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THE UNIFORM STRUCTURE OF BOUNDEDLY COMPACT SPACES.

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A uniform space can be given a boundedly compact compatible psuedometric if and only if it is uniformly locally compact and second countable and has a countable base for its entourages.

A psuedometric space is said to be *boundedly compact* if every bounded subset is relatively compact or, equivalently, if every bounded sequence has a convergent subsequence. The topological structure of boundedly compact spaces is known: a topological space can be given a boundedly compact psuedometric compatible with its topology if and only if it is locally compact, regular and second countable (Vaughan [**6**], Busemann [**2**]). Here we give an analogous description of the uniform structure of boundedly compact spaces from which Vaughan and Busemann's result follows as an easy corollary. Not suprisingly, the proof of the uniform result is similar to that of the weaker topological one; however, we must exercise rather more care in the construction in order to ensure that the function f which we define is uniformly continuous. No separation axioms will be assumed except those explicitly stated.

THEOREM. *A uniform space can be given a boundedly compact psuedometric compatible with its uniformity if and only if it is uniformly locally compact and second countable and has a countable base for its entourages.*

Necessity: straightforward.

Sufficiency: suppose that X is a uniform space of the type described. Since it is uniformly locally compact, there is an entourage V such that $V(x)$ is relatively compact for all $x \in X$. Choose a symmetric entourage $U \subseteq V$. Since X is σ-compact, we can find a covering $(A_n)_{n \geqslant 0}$ of X by relatively compact open sets such that $\bar{A}_n \subseteq A_{n+1}$ for all n. If we define a sequence $(D_n)_{n \geqslant 0}$ of subsets of X recursively so that $D_0 = \{x_0\}$ for some $x_0 \in X$ and $D_{n+1} = \overline{U(D_n) \cap A_n}$ for all n, then it is not difficult to show that each D_n is compact.

Now since there is a countable base for the entourages of X we can choose a psuedometric d_1 compatible with the uniformity of X in such a way that $d_1(x, y) \leqslant$

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$1 \implies (x,y) \in U$ for all $x, y \in X$. Let $f_n(x) = \min(1, d_1(x, D_n))$. We intend to prove that the function $f = \sum_{n \geqslant 0} f_n$ (which is well-defined since at each point of X only finitely many of the f_n are non-zero) is uniformly continuous. So suppose that $0 < \varepsilon \leqslant 1$ and $d_1(x,y) < \varepsilon$. Now there exists $n_0 \in \mathbb{N}$ such that $x \in D_{n_0+1} - D_{n_0}$. So if $n > n_0 + 1$ then $x \in D_{n_0+1} \subseteq D_n$ and $y \in U(D_{n_0+1}) \subseteq D_{n_0+2} \subseteq D_n$, so that $f_n(x) = f_n(y) = 0$. Similarly, if $n < n_0 - 1$ then $f_n(x) = f_n(y) = 1$. Moreover, for any n

$$|f_n(x) - f_n(y)| \leqslant |d_1(x, D_n) - d_1(y, D_n)| \leqslant d_1(x,y) < \varepsilon.$$

Therefore

$$|f(x) - f(y)| \leqslant \sum_n |f_n(x) - f_n(y)| = \sum_{n_0 - 1 \leqslant n \leqslant n_0 + 1} |f_n(x) - f_n(y)| < 3\varepsilon.$$

This shows that f is indeed uniformly continuous.

So if we let $d_2(x,y) = |f(x) - f(y)|$, then d_2 is a psuedometric which generates a uniformity coarser than that of X. Hence $d = \sup(d_1, d_2)$ is a psuedometric compatible with the uniformity of X. Moreover, if B is any d-bounded subset of X, then there exists $n \in \mathbb{N}$ such that

$$x \in B \implies d(x, x_0) \leqslant n \implies f(x) \leqslant n \implies x \in D_n;$$

in other words, B is a subset of the compact set D_n and is therefore relatively compact. So d is the psuedometric we require.

LEMMA. *If $\mathcal{U}$ is an open covering of a psuedometrisable topological space X, then there exists a psuedometric on X such that the family $\{B_1(x)\}_{x \in X}$ of open unit balls of X is a refinement of $\mathcal{U}$.*

PROOF: See Michael [**3**]. ∎

COROLLARY. *A topological space can be given a boundedly compact psuedometric compatible with its topology if and only if it is locally compact, regular and second countable.*

Necessity: straightforward.

Sufficiency: X is certainly psuedometrisable. We can therefore apply the Lemma to a relatively compact open covering of X to obtain a psuedometric on X whose uniformity is uniformly locally compact. The result now follows from the Theorem.

Remark. The condition in the Theorem that there should be a countable base for the entourages of X may be replaced by the apparently weaker requirement that there should exist a sequence $(V_n)_{n\in\mathbf{N}}$ of entourages of X such that $\bigcap_{n\in\mathbf{N}} V_n = \{(x,y)\colon \bar{x} = \bar{y}\}$ (see Umegaki [5]). However, the condition cannot be dropped entirely, as is shown by considering the finest uniformity compatible with the usual topology on $\mathbf{R}$: this uniformity is evidently uniformly locally compact and second countable, but has been shown by Nagata [4] not to be metrisable.

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