
\omega_{w\bar z}&=-\sqrt{-1}\frac{1}{w}f''(s)\overline{\partial}h=O(r^{2\beta_0-1}),\\
\omega_{z\bar w}&=\sqrt{-1}\frac{1}{\overline{w}}f''(s)\partial h=O(r^{2\beta_0-1}),\\
\omega_{w\bar w}&=\sqrt{-1}r^{2\beta_0-2}\left(1+A(r)/r^{2\beta_0}\right);
\end{aligned}$$

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hence, the metric ω given by the momentum construction has a conical singularity along E_0 of angle $2\pi\beta_0$. Since $d/ds = \frac{r}{2}d/dr$, $f''(s) = \phi(\tau)$, and $f'''(s) = \phi(\tau)\phi'(\tau)$, this implies

$$\lim_{\tau\to-1}\phi(\tau) = 0$$

and

$$\lim_{\tau\to-1}\phi'(\tau) = \beta_0.$$

To proof for E_∞ is the same up to a change of variable. ∎

As in the previous section, it is convenient to consider the reparametrization

$$x = \frac{k}{k+k'}$$

for $k' > 0$. Similarly, we introduce the $(1,1)$-forms

$$\beta = \frac{x^2}{(1-xf'(s))^2}\left(\frac{1-xf'(s)}{x}\omega_\Sigma - \sqrt{-1}f''(s)\frac{dw\wedge d\bar{w}}{|w|^2}\right),$$
$$F = k_1\omega + \frac{1-x^2}{x^2}k_2\beta,$$

as well as the ansatz, extending the momentum construction

$$F_g = F + \sqrt{-1}\partial\bar{\partial}g(s).$$

We also denote by $v(\tau)$ the image of $g'(s)$ under the Legendre transform diffeomorphism relative to $f(s)$. The proof of the following result is almost identical to the smooth case, and we leave it to the reader.

Lemma 20 *A Kähler form ω with conical singularities as above is a closed $(1,1)$-current on X, with cohomology class*

$$[\omega] = 2\pi[2E_0 + k'C].$$

Similarly, F is a closed $(1,1)$-current on X with cohomology class

$$[F] = 2\pi[2(k_1-k_2)E_0 + (2kk_2 + k'(k_1+k_2)C].$$

Moreover, $\sqrt{-1}\partial\bar{\partial}g(s)$ extends to a closed $(1,1)$-current on X, which has vanishing cohomology class iff $v(\tau)$ satisfies the boundary conditions

$$\lim_{\tau\to\pm1}v(\tau) = 0.$$

We are now in a position to complete the proof of Theorem 3. Let us first note that, precisely as in the proof of Theorem 2, under the momentum construction, the dHYM equation for ω and F becomes the ODE (3.4), together with the boundary conditions (3.5). By Lemma 20, the cone angles do not play a role in this reduction. It follows that the second of our coupled equations (1.2) also reduces to the same ODE (4.2) for a single function $\psi(t) > 0$ of the variable

$$t = 1/x - \tau$$

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appearing in the proof of Theorem 2. By Lemma 19, the boundary conditions corresponding to general cone angles β_0, β_∞ are

$$\lim_{t\to\pm 1}\psi(t) = 0, \lim_{t\to\frac{1}{x}+1}\psi'(t) = -2\beta_0\left(\frac{1}{x}+1\right), \lim_{t\to\frac{1}{x}-1}\psi'(t) = 2\beta_\infty\left(\frac{1}{x}-1\right).$$

However, as (4.2) is second-order ODE, this problem is overdetermined. If we consider the general solution (4.5) and impose the boundary condition

$$\lim_{t\to\frac{1}{x}+1}\psi'(t) = -2\beta_0\left(\frac{1}{x}+1\right)$$

corresponding to a cone angle $2\pi\beta_0$ along E_0, we find that the integration constant d_1 can be expressed in terms of β_0 and the coupling constant α as

$$\begin{aligned} d_1 = {}& \frac{(x+1)(2k_1(x(-2\beta_0+s_\Sigma(x-1)+1)+1)}{2k_1x^2} \\ & -\alpha\frac{k_2(x-1)\sqrt{k_1^4-2k_1^2(k_2^2-1)+(k_2^2+1)^2)}}{2k_1x^2}. \end{aligned} \tag{5.1}$$

Similarly, imposing the condition

$$\lim_{t\to\frac{1}{x}+1}\psi(t) = 0$$

and using our expression for d_1 gives the relation

$$\begin{aligned} d_0 = {}& 4\alpha\frac{k_2^2\left(x^3(k_1-k_2)^2+(k_1+k_2)^2+x^3+1\right)}{3x^3\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}} \\ & -\frac{(x+1)^2\sqrt{k_1^4-2k_1^2\left(k_2^2-1\right)+\left(k_2^2+1\right)^2}(x(-6\beta_0+s_\Sigma(2x-1)+2)+2)}{3x^3\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}}. \end{aligned}$$

Further, imposing the condition

$$\lim_{t\to\frac{1}{x}-1}\psi(t) = 0$$

and using our expressions for d_0, d_1 determines the coupling constant uniquely as

$$\alpha = \frac{\sqrt{\left(-k_1^2+k_2^2+1\right)^2+4k_1^2}\left(s_\Sigma x^2-3\beta_0(x+1)+x+3\right)}{2k_2^2x\left((k_1-k_2)^2+1\right)}. \tag{5.2}$$

We can now compute directly that a solution $\psi(t)$ corresponding to a cone angle $2\pi\beta_0$ along E_0 satisfies

$$\lim_{t\to\frac{1}{x}-1}\psi'(t) = -\frac{2(\beta_0+\beta_0x-2)}{x} = 2\frac{-2+\beta_0(1+x)}{-1+x}\left(\frac{1}{x}-1\right),$$

which yields a cone angle $2\pi\beta_\infty$ along E_∞, with

$$\beta_\infty = \frac{-2+\beta_0(1+x)}{-1+x}.$$

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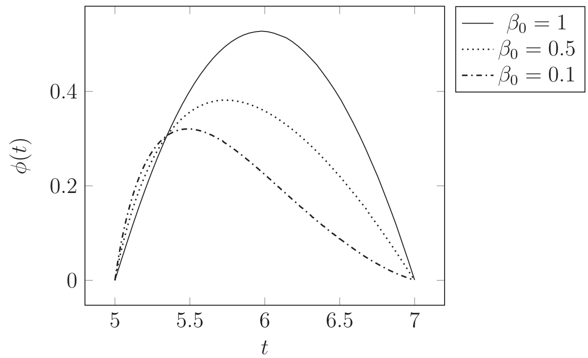

Figure 1. The momentum profile $\phi(t)$ of the solution when $k_2 = -k_1 = 1$, $h = 0$ and $x = 1/6$.

In order to prove the positivity of $\psi(t)$, we consider again (4.6), with the coupling constant α given by (5.2). When

$$3(1+x)\beta_0 - 3 > x(1+s_\Sigma x),$$

we construct solutions for $\alpha < 0$ and hence $\frac{d^4\psi}{dt^4} > 0$. Moreover,

$$\psi''(t_-) - \psi''(t_+) = 4\frac{(3(1+x)\beta_0 - 3)\left(1+(k_1+k_2)^2\right)^2 - x^2\left(1+(k_1-k_2)^2\right)^2(s_\Sigma x^2 + x)}{\left(1+(k_1+k_2)^2\right)^2 x - \left(1+(k_1-k_2)^2\right)^2 x^3} > 0,$$

and we can use essentially the same argument given in the proof of Theorem 2. When $\alpha > 0$, an explicit analysis of the momentum profile is more complicated and the positivity of $\psi(t)$ is best checked with the assistance of a numerical software package (see Figure 1). This completes the proof of Theorem 3.

6 Twisted Kähler–Einstein equation

This section is devoted to the proof of Proposition 5, which states explicitly when the equation in (1.2) involving the scalar curvature of ω reduces to a twisted Kähler–Einstein equation. For a general complex surface, we should require that

$$[\mathrm{Ric}(\omega)] + \frac{\alpha}{2\sin\hat{\theta}}[F] = \frac{\hat{s} - \alpha\hat{r}}{4}[\omega], \tag{6.1}$$

and we will make this condition explicit in our current setting.

Lemma 21 *For any Kähler form ω on X given by the momentum construction, with cone angle $2\pi\beta_0$ along E_0 (resp. $2\pi\beta_\infty$ along E_∞), the cohomology class of* Ric(ω) *is given by*

$$\left[\frac{\mathrm{Ric}(\omega)}{2\pi}\right] = (\beta_0 + \beta_\infty)[E_0] + (2(1-h) - k\beta_\infty)[C].$$

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Proof We recall that

$$\begin{aligned}\mathrm{Ric}(\omega) &= -\sqrt{-1}\partial\bar{\partial}\log\det\omega\\ &= -\sqrt{-1}\partial\bar{\partial}\log\det\left(\frac{2}{|w|^2}\left(\frac{1}{x}-f'\right)f''\omega_\Sigma\right)\\ &= -\sqrt{-1}\partial\bar{\partial}\log\det\left(\left(\frac{1}{x}-f'\right)f''\omega_\Sigma\right);\end{aligned}$$

hence, by a straightforward calculation, we get

$$-\sqrt{-1}\partial_z\partial_{\bar{z}}\log\det\left(\left(\frac{1}{x}-f'\right)f''\omega_\Sigma\right)=\left(\frac{f'''}{f''}+\rho_\Sigma-\frac{x}{(1-xf')}f''\right)\omega_\Sigma$$

and

$$-\sqrt{-1}\partial_w\partial_{\bar{w}}\log\det\left(\left(\frac{1}{x}-f'\right)f''\omega_\Sigma\right)=\sqrt{-1}\frac{1}{|w|^2}\frac{d}{ds}\left(\frac{x}{(1-xf')}f''-\frac{f'''}{f''}\right)dw\wedge d\bar{w}.$$

Using the identities $f'''(s)/f''(s)=\phi'(\tau), f''(s)=\phi(\tau)$ and the boundary conditions required for the momentum profile and its derivative, we compute

$$\begin{aligned}\int_{E_0}\mathrm{Ric}(\omega) &= \left(\phi'(-1)+\frac{2(1-h)}{k}-\frac{\phi(-1)}{x^{-1}-1}\right)\int_\Sigma\omega_\Sigma dz\wedge d\bar{z}\\ &= 2\pi\left(2(1-h)+k\beta_0\right)\end{aligned}$$

and

$$\begin{aligned}\int_C\mathrm{Ric}(\omega) &= \int_{\mathbb{C}\setminus 0}\sqrt{-1}\frac{dw\wedge d\bar{w}}{|w|^2}\frac{d}{ds}\left(\frac{x}{(1-xf')}f''-\frac{f'''}{f''}\right)\\ &= \int_{-\infty}^{\infty}\int_0^{2\pi}\frac{d}{dr}\left(\frac{x}{(1-xf')}f''-\frac{f'''}{f''}\right)dr\wedge d\theta\\ &= 2\pi\left(\beta_0+\beta_\infty\right),\end{aligned}$$

so our claim follows directly from (2.6). ∎

For the following computations, it is convenient to introduce the quantity

$$\Gamma=\frac{3+x+s_\Sigma x^2-3(1+x)\beta_0}{x}=4-6\beta_0+3\frac{k'}{k}(1-\beta_0)+2\frac{1-h}{k+k'}.$$

Using Lemmas 20 and 21, we can then rephrase the general condition (6.1) as the system of equations

$$\begin{cases}\frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma=2+4\frac{1-h}{k+k'}-2\left(\beta_0+\beta_\infty\right),\\ \frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma\left(\frac{k'}{2}+k\right)=2\left(1-h\right)\frac{2k+k'}{k+k'}-2k\beta_\infty-k'.\end{cases}\tag{6.2}$$

Notice that the two equations in (6.2) actually coincide when

$$-2\left(1-h\right)\frac{2k+k'}{k+k'}+2k\beta_\infty+k'=(2k+k')\left(\beta_0+\beta_\infty-1-2\frac{1-h}{k+k'}\right),$$

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or, equivalently,

$$2\left(k'+k\right)=\left(2k+k'\right)\beta_0+k'\beta_\infty. \tag{6.3}$$

Recall, however, that in order to have solutions to our equations in the momentum construction, the cone angle $2\pi\beta_0$ is not arbitrary but must satisfy

$$\beta_\infty=\frac{-2+\beta_0(1+x)}{(-1+x)},$$

in which case (6.3) holds automatically. Then the general condition (6.1) corresponds to

$$\begin{aligned}\frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma &= 2+4\frac{1-h}{k+k'}-2\left(\beta_0+\beta_\infty\right)\\ &= 2\left(2\frac{1-h}{k+k'}+2\frac{k}{k'}\left(\beta_0-1\right)-1\right).\end{aligned} \tag{6.4}$$

In order to show that this coincides with the condition (1.6) spelled out in Proposition 5, we rewrite the latter as

$$(-1+x)\left(1+(k_1+k_2)^2\right)\Gamma-4k_1k_2\left(1+s_\Sigma-x\left(-1+s_\Sigma+2\beta_0\right)\right)=0,$$

which implies

$$\begin{aligned}\frac{1+(k_1+k_2)^2}{2k_1k_2}\Gamma &= 2\frac{1}{-1+x}\left(1+s_\Sigma-x\left(-1+s_\Sigma+2\beta_0\right)\right)\\ &= 2\left(s_\Sigma x+2\beta_0\frac{x}{1-x}+\frac{1+x}{-1+x}\right)\\ &= 2\left(s_\Sigma x+2\frac{x}{1-x}\left(\beta_0-1\right)-1\right)\\ &= 2\left(2\frac{1-h}{k+k'}+2\frac{k}{k'}\left(\beta_0-1\right)-1\right).\end{aligned}$$

Reading these identities backward shows that the two conditions (1.6) and (6.4) are indeed equivalent.

It remains to establish the second claim of Proposition 5, namely that the condition (6.4) actually holds for infinitely many solutions of the system (1.2). It is convenient to rewrite (6.4) in the form

$$F(k_1,k_2)=H(k,k',h,\beta_0), \tag{6.5}$$

with

$$F(k_1,k_2)=\frac{1+(k_1+k_2)^2}{2k_1k_2}$$

and

$$\begin{aligned}H(k,k',h,\beta_0) &= \frac{2}{\Gamma}\left(2\frac{1-h}{k+k'}+1-\beta_0-\beta_\infty\right)\\ &= 2\frac{2\frac{1-h}{k+k'}+2(\beta_0-1)\frac{k}{k'}-1}{2\frac{1-h}{k+k'}+3\frac{k'}{k}(1-\beta_0)+4-6\beta_0}.\end{aligned}$$

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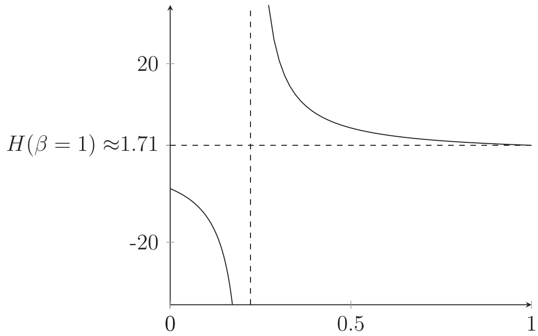

Figure 2. $H(k, k', h, \beta)$ for $k = k' = 1$ and $h = 6$.

We assume that $k_2 < 0$, so the stability condition (1.4) is automatically satisfied, and the system (1.2) is solvable. We observe that, under this assumption, the l.h.s. of (6.5) satisfies

$$F(k_1, k_2) > 2.$$

On the other hand, $H(k, k', h, \beta)$, as a function of the single variable β, has a vertical asymptote at

$$\overline{\beta} = \frac{\frac{4}{3}k + k' + \frac{2(1-h)k}{3(k+k')}}{k' + 2k}$$

and it is easy to check that $0 < \overline{\beta} < 1$, for $k' > M(k, h) > 0$. Moreover, at $\beta = 1$, we have

$$0 < H(k, k', h, 1) = \frac{k + k' + 2\,(h-1)}{k + k' + h - 1} < 2$$

and

$$\frac{d}{d\beta}H(k, k', h, 1) = \frac{4k}{k'\left(-2 + \frac{2(1-h)}{k+k'}\right)} + \frac{2\left(6 + \frac{3k}{k'}\right)\left(-1 + \frac{2(1-h)}{k+k'}\right)}{\left(-2 + \frac{2(1-h)}{k+k'}\right)^2} < 0$$

(Figure 2 shows the graph of $H\,(\beta)$ for $k = k' = 1$ and $h = 6$).

This implies that

$$F(k_1, k_2) \in (2, \infty) \subset \operatorname{im} H|_{\beta \in (0,1)},$$

which completes the proof of Proposition 5.

Remark 22 A direct computation using (5.1) shows that the condition (1.6) holds precisely when the coefficient of the linear term d_1 in $\psi(t)$ vanishes, i.e., a solution ω is twisted Kähler–Einstein precisely when the linear term is missing from the momentum profile.

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7 Large and small radius limits

Let us first prove Theorem 7. As we already observed, the "slope parameter" α' appears in the coupled equations (1.7) simply as a scale factor for the curvature form F. In other words, a pair ω, F solves (1.7) iff the pair $\omega, \alpha' F$ solves (1.2): the cohomology parameters are simply rescaled $(k_1, k_2) \mapsto (\alpha' k_1, \alpha' k_2)$. Thus, according to Theorem 2, there exists a (unique) solution of (1.7) given by the momentum construction iff the "stability condition"

$$1 + (\alpha')^2 (k_1 + k_2)^2 > x \left(1 + (\alpha')^2 (k_1 - k_2)^2\right)$$

holds. Since $x < 1$ by construction, this inequality holds for all sufficiently small α', depending only on k_1, k_2, and k'. Let us write $\omega_{\alpha'}, F_{\alpha'}$ for the corresponding family of solutions.

The attached function $H_{\alpha'}(t) = H_-(t)$ appearing in (3.4) is also obtained from (3.8) simply by rescaling $(k_1, k_2) \mapsto (\alpha' k_1, \alpha' k_2)$, and so it can be computed explicitly as

$$\begin{aligned} H_{\alpha'}(t) &= t \cot \hat{\theta}_{\alpha'} - \sqrt{\left(\cot^2 \hat{\theta}_{\alpha'} + 1\right)\left(t^2 + C'_{\alpha'}\right)}, \\ e^{\sqrt{-1}\hat{\theta}_{\alpha'}} &= \frac{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2 - 2\sqrt{-1}\alpha' k_1\right)}{\sqrt{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2 + \left(2(\alpha')^2 k_1\right)^2}}, \\ \cot \hat{\theta}_{\alpha'} &= -\frac{-(\alpha')^2 k_1^2 + (\alpha')^2 k_2^2 + 1}{2\alpha' k_1}, \\ C'_\alpha &= 4(\alpha')^2 k_1 k_2 \left(\frac{1}{x^2\left((\alpha' k_1 - \alpha' k_2)^2 + 1\right)} - \frac{1}{(\alpha' k_1 + \alpha' k_2)^2 + 1}\right). \end{aligned} \tag{7.1}$$

By elementary computations using these explicit formulae, recalling that we also have $k_1 < 0$, we find

$$H_{\alpha'}(t) = \left(k_1 t + \frac{k_2}{t}\left(-1 + \frac{1}{x^2}\right)\right)\alpha' + (\alpha')^2 R(\alpha', t) \tag{7.2}$$

for some function $R(\alpha', t)$, smooth up to $\alpha' = 0$.

As a first consequence, we can show that the sequence of Kähler forms $\omega_{\alpha'}$ converges smoothly to a Kähler form ω as $\alpha' \to 0$. It will be enough to show the smooth convergence of the momentum profiles $\phi_{\alpha'}(t)$. According to Lemma 17 and the subsequent explicit formulae for the coupling constant α and average radius $\hat{r}$, the profile $\phi_{\alpha'}(t)$ is obtained by integrating twice the identity

$$\begin{aligned} \frac{2s_\Sigma}{t} - \frac{1}{t}(2t\phi(t))'' = \alpha \left(\cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right)\right) \\ + \hat{s} - \alpha\hat{r}, \end{aligned} \tag{7.3}$$

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where all quantities are understood as evaluated at $(\alpha' k_1, \alpha' k_2)$, and in particular

$$\alpha = \frac{\sqrt{4(\alpha')^2 k_1^2 + \left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2}}{2\left(1 + (\alpha' k_1 - \alpha' k_2)^2\right)(\alpha')^2 k_2^2}(-2 + s_\Sigma x),$$

$$\hat{r} = \sqrt{\left(1 - (\alpha')^2 k_1^2 + (\alpha')^2 k_2^2\right)^2 + 4(\alpha')^2 k_1^2}. \tag{7.4}$$

By the latter explicit formulae and (7.2), the quantity

$$\alpha\left(\cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right) - \hat{r}\right)$$

has a smooth limit as $\alpha' \to 0$, so the same holds for the right-hand side of (7.3) and for the momentum profile $\phi(t) = \phi_{\alpha'}(t)$. The positivity of $\phi_{\alpha'}(t)$ and its limit for $\alpha' \to 0$ follows from Remark 18. We can now show that the curvature forms $F_{\alpha'}$ also converge smoothly as $\alpha' \to 0$. By construction, we have $F_{\alpha'} = F_{0,\alpha'} + \sqrt{-1}\partial\bar{\partial}(\alpha')^{-1} g_{\alpha'}(s)$, where $F_0 = c_1\omega_{\alpha'} + c_2\beta_{\alpha'}$ and the potential $g_{\alpha'}(s)$ corresponds to the solution for the parameters $(\alpha' k_1, \alpha' k_2)$ (i.e., for the cohomology class $\alpha'[F_0]$). By the smooth convergence of the Kähler forms $\omega_{\alpha'}$, which we just established, it will be enough to show that the potentials $(\alpha')^{-1} g_{\alpha'}(s)$ converge smoothly. In fact, they converge smoothly to the zero potential. Indeed, by (3.3) and (7.2), we have

$$\begin{aligned}(\alpha')^{-1} g_{\alpha'}(s) &= (\alpha')^{-1} v_{\alpha'}(\tau)\\ &= (\alpha')^{-1}\left(\alpha' k_1 t + \alpha' \frac{k_2}{t}\frac{1 - x^2}{x^2} - H_{\alpha'}(t)\right)\\ &= \alpha' R(\alpha', t),\end{aligned}$$

where $R(\alpha', t)$ is smooth in a neighborhood of $\alpha' = 0$. It follows that we have, smoothly as $\alpha' \to 0$,

$$F_{\alpha'} \to F_0 = c_1\omega + c_2\beta,$$

which is indeed a solution of the HYM equation $\Lambda_\omega F = \mu$.

Finally, this allows to write down the equation satisfied by the limit Kähler form ω. Recall that $\omega_{\alpha'}$ solves the equation

$$s(\omega_{\alpha'}) - \alpha_{\alpha'} \operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}} \frac{\left(\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'}\right)^2}{\omega_{\alpha'}^2}\right) = \hat{s} - \alpha_{\alpha'}\hat{r}_{\alpha'}.$$

Expanding around $\alpha' = 0$, we find

$$\begin{aligned}&\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}}(\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'})^2\right)\\ &= \omega_{\alpha'}^2 - \left(F_{\alpha'} \wedge F_{\alpha'} - z_1\omega_{\alpha'} \wedge F_{\alpha'} + z_2\omega_{\alpha'}^2\right)(\alpha')^2 + O(\alpha'^4)\end{aligned}$$

for certain cohomological constants z_1, z_2. Similarly,

$$\begin{aligned}\alpha_{\alpha'} &= \frac{1}{(\alpha')^2}\left(\frac{-2 + s_\Sigma x}{2k_2^2} + O(\alpha')\right),\\ \hat{r}_{\alpha'} &= 1 + O(\alpha').\end{aligned}$$

Thus, taking the smooth limit as $\alpha' \to 0$, and using our result that the limit curvature form satisfies $\Lambda_\omega F = \mu$, we see that ω satisfies

$$s(\omega) + \tilde{\alpha}\Lambda_\omega^2(F \wedge F) = c,$$

where

$$\tilde{\alpha} = \frac{-2 + s_\Sigma x}{2k_2^2}$$

and c is a cohomological constant. This completes the proof of Theorem 7.

The proof of Theorem 8 is quite similar. Our assumption

$$(k_1 + k_2)^2 > x(k_1 - k_2)^2$$

implies that, for any $\alpha' > 0$, the "stability condition"

$$1 + (\alpha' k_1 + \alpha' k_2)^2 > x(1 + (\alpha' k_1 - \alpha' k_2))^2$$

holds. Thus, by Theorem 2, the coupled equations (1.7) are uniquely solvable with the momentum construction. We denote the corresponding solutions by $\omega_{\alpha'}$, $F_{\alpha'}$ as before. By (7.1), as $\alpha' \to \infty$, we have an expansion

$$H_{\alpha'}(t) = \frac{\sqrt{(k_1^2 - k_2^2)^2\left(k_1k_2\left(\frac{4}{x^2(k_1-k_2)^2} - \frac{4}{(k_1+k_2)^2}\right) + t^2\right)} + k_1^2 t - k_2^2 t}{2k_1}\alpha' + S((\alpha')^{-1}, t), \tag{7.5}$$

where $S(y, t)$ is a smooth function near $y = 0$. By this expansion and (7.4), the quantity

$$\alpha\left(\cos\hat{\theta}\left(1 - \frac{H(t)H'(t)}{t}\right) - \sin\hat{\theta}\left(H'(t) + \frac{H(t)}{t}\right) - \hat{r}\right)$$

has a smooth limit as $\alpha' \to \infty$, so the same holds for the right-hand side of (7.3) and for the momentum profile $\phi(t) = \phi_{\alpha'}(t)$. Since we are assuming $(k_1 + k_2)^2 > x(k_1 - k_2)^2$, $\phi_{\alpha'}(t)$ and its limit satisfy the positivity condition, by Remark 18. Thus, the sequence of Kähler forms $\omega_{\alpha'}$ converges smoothly to a Kähler form ω as $\alpha' \to \infty$.

Considering now the curvature forms $F_{\alpha'} = F_{0,\alpha'} + \sqrt{-1}\partial\bar{\partial}(\alpha')^{-1}g_{\alpha'}(s)$ as before, we find

$$\begin{aligned}(\alpha')^{-1}g_{\alpha'}(s) &= (\alpha')^{-1}v_{\alpha'}(\tau)\\ &= (\alpha')^{-1}\left(\alpha' k_1 t + \alpha'\frac{k_2}{t}\frac{1-x^2}{x^2} - H_{\alpha'}(t)\right)\\ &= k_1 t + \frac{k_2}{t}\frac{1-x^2}{x^2} - (\alpha')^{-1}H_{\alpha'}(t),\end{aligned}$$

where by (7.5) we have the smooth convergence, as $\alpha' \to \infty$,

$$(\alpha')^{-1}H_{\alpha'}(t) \to \frac{k_1^2 - k_2^2}{2k_1}K_\pm(t),$$

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where

$$K_{\pm}(t) = t \pm \sqrt{t^2 + \hat{C}},$$
$$\hat{C} = 4k_1k_2\left(\frac{1}{x^2(k_1-k_2)^2} - \frac{1}{(k_1+k_2)^2}\right),$$

and the sign $\pm$ is that of the quantity $k_1^2 - k_2^2$. Thus, by the convergence of the Kähler forms $\omega_{\alpha'}$, the curvature forms $F_{\alpha'}$ also have a smooth limit F as $\alpha' \to \infty$.

Finally, we may write down the equations satisfied by the limit Kähler form ω and curvature form F. By our previous results, we have expansions, as $\alpha' \to \infty$,

$$\operatorname{Im}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}}(\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'})^2\right) = \left(Z_1\omega_{\alpha'} \wedge F_{\alpha'} - Z_2 F^2_{\alpha'}\right)\alpha' + O(1),$$
$$\operatorname{Re}\left(e^{-\sqrt{-1}\hat{\theta}_{\alpha'}}(\omega_{\alpha'} - \sqrt{-1}\alpha' F_{\alpha'})^2\right) = F^2_{\alpha'}(\alpha')^2 + O(1),$$

for some cohomological constants Z_1, Z_2. Similarly,

$$\alpha_{\alpha'} = \frac{|k_1^2 - k_2^2|}{2(k_1-k_2)^2k_2^2}\frac{(-2 + s_\Sigma x)}{(\alpha')^2} + O\left(\frac{1}{(\alpha')^3}\right),$$
$$\hat{r}_{\alpha'} = |k_1^2 - k_2^2|(\alpha')^2 + O(\alpha').$$

Thus, passing to the limit as $\alpha' \to \infty$ in equations (1.7), we find that ω and F satisfy the equations

$$\begin{cases} F \wedge \omega = c_1 F^2 \\ s(\omega) - \alpha_\infty \frac{F^2}{\omega^2} = c_2 \end{cases}$$

for a unique α_∞ and cohomological constants c_1 and c_2. Using the first equation, the second can also be written in the twisted cscK form as

$$s(\omega) - \hat{\alpha}\Lambda_\omega F = c_2$$

for some unique $\hat{\alpha}$. This completes the proof of Theorem 8.

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