THE CURRENT STATE OF STELLAR MIXING-LENGTH THEORY

Douglas Gough

Institute of Astronomy & Department of Applied Mathematics
and Theoretical Physics, University of Cambridge

SUMMARY

The basic assumptions of the mixing-length formalism are described, and the theory is developed with a view to representing convection in stars. Directions in which the results might be improved and extended are indicated.

1. INTRODUCTION

Aside from some recent pioneering work by Latour, Spiegel, Toomre and Zahn (1976 a,b), the mixing-length formalism in one or other of its guises remains the sole method for computing the stratification of convection zones in stellar models. Little attention is usually paid to assessing the accuracy of the models, partly because there is a general feeling that mixing-length theory is so uncertain that the task would be fruitless, and partly, perhaps, because of an optimism that the theory will soon be superseded by something better. There appears to be no better convection theory emerging that might be applicable to stars in the foreseeable future, however; the mixing-length is likely to stay with us for some time. It is perhaps time, therefore, to take stock of the situation, and to ask whether the methods currently employed can be made more reliable.

The first stage of any enquiry of this kind must be a definition of the physical model upon which the theory is based. What started as little more than an order-of-magnitude estimate of turbulent transport processes has subsequently

been taken rather literally in some contexts. It is therefore important to
appreciate what the assumptions are, and where the uncertainties lie. Only after
that can there be some hope of improving the representation of the physics. And
as a byproduct, one might see how best the theory might be extended to describe
more general situations than those for which it is customarily employed in stellar
physics.

But perhaps most important of all is to appreciate the degree of contact
with reality. Astronomical verification of the mixing-length prescription is at a
very primitive level, and permits only a poor assessment of the validity of the
functional forms of the formulae describing the convective transport processes.

2. THE IDEAS BEHIND MIXING-LENGTH THEORIES

The mixing-length idea was introduced independently by Taylor (1915), Schmidt
(1917) and Prandtl (1925) to provide a means of understanding the transport of vor-
ticity, heat and momentum in turbulent fluid. By analogy with gas kinetic theory the
fluid is considered to be composed of turbulent 'eddies', 'parcels' or 'elements'
which advect properties such as heat, in the case of thermal convection, and
vorticity or momentum, in the case of shear turbulence. An element arises as a
result of instability, with about the same properties as its immediate environment.
It travels with a characteristic speed w through a mean-free-path or mixing
length ℓ , and finally breaks up because it becomes unstable itself, and merges
with its new surroundings. This breakup into smaller scales of motion is
considered to be instantaneous. It is the mixing-length description of the
beginning of the turbulent cascade; velocity components of the consequent small
scale motion and the associated temperature fluctuations are assumed to be un-
correlated so there is no contribution from them to the overall transport of heat
and transverse momentum. From such a description it is a straightforward matter
to estimate the mean heat flux or shear stress in terms of ℓ , w and the
structure of the mean environment. To complete the theory a procedure for
obtaining ℓ and w must be found.

In the case of shear flow Prandtl (1925) assumed the turbulence to be more
or less isotropic and so equated the velocity w perpendicular to the mean motion
to the velocity fluctuation in the mean flow direction induced by the shear.
Prandtl assumed the turbulent elements to be momentum conserving and obtained an
expression for the shear stress in the form of a product of the mean velocity
gradient and a turbulent transport or exchange coefficient $\rho w \ell$ (Austausch
coefficient) where ρ is density. Thus in his form of the theory, turbulent shear
stresses (Reynolds stresses) behave like viscous stresses with the Austausch
coefficient being a sort of turbulent viscosity, a concept that had been discussed
previously by Boussinesq (1877). The Reynolds stresses take on a somewhat

different form if it is assumed that turbulent elements conserve vorticity (Taylor 1915, 1932) and sometimes this yields better agreement with experiment (e.g. Prandtl, 1952). In either case it is only the mixing length ℓ that remains undetermined.

In the case of free convection there is no externally imposed velocity scale as in shear flow, and it is necessary to consider the dynamics of the turbulent elements in greater detail. This can be done only after the mixing-length model is more precisely defined. During its existence a turbulent element is accelerated by the imbalance between buoyancy forces, pressure gradients and nonlinear advection processes. In addition it can gain or lose mass by entrainment or erosion. As a result of ignoring different combinations of these processes, approximating the remaining ones in slightly different ways and making slightly different assumptions about the geometry of the flow, different physical models have emerged. They all predict similar heat transports when the mean atmospheric structure is time independent, which is hardly surprising because the formulae can be obtained from barely more than dimensional reasoning. As a consequence, the differences between the physical assumptions are not usually emphasized in the astrophysical literature, perhaps because it is difficult to differentiate astronomically between rather gross variations in the functional form of the turbulent heat flux.

It was pointed out by Prandtl (1926) in a discussion of turbulent shear flow that, in the absence of a driving force, turbulent drag would cause an element of characteristic size ℓ to lose its kinetic energy after travelling a distance of about ℓ . This is simply because turbulent drag at high Reynolds number is proportional to the square of the velocity, and hence also to the kinetic energy. Thus if the mixing length represents both the element size and the mean-free-path it is immaterial whether one postulates unimpeded motion followed by instantaneous annihilation, as would be natural by direct analogy with gas kinetic theory, or continuous momentum exchange between the element and its surroundings. This led to the first and perhaps the simplest description of the dynamics of thermal convection : namely an exact balance between buoyancy force and turbulent drag (Prandtl, 1932). Convective elements are assumed to achieve this balance instantaneously, which implies that their inertia is unimportant. They move through a distance ℓ comparable with their own diameter, conserving their heat, and then instantaneously mix their excess heat with the new surroundings. These ideas were applied to stellar convection by Biermann (1932, 1937, 1943) and Siedentopf (1933 a,b, 1935).

The model can be made more consistent by assuming interchange of heat between the element and its surroundings to be continuous too, as was emphasized by Öpik (1950). Then heat and momentum exchange are treated similarly. Since there is always an exact balance between buoyancy force and turbulent drag, and between the

rate of increase of a temperature fluctuation and the diminution of that fluctuation by heat exchange, it doesn't matter where the element came from, and the mixing-length description of annihilation of elements can be dispensed with entirely. This is not the case, however, when the star is pulsating (Unno 1967, Gough 1977).

The alternative approach of considering elements to be accelerated adiabatically from rest by buoyancy alone was adopted in the later papers by Biermann (e.g. 1948 a,b). Pressure forces and turbulent drag can be incorporated approximately into the dynamics without changing the functional forms of the equations used, though they introduce different factors of order unity. In more recent work that includes radiative heat exchange (e.g. Vitense 1953, Böhm-Vitense 1958) it is common to ignore turbulent exchange during the life of an element and invoke instantaneous breakup to account for all the nonlinearities that occur in the equations governing the turbulent fluctuations.

3. EQUATIONS OF MOTION

To simplify the presentation attention will be restricted to a plane parallel fluid layer. It will be assumed that horizontal averages, which will be denoted by overbars, are independent of time and that there is no mean mass transport through the convection zone. The horizontally averaged momentum and total energy equations can then be written

$$\frac{d}{dz}\left(\overline{p} + \overline{\rho w^2} + \overline{\sigma_{zz}}\right) + g = 0, \tag{3.1}$$

$$\frac{d}{dz}\left[\overline{F_{rz}} + \overline{\rho h w} + \tfrac{1}{2}\overline{\rho \underset{\sim}{u}\cdot\underset{\sim}{u} w} + \overline{(\underset{\sim}{u}\cdot\underset{\sim}{\sigma})_z}\right] = 0, \tag{3.2}$$

where z is the vertical co-ordinate of a Cartesian system (x, y, z), p, ρ, h are gas plus radiation pressure, density and specific enthalpy; $\underset{\sim}{u} = (u, v, w)$ is the fluid velocity, σ_{zz} is the z,z component of the viscous stress $\underset{\sim}{\sigma}$ and F_{rz} is the vertical component of the radiative energy flux $\underset{\sim}{F_r}$ which will be assumed, again for simplicity, to be given by the diffusion approximation

$$\underset{\sim}{F_r} = - K \nabla T, \tag{3.3}$$

where T is temperature and K $= 4acT^3/3\chi\rho$, a being the radiation density constant, c the speed of light and χ the Rosseland mean opacity. Perturbations in the gravitational acceleration $\underset{\sim}{g} = (0, 0, -g)$ have been ignored. Equation

(3.3) can be replaced by a more general though local representation of radiative transfer, such as the Eddington approximation (Unno & Spiegel 1966), without adding unduly to the complexity of the analysis; adding a nuclear source term to equation (3.2) or casting the problem in spherical geometry introduces no new conceptual difficulties.

Equations (3.1) - (3.3) must be supplemented with a continuity equation and an equation of state. The system is then completed with formulae for the convective fluxes. It is these that the mixing-length theory must provide.

In studying the dynamics of convection it is usual to separate all quantities into mean (horizontally averaged) and Eulerian fluctuating parts, as in

$$ p \;=\; \bar{p}(z) \;+\; p'(x,y,z,t), \tag{3.4} $$

where t is time, and to subtract the mean equations from the full equations of motion from which they were derived to obtain equations for the fluctuations. It is at this point that serious assumptions are first introduced. Though it is rarely stated explicitly, in almost all attempts to model stellar convection the Boussinesq (1903) approximation is used; this can be justified only when the scale ℓ of the motion is much less than the pressure and density scale heights of the layer (Spiegel & Veronis 1960, Malkus 1964). In this approximation the viscous terms and the kinetic energy flux $\tfrac{1}{2}\overline{\rho\, \underline{u}\cdot\underline{u}\, w}$ in equations (3.1) & (3.2) are neglected which renders (3.2) indistinguishable from the mean thermal energy equation. The equations for the fluctuations, in this approximation, are

$$ \bar{\rho}\left(\frac{D\underline{u}}{Dt}\right)' \;\equiv\; \bar{\rho}\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} - \overline{\underline{u}\cdot\nabla\underline{u}}\right) = -\,\nabla p' - g\,\frac{\bar{\rho}\,\delta}{T}T', \tag{3.5} $$

$$ \mathrm{div}\ \underline{u} \;=\; 0, \tag{3.6} $$

$$ \left(\frac{DT}{Dt}\right)' \;\equiv\; \frac{\partial T'}{\partial t} + \underline{u}\cdot\nabla T' - \overline{\underline{u}\cdot\nabla T'} - \beta w \;=\; -\frac{1}{\bar{\rho}\,\bar{c}_p}\,\mathrm{div}\,\underline{F}'_r, \tag{3.7} $$

$$\frac{\rho'}{\bar{\rho}} = -\bar{\delta}\frac{T'}{\bar{T}} , \tag{3.8}$$

where, with the exception of β, all mean quantities are considered to be constant over the scale ℓ of the motion. In these equations c_{ρ} is the specific heat at constant pressure, $\delta = -(\partial \ln \rho / \partial \ln T)_{\rho}$ and

$$\beta = -\left(\frac{d\bar{T}}{dz} - \frac{\bar{\delta}}{\bar{\rho}\bar{c}_{\rho}}\frac{d\bar{\rho}}{dz}\right) . \tag{3.9}$$

Moreover, the convective momentum and heat fluxes in equations (3.1) and (3.2) simplify to

$$p_t \equiv \overline{\rho w^2} \simeq \bar{\rho}\,\overline{w^2} , \tag{3.10}$$

$$F_c \equiv \overline{\rho h w} \simeq \bar{\rho}\,\bar{c}_{\rho}\,\overline{wT'} . \tag{3.11}$$

Quantities such as χ and δ are considered to be functions of the thermodynamical state variables ρ and T whose fluctuations are related by (3.8). The pressure fluctuation appears only in the momentum equation, and has no thermodynamical significance. Indeed it can be eliminated by taking the double curl of equation (3.5), the vertical component of which, after use of (3.6), becomes

$$\partial_t \nabla^2 w + \nabla^2 U_z - \partial_z \operatorname{div} \underset{\sim}{U} - \frac{\bar{g}\bar{\delta}}{\bar{T}}\nabla_1^2 T' = 0 , \tag{3.12}$$

where

$$\underset{\sim}{U} \equiv (U_x, U_y, U_z) \equiv \underset{\sim}{u}.\nabla\underset{\sim}{u} - \overline{\underset{\sim}{u}.\nabla\underset{\sim}{u}} , \tag{3.13}$$

$\partial_t \equiv \partial/\partial t$, $\partial_z \equiv \partial/\partial z$ and $\nabla_1^2 \equiv \nabla^2 - \partial_z^2$.

The viscous stress has been omitted from the momentum equation (3.5), and hence from (3.12). This is justifiable in stars because the Reynolds number characteristic of the largest convective eddies, which are the only motions

treated explicitly in stellar mixing-length theories, is large. The continuity equation (3.6) indicates that dynamically the fluid behaves as though it were incompressible; thus the occurrence of acoustic waves is prohibited. Density fluctuations that do arise serve only to provide buoyancy. Equation (3.7) is the fluctuating thermal energy equation, and not the equation for fluctuations in total energy.

It should be emphasized that the Boussinesq approximation is not an essential component of the assumptions of mixing-length theory. Other less restrictive though more complicated approximations to the equations of motion could be used, such as the anelastic approximation (Ogura and Phillips 1962, Gough 1969) which holds for low Mach number convection in deep layers of gas. Like the Boussinesq approximation it filters out the possibility of acoustic waves, whilst retaining some of the features of compressibility. No attempt to develop a mixing-length theory with consistent use of such an approximation seems to have been made.

4. LOCAL MIXING-LENGTH FORMALISMS FOR A STATIONARY ENVELOPE

In the Boussinesq approximation, mean thermodynamical state variables are considered to be constant over the assumed scale ℓ of the motion. Another approximation, common to the formulation of most mixing-length theories used in stellar structure computations, is to treat the superadiabatic lapse rate β in the thermal energy equation as though it were constant. Though in practice it is found that this approximation is poor, because at the top of the hydrogen ionization zone in particular β varies on a scale much shorter than ℓ , it is usually retained because it brings great simplicity to the mixing-length formulae. It permits the heat flux and Reynolds stress to be expressed at any level in the envelope solely in terms of the mean conditions at that level. For this reason the resulting theories are called local.

Formulation under the assumption of balance between buoyancy and turbulent drag

Most mixing-length descriptions are formulated in terms of rising and falling fluid parcels having typical radius $\frac{1}{2}\ell$ and which at any instant can be characterized by a single vertical velocity w and temperature fluctuation T' . In the early discussions by Prandtl, Biermann and Siedentopf the parcels were presumed to travel at their terminal velocity, buoyant driving being balanced exactly by turbulent drag. Thus in the vertical component of the momentum equation (3.5) the time derivative was essentially ignored and the nonlinear advection terms were replaced by a drag force of the form $w^{2}/(\frac{1}{2}\ell)$. If the pressure fluctuation is ignored the coupling between vertical and horizontal motion is removed, and a relation between the vertical speed and the temperature fluctuation results:

$$\omega^2 = (g\delta\ell/2T)|T'|. \tag{4.1}$$

From here on overbars are omitted from mean quantities where no ambiguity results. Initially it was common to consider the fluid motion to be adiabatic. Equation (3.7), with the right hand side set to zero and $\overline{\underline{u}.\nabla T'}$ ignored, can then be integrated along the trajectory of the parcel. Taking β to be constant over the distance the parcel moves, one obtains

$$T' = \beta\xi, \tag{4.2}$$

where ξ is the vertical displacement of the parcel above its point of origin. The constant of integration is zero since T' is assumed to be zero at $\xi = 0$. A typical parcel at any instant might have travelled say half the distance ℓ, so a typical temperature fluctuation θ and velocity w can be obtained from (4.1) and (4.2) by setting $\xi = \frac{1}{2}\ell$. Noting furthermore that parcels with $T' > 0$ rise and parcels with $T' < 0$ fall, the heat flux F_c can be estimated by replacing $\overline{wT'}$ with $w\theta$. Then

$$F_c \simeq \frac{1}{4}\rho c_r (g\delta/T)^{1/2} \ell^2 \beta^{3/2}. \tag{4.3}$$

The Reynolds stress may be estimated in a similar way to be

$$p_t \simeq \frac{1}{4}\rho (g\delta/T) \ell^2\beta. \tag{4.4}$$

The numerical factors in front of these formulae vary from paper to paper, because the precise definition of ℓ and in particular the relation between parcel size and mean-free-path is not universal, and because factors of order unity can be introduced to account for effects of pressure fluctuation or imperfect correlation between w and T'.

Kinetic theory of accelerating fluid elements

The alternative approach is to imagine the fluid parcel to accelerate from rest. It is usual then to ignore the nonlinear terms in the momentum equation. The influence of pressure fluctuations can be estimated by working from equation (3.12), and introducing typical horizontal and vertical wavenumbers by setting $\nabla_1^2 = -k_1^2$ and $\partial_z^2 = -k_z^2$. This is perhaps not quite as crude an approximation as one might first imagine, because these relationships are satisfied by the convective eddies of linear stability theory whose visual

appearance is not wholly dissimilar to the eddies of intensely turbulent
convection. The linearized form of (3.12) then becomes

$$\partial_t \omega - (g\delta/\Phi T) T' = 0, \qquad (4.5)$$

where

$$\Phi = 1 + k_2^2/k_1^2 . \qquad (4.6)$$

The only difference between equation (4.5) and what one would have obtained from
the linearized vertical component of (3.5) with the pressure fluctuations ignored
is the factor Φ . The pressure fluctuations divert vertical motion into
horizontal flow, thereby decreasing the efficiency with which the motion might
otherwise have released potential energy. The effect in this approximation is
simply to increase the apparent inertia of the vertically moving fluid, without
changing the functional form of the equation of motion. In some derivations
equation (4.5) is obtained directly from (3.5), the factor Φ being introduced by
analogy with the virtual inertia of a body moving in a potential flow.

When integrating equation (4.5) it is usual to regard the temperature
fluctuation as a function of the parcel displacement ξ , and approximate it by
the leading term in its Taylor expansion. Of course for adiabatic motion
equation (4.2) indicates that the leading term is the only term present. The
operator ∂_t in equation (4.5) can be replaced by $\omega \partial/\partial\xi$ without further
assumption, since in linear theory there is no distinction between Eulerian and
Lagrangian time derivatives of perturbation quantities. The equation can then be
integrated to yield

$$\omega^2 \simeq (g\delta/\Phi T) T'\xi . \qquad (4.7)$$

For adiabatic motion, (4.7) together with (4.2) complete the description of the
dynamics. If typical velocity and temperature fluctuations defined by setting
$\xi = \pm\ell$ are used as before to estimate F_c and p_t , the same equations (4.3)
and (4.4) are obtained, aside from factors involving Φ . Note that pressure
fluctuations could have been incorporated into the original formulation of the
theory by dividing the right hand side of (4.1) by Φ .

Heat exchange between fluid parcels and the environment

Heat exchange between fluid parcels and their surroundings is most simply
accounted for by treating equation (3.7) in an analogous way to the momentum
equation. Retaining only the leading term in the Taylor expansion of $\omega(\xi)$ in
the linearized version of (3.7) and integrating along the trajectory yields

$$T' \simeq [\beta - (K/\rho c_p) k^2 T'/\omega] \, \xi , \qquad (4.8)$$

where $k^2 = k_1^2 + k_z^2$. When $\xi = \frac{1}{2}\ell$ this is precisely the same relation that one would obtain by neglecting the time derivative in (3.7) and replacing $\underline{\psi} \cdot \nabla T' - \overline{\underline{\psi} \cdot \nabla T'}$ by the estimate $\omega T'/(\frac{1}{2}\ell)$ for the turbulent heat exchange. In deriving this equation the fluctuating part of (3.3) was used; fluctuations in K do not arise because, consistently with assuming β constant, the gradient of $\overline{T}$ is small compared with $|\nabla T'|$. Whereas for adiabatic motion the wavenumbers entered only in their ratio, in the nonadiabatic theory their magnitudes are also required for estimating $\text{div} \, \underline{F}_r'$. Taking the mixing length to be a measure of the vertical extent of the eddy suggests

$$k_v = \pi/\ell . \qquad (4.9)$$

Proceeding as in the adiabatic case, but with (4.8) replacing (4.2), one is led to

$$F_c = \frac{1}{4} \Phi^{-1/2} \eta^{-3} S^{-1} [(1 + \eta^2 S)^{1/2} - 1]^3 \, K\beta , \qquad (4.10)$$

$$P_t = \frac{1}{4} \Phi^{-1} \eta^{-2} S^{-1} [(1 + \eta^2 S)^{1/2} - 1]^2 (g\rho\delta/T) \, \ell^2 \beta , \qquad (4.11)$$

where

$$S \equiv \frac{g(\delta/T)\beta \ell^4}{(K/\rho c_p)^2} \qquad (4.12)$$

is the product of the Prandtl number and a Rayleigh number based on ℓ , and

$$\eta \equiv 2\pi^{-2} \Phi^{-3/2} (\Phi - 1) \qquad (4.13)$$

is a geometrical factor of order unity.

When $\eta^2 S \gg 1$ the convective motion is almost adiabatic and

$$F_c \simeq \frac{1}{4} \Phi^{-1/2} S^{1/2} \, K\beta , \qquad (4.14)$$

$$P_t \;\simeq\; \tfrac{1}{4}\left(g\delta/\Phi T\right)\rho\,\ell^2\beta\,, \tag{4.15}$$

which are the same as (4.3) and (4.4) aside from the factors involving Φ. In the solar convection zone this condition is satisfied everywhere except in extremely thin layers at the top and bottom, the latter and sometimes even the former being too thin to be resolved in normal stellar structure computations. When $\eta^2 S \ll 1$,

$$F_c \;\simeq\; \tfrac{1}{32}\,\Phi^{-1/2}\eta^3\,S^2\,K\beta, \tag{4.16}$$

$$P_t \;\simeq\; \tfrac{1}{16}\,\eta^2 S\left(g\delta/\Phi T\right)\rho\,\ell^2\beta, \tag{4.17}$$

which are relevant in substantial regions of some red giant envelopes.

Other formulations

In essence, the derivation presented above in terms of accelerated and subsequently annihilated fluid parcels is the same as that of Vitense (1953; Böhm-Vitense 1958) whose prescription has been the most widely used for computing stellar models. Her 1958 formulae for the heat flux imply (4.10) with $\Phi = 2$ and $\eta = \sqrt{2}/9$. Vitense also studied the case when fluid parcels are optically thin, and adjusted numerical constants so that the optically thick and optically thin formulae gave the same result at unit optical thickness. A smoother and probably more accurate transition between the two limits can be obtained by using the Eddington approximation to radiative transfer (Unno & Spiegel 1966, Unno 1967, Gough 1977).

The derivation in terms of continuous turbulent exchange of heat and momentum has been adopted by Unno (1967) and is similar to an earlier approach of Öpik (1950) in terms of convective cells. Öpik's formula for F_c is mathematically somewhat more complicated than (4.10), but it takes similar values if Φ and η are chosen suitably (Gough & Weiss, 1976). The differences between the values of F_c predicted by Vitense and Unno arise mainly from the different assumptions about flow geometry and the slightly different constants of order unity appearing in the approximations to the equations of motion, rather than from apparent variances in the physical models.

5. REMARKS ON THE ASSUMED STRUCTURE OF THE CONVECTIVE FLOW

In order to complete the prescription for F_c and P_t the parameter Φ ,

which depends on the geometry of the flow, and the mixing length ℓ must be determined. The latter is one of the most difficult parts of the theory. Here the criteria that might be most important in affecting a choice of Φ are discussed; the mixing length is postponed to a later section.

What one might consider the most natural choice of Φ depends very much on the mixing-length model that has been adopted. Thinking in terms of approximately spherical parcels of fluid rising and sinking through the background medium suggests adopting the formula for the virtual inertia of a sphere. The particle is thus considered in isolation, and $\Phi = 3/2$. Different values would be obtained if the parcel were thought to be aspherical.

An alternative approach is to assume the return flow around a moving parcel is confined to the immediate environment by the interaction with neighbouring eddies, so that a relatively compact convective eddy or cell is formed. To determine Φ the shape of the eddy must be specified. For want of a more reliable procedure, the marginal or unstable modes of linear theory might be used to describe the flow in convective cells. This has the computational advantages of being simple to calculate for local mixing-length theory, and of providing a basis on which to generalize to nonlocal theories or theories that one might hope to apply to convection in more complicated situations. Of course it is not clear what boundary conditions are the best to adopt, but that is unlikely to be crucial; it is expedient to choose those that yield the simplest solutions. Thus for the relatively simple theory discussed in the previous section one might set (cf. Chandrasekhar 1961)

$$\underset{\sim}{u} = \left(k_1^{-2} \partial_z \tilde{W} \partial_x f, \; k_1^{-2} \partial_z \tilde{W} \partial_y f, \; \tilde{W} f \right), \tag{5.1}$$

$$T' = \tilde{\Theta} f, \tag{5.2}$$

in an obvious notation, where $\tilde{W}$ and $\tilde{\Theta}$ are sinusoidal in z and depend possibly on t, and the planform f depends only on x and y and satisfies

$$\nabla_1^2 f = -k_1^2 f, \qquad \overline{f^2} = 1. \tag{5.3}$$

Equation (4.5) was derived from (3.12) with a flow such as this in mind.

If the mixing-length model is one in which a statistically steady eddy is maintained, the continual turbulent interchange of momentum and heat may be regarded as being due to an eddy viscosity and conductivity. The convective cell is thus like the marginal mode of linear stability theory, for which $\Phi = 3$. If, on the other hand, the model is one in which the eddy grows and subsequently breaks up, Spiegel's (1963) suggestion of choosing the most unstable mode is more

appropriate. The rationale for this is that it is the most rapidly growing mode that would dominate the flow. Its shape is approximated by (Gough, 1977)

$$ k^2/k_z^2 \;\simeq\; \tfrac{1}{2}\left\{ 1 + \left[3 + 2\,(4\pi^{-4} S + 9)^{1/2} \right]^{1/2} \right\} , \tag{5.4} $$

given that its vertical extent $\pi/k_z = \ell$ is fixed. Thus Φ varies monotonically from 2 to 1 as S varies from 0 to ∞ .

Other possibilities come to mind, such as choosing the mode that maximizes F_c , though that results in a shape not very different from the one implied by (5.4), with Φ varying between 5/2 and 1. One might consider averaging over an ensemble of different eddies, but then one is faced with the problem of choosing the distribution function.

The flow geometry also enters directly into the mean equation for the hydrostatic support of a stellar envelope. In a plane parallel layer only vertical transport of vertical momentum matters, but in a star vertical support is provided partly by horizontal stress. If the star is spherically symmetrical, the Reynolds stress tensor is axisymmetrical about the vertical and has only two distinct eigenvalues. Thus in spherical polar co-ordinates the components depend on only two quantities, which may be taken to be p_t and Φ . The equation for hydrostatic support, which is nontrivial only in the radial direction, may be written

$$ \frac{d}{dr}\,(p + p_t) \;+\; (3 - \Phi)\,p_t/(r\rho) \;+\; g \;=\; 0 , \tag{5.5} $$

where r is the radial co-ordinate. If the turbulence is isotropic, $\Phi = 3$ and the Reynolds stress behaves like a pressure of magnitude p_t .

6. FURTHER REMARKS ON THE DYNAMICS OF CONVECTIVE EDDIES

When the growth and subsequent annihilation of convective elements or eddies is taken into account, mean transports are usually estimated by using characteristic values of the velocity and temperature fluctuations. These values are normally taken to be the actual fluctuations at $\xi = \tfrac{1}{2}\ell$, when the element has moved half its mean-free-path; Vitense took a space average over an element's trajectory, which in local theory is equivalent. The approximation implies certain assumptions about the creation of eddies, which become apparent as soon as an explicit attempt to average over all turbulent elements is made. The discussion that follows is based on the ideas behind Spiegel's (1963) formulation of the theory, though the details are somewhat different from that approach.

A specific mixing-length model

Consider, for example, the evaluation of the heat flux $F_c = \rho c_p \,\overline{w\,T'}$. It is convenient to have a specific model in mind, and to this end a flow field of

the kind (5.1) - (5.3) will be assumed. Thus the flow is represented by a
conglomerate of cells or eddies that form, grow and subsequently break up. Though
it is envisaged that there is a degree of randomness in the positions at which
the eddies form, each is presumed to stay in the same place during its existence.
An eddy centred at height z_0 that originated at time t_0 is thus described by

$$w = W(t; t_0, z_0) f(x, y) \cos[\pi(z - z_0)/\ell], \qquad (6.1)$$

$$T' = \Theta(t; t_0, z_0) f(x, y) \cos[\pi(z - z_0)/\ell], \qquad (6.2)$$

until the moment of annihilation, the functions W and Θ being determined by the
linearized forms of equations (3.7) and (3.12). If $p(z_0, t_0; t)$ is the
probability that the eddy has not yet been destroyed, the average of any function
over all eddies at height z_0 is obtained by weighting that function with p
and integrating over all possible times of creation. Thus if eddies have mass m
and are created at a rate n per unit volume per unit time, the heat flux at
$z = z_0$ is

$$F_c = n m c_p \int_{-\infty}^{t} W \Theta p \, dt_0 . \qquad (6.3)$$

Here all the eddies have been assumed to be centred at z_0 where they contribute
maximally to F_c. In principal an average over eddies centred in the range
$(z_0 - \frac{1}{4}\ell, z_0 + \frac{1}{4}\ell)$ ought really to be taken. In local theories this does no
more than multiply the right hand side of (6.3) by a constant factor of order
unity.

The growth of convective eddies

The velocity and temperature fluctuation amplitudes depend on the initial
conditions of the eddy. One of the assumptions of the mixing-length approach is
that turbulent fluid elements originate with the same properties on average as
their immediate environment, though in any individual eddy there must be some
deviation from the mean state for otherwise that eddy would have no identity. It
must be assumed that the eddy grew from a non-zero perturbation, but provided the
initial amplitude is much smaller then the average value, the precise details of
the initial conditions are unimportant. In this discussion the conditions that
lead to pure exponential time dependence of W and Θ will be chosen, merely to
simplify the mathematics. Then

$$W = W_0 \, e^{\sigma_\pm (t-t_0)} \; ,$$

$$\Theta = \Theta_0 \, e^{\sigma_\pm (t-t_0)} \; , \tag{6.4}$$

where W_0 and Θ_0 are the initial values of W and Θ , and are related by

$$\Theta_0 = \left(\sigma_\pm \, \Phi T / g \delta \right) W_0 \; . \tag{6.5}$$

The growth rates $\sigma_\pm$ are given by

$$\sigma_\pm = \pm \sqrt{p^2 + \kappa^2} - \kappa = p \eta^{-1} S^{-1/2} \left[\pm (1 + \eta^2 S)^{1/2} - 1 \right] , \tag{6.6}$$

where $p^2 \equiv g \delta \beta / \Phi T$ and $\kappa \equiv k^2 K / 2 \rho c_p$. Note that if W_0 and Θ_0 are of the same sign the eddy grows, but if they are of reverse sign it decays.

It is presumed that in the initial perturbations, which arise out of the smaller scale turbulence resulting from the breakup of both the major heat carrying eddies and possibly also from larger eddies that make lesser contributions to the heat flux, the velocity and temperature fluctuations are uncorrelated. Thus only half the eddies have W_0 and Θ_0 of the same sign and subsequently grow. The other half make an insignificant contribution to the heat flux and are ignored.

Eddy creation rate and initial conditions

The expression for the heat flux depends explicitly on the eddy creation rate n and the amplitudes W_0 , Θ_0 . These are governed by the background turbulence. In a statistically steady state, however, n can be evaluated from the statement that the entire fluid volume (or some constant fraction of it) is occupied by eddies. Thus

$$n \, m \int_{-\infty}^{t} p \, dt_0 = \rho \; . \tag{6.7}$$

It is much more difficult to specify the initial amplitudes, and at this point it will be observed only that if the flow is to be controlled for most of the time by the linearized dynamics leading to (6.4), and not by the eddy breakup process, then W_0 and Θ_0 should be small compared with the average amplitudes. Thus $\exp(\sigma \tau) \gg 1$, where $\sigma \equiv \sigma_+$ and τ is the mean lifetime of an eddy. If τ is

estimated by integrating W in (6.4) and setting the resulting displacement ξ equal to ℓ , this condition becomes

$$\lambda \equiv \sigma \ell / W_o \gg 1 .\tag{6.8}$$

Eddy annihilation hypothesis

Finally it is necessary to obtain $\not\!\!\!\rho$, which depends on the disintegration of eddies. The most natural interpretation of the mixing–length annihilation hypothesis is that a fluid element is considered to break up as it is displaced through $d\xi$ with probability $d\xi / \ell = (W/\ell) dt$. In other words the element has a mean–free–path ℓ , and the probability of its annihilation is proportional to the shear in the eddy and is not explicitly dependent on the details of its past history. It follows that

$$\begin{aligned}
\not\!\!\!\rho &= \exp \left\{ - \int_{t_o}^{t} W_o \ell^{-1} e^{\sigma(t'-t_o)} dt' \right\} \\
&= \exp \left\{ - \lambda^{-1} e^{\sigma(t-t_o)} \right\} \left\{ 1 + O(\lambda) \right\} .
\end{aligned}\tag{6.9}$$

The turbulent fluxes

It is now straightforward to evaluate F_c . If terms $O(\lambda)$ of the leading terms are ignored, equation (6.7) for the eddy creation rate yields

$$nm = \sigma \rho / (\ln \lambda - \gamma) ,\tag{6.10}$$

where γ is Euler's constant. Equation (6.3) then becomes

$$F_c = \frac{\rho c_p \sigma W_o \Theta_o}{2 (\ln \lambda - \gamma)} \int_{-\infty}^{t} \exp \left\{ 2\sigma (t - t_o) - \lambda^{-1} e^{\sigma(t-t_o)} \right\} dt_o$$

$$\simeq \frac{\rho c_p \Phi T \ell^2 \sigma^3}{2 (\ln \lambda - \gamma) g \delta}\tag{6.11}$$

$$= \left[1/2 (\ln \lambda - \gamma) \right] \Phi^{-1/2} \eta^{-3} S^{-1} \left[(1 + \eta^2 S)^{1/2} - 1 \right]^3 K \beta ,\tag{6.12}$$

the factor 2 in the denominator arising because only half the eddies have positive growth rates. This expression has the same form as (4.10), and can be made equal to it by setting

$$\lambda \;=\; e^{2\cdot58} \;\simeq\; 13.$$ (6.13)

The stress p_t can be computed similarly, and yields

$$p_t \;=\; \frac{\rho\, \ell^2 \sigma^2}{2\,(\ln\lambda - \gamma)} \;,$$ (6.14)

which reduces to (4.11) when (6.13) is again used for λ .

Remarks

It is now evident that the rough estimates (4.10) and (4.11) imply a value for λ , which approximates the degree of amplification of the velocity and temperature fluctuations during the lifetime of a convective eddy. Of course the precise value (6.13) must not be taken seriously, especially since it was obtained by exponentiating a poorly estimated quantity, though one may be tempted to take comfort in the fact that it is at least reasonably consistent with (6.8). It would be preferable if some independent method of determining the initial conditions could be found. It is worth noting, however, that (6.12) and the corresponding expression for p_t depend only logarithmically on λ , so the method of estimating the initial conditions need not be very accurate. The constraint (6.5), which was applied only to minimize algebraic complexity, is not an essential part of the formulation; other relations between W_0 and Θ_0 lead to expressions for F_c similar to (6.12). These also have multiplicative factors that contain the logarithm of the amplification λ in the denominator.

The theory can also be formulated in terms of rising and falling fluid parcels, with the only difference that the integrals in (6.3), (6.7) and (6.9) are now considered to be evaluated along the parcel trajectories. In the local approximation the two approaches are identical.

The discussion in this section has not led to any modification in the mixing-length formulae. Its purpose has been to highlight the role of the eddy creation process in determining F_c and p_t .

7. THE CHOICE OF MIXING LENGTH

Expressions (4.10) and (4.11) for F_c and p_t are both increasing functions of ℓ . Since the philosophy of the mixing-length approach is to concentrate on only one scale of motion at any level in the fluid, namely the scale that contributes most to the fluxes, the largest value of ℓ compatible with the dynamics must be chosen.

When modelling laboratory flows the choice of ℓ seems straightforward. The largest eddy that one can imagine is determined by the geometry of the vessel containing the fluid. Thus at any point it is usual to take

$$\ell = \alpha x , \tag{7.1}$$

where x is the distance to the nearest boundary of the container and α is a constant scaling factor of order unity which is determined by comparison of theory with experiment.

Of course the dynamics of the flow could be such as to prevent this largest conceivable eddy from growing, so that ℓ is determined mainly by other factors. Thus von Kármán (1930) suggested that for turbulent shear flow the mixing length should be taken to be a multiple of a scale length of the mean shear. It seems that for most laboratory applications this yields results that agree with experiment less well than (7.1).

A stellar convection zone is not bounded by a rigid container, and must therefore determine its own length scales. But just how ℓ should be specified is not clear. It is most common to follow von Kármán's philosophy and choose

$$\ell = \alpha H , \tag{7.2}$$

where H is a scale height of the mean stratification, though some stellar models have been computed using the lesser of (7.1) and (7.2) (Hofmeister & Weigert 1964; Böhm & Stückl 1967). Öpik (1938) took H to be a scale height of density. This choice has been favoured also by Biermann (1943), and by Schwarzschild (1961) who argued that it is the distortion of expanding or contracting fluid elements as they experience substantial changes in mean density that limits the size of an eddy. This reasoning is not Boussinesq, and introduces some representation of the effect of compressibility into the prescription. Vitense (1953) set H to a pressure scale height. This has been widely used since, presumably for reasons of computational convenience.

It is unfortunate that the attempt to incorporate compressibility into the local theory results in a choice of ℓ that does not reduce continuously in any natural way to the kind of value that is favoured for laboratory applications. Stellar convection zones are often many scale heights deep, which is currently unattainable in the laboratory, but it would have been encouraging had some experimental verification of the theory been feasible, even if it were in a parameter range inappropriate to astrophysics. Only astronomical checks are available at present, but these appear to provide little support for the details of the theory.

8. CALIBRATION OF THE HEAT FLUX FORMULA

Stellar models are usually computed using equation (4.10) for the convective heat flux, with (7.2) defining the mixing length. The Reynolds stress p_t is rarely included. The constant α is calibrated by comparison with observation and then taken to be a universal constant, though some authors do not bother with this nicety on the grounds that the mixing-length formulae are too unreliable for the calibration to be meaningful. The determination of α can be affected either by constructing a solar model with the correct luminosity and effective temperature or by comparing the slope of a theoretical lower main sequence with observation. The two methods give results in reasonable agreement with one another when H is a pressure scale height, though they are subject to considerable uncertainty. In particular, uncertainties in the solar composition are reflected in the calibration, as are uncertainties in the opacities, equation of state and nuclear reaction rates. The results depend also on assumptions concerning the mixing of material that has been processed in the core. In addition to these, additional uncertainty is introduced by inaccuracies in the numerical techniques, whose existence is indicated by discrepances between the results of different investigators.

It should be noticed that the conclusion $\alpha \simeq 1$ does not satisfy the condition $\alpha \ll 1$ upon which the Boussinesq approximation depends. The theory is therefore not internally consistent. However α is not much greater than unity, and the effects of compressibility may be insufficient to modify the heat flux severely. More serious is the local approximation that regards β as being approximately constant over the length scale ℓ. It is one of the aims of non-local mixing-length theories to rectify this flaw.

The calibration of α rationalizes the astronomical data, but it does not provide a test of the mixing-length theory. The reason is partly that convective envelopes of solar type stars are approximately adiabatically stratified everywhere except in thin transition regions above the hydrogen ionization zone. The sole function of the convection theory in determining the gross structure of the star, therefore, is merely to prescribe which adiabat characterizes the bulk of the convection zone. This depends on the jump in temperature across the transition region, but that hardly depends on the detailed functional form of the expression for F_c. Indeed if (4.10) is replaced by (4.16), even though $\eta^2 S \gg 1$ throughout almost the entire convection zone, the solar calibration requires $\alpha = 1.35 \times 10^{-3}$ when Φ and η take the values implied by Böhm-Vitense's (1958) formulae (Gough & Weiss 1976). The resulting solar model is barely distinguishable from that computed in the usual way. In red giants there are regions where $\eta^2 S < 1$, but the envelope models are insensitive to details of how the two asymptotic limits (4.14) and (4.16) meet.

The calibration of α does not even determine both asymptotic limits, since there is considerable uncertainty in the geometry of the convective motion. The discussion in §5 suggests that $\Phi^{1/2}$ does not vary greatly, but the possible range for η is very wide. This influences only the limit (4.16), which hardly matters for solar type stars. One might anticipate, however, that a calibration of η would be possible by comparing theoretical red giants with observation.

An investigation of the sensitivity of red giant evolution to changes in the constants in the mixing-length theory has been reported by Henyey, Vardya and Bodenheimer (1965), and interpreted in terms of the asymptotic limits (4.14) and (4.16) by Gough and Weiss (1976). Plausible variations in η do induce noticeable changes in the evolutionary tracks on the H-R diagram, but it appears that other uncertainties in both the theory and observation at present prohibit calibration of η by this method.

A potentially more sensitive test for η might have been a measure of the maximum depth of penetration of the convection zone. In some red giants the convection zone extends deep enough to mix the products of the nuclear reactions to the surface. The extent to which the convection zone has penetrated in such a star could be determined in principle from observations of O^{17}/O^{18} ratios in red giant atmospheres (Dearborn & Eggleton 1976). Computations by D.S.P. Dearborn and myself, however, have revealed that such a test would probably not provide the required information, for at its maximum depth the convection zone of a red giant envelope is adiabatically stratified almost throughout, and the heat flux is determined by (4.14). But this does not necessarily imply that there is nothing to be learnt. Conditions may be sufficiently different in red giants that a recalibration of (4.14) would yield a different result. This might shed some light on the variation of Φ , whether density or pressure scale heights are appropriate, or whether (7.2) is even a useful assumption.

9. THE REYNOLDS STRESS

It is common practice to ignore the turbulent stress $p_t = \overline{\rho w^2}$ in the mean momentum equation (3.1). One reason, perhaps, is that to justify the Boussinesq equations upon which mixing-length theory is based the convective velocities must always be substantially less than the sound speed. This implies $p_t \ll p$. However, it is the derivative of p_t that appears in (3.1), and if $|dp_t/dr|$ is evaluated in a stellar model that has been computed without that term, it can be the case that it exceeds $|dp/dr|$ by a considerable degree in the transition region at the top of the convection zone, even though p_t might be small compared with p . Another reason for ignoring p_t is because to do so removes singularities from the equations of stellar structure and thus makes them much easier to solve.

The equations, in the plane layer approximation, are

$$\frac{d}{dz}(p + p_t) = -g\rho \tag{9.1}$$

and

$$-K\frac{dT}{dz} + F_c = F = \text{constant}. \tag{9.2}$$

The heat flux and Reynolds stress are given by (4.10) and (4.11) in terms of β . Equations (9.1), (9.2) and the equation (3.9) defining β can be rewritten as a system of first order equations for p_t, ρ and T thus:

$$\frac{dp_t}{dz} = \left(\frac{\rho c_\rho}{\delta}\right)\left\{\left(\frac{F}{K} - \frac{g\delta}{c_\rho}\right) - \beta - \frac{F_c}{K}\right\} \equiv \mathcal{H} , \tag{9.3}$$

$$\frac{d\rho}{dz} = -\left(g\rho + \mathcal{H}\right) , \tag{9.4}$$

$$\frac{dT}{dz} = \frac{1}{K}\left(F_c - F\right). \tag{9.5}$$

Here β and F_c are regarded as functions of p_t . The usual stellar structure computations are governed by the spherical analogues of (9.1) and (9.2) with p_t ignored, which is of one order lower than the system considered here. Solutions of (9.3) − (9.5) are to be sought satisfying $p_t = 0$ at the edges of the convection zones, where $F/K - g\delta/c_\rho$ also vanishes.

To analyse the nature of the singularities it is sufficient to consider the structure of equation (9.3) near a boundary of a convection zone. Since $S \to 0$ as the boundary is approached, the asymptotic forms of (4.10) and (4.11)

$$F_c \sim A\,\ell^8\beta^3 , \tag{9.6}$$

$$p_t \sim B\,\ell^6\beta^2 \tag{9.7}$$

may be used, where A and B are nonvanishing functions. If the origin of z is taken to be at the base of the convection zone, and all the coefficients in (9.3) are expanded in a Taylor series about the origin, only the leading terms being

retained, the resulting equation is found to have the structure

$$\frac{d p_t}{dz} \simeq Mz - N p_t^{1/2} - Q p_t^{3/2} , \tag{9.8}$$

where M, N and Q are positive constants. In deriving this it was assumed that the mixing length does not vanish at the boundary of the convection zone. The singularity introduced by the $p_t^{1/2}$ term is best analyzed by setting $u^2 = p_t$ and writing (9.8), with the small last term neglected, as a linear system in terms of a new independent variable $\int$:

$$\frac{du}{d\int} = -Nu + Mz , \tag{9.9}$$

$$\frac{dz}{d\int} = 2u .$$

The coefficients matrix of the right hand side has eigenvalues

$$\lambda_{\pm} = -\tfrac{1}{2}N \pm \sqrt{\tfrac{1}{4}N^2 + 2M} , \tag{9.10}$$

which are real and of opposite sign, indicating a saddle point at the origin.

A similar analysis may be performed at an upper boundary. The resulting equation for p_t has the same structure as (9.8), except that now the coefficients M and N are negative. Provided $N^2 \geqslant 8|M|$, both eigenvalues $\lambda_{\pm}$ are real and of the same sign, indicating that the origin is a nodal point, though if $N^2 < 8|M|$ the solution is a spiral that cannot satisfy the condition that u vanishes at the boundary of the zone. This latter situation might arise if too large a mixing length is chosen.

It is because the upper singularity is either a node or does not permit a physically acceptable solution that inward numerical integrations from the atmosphere of a star cannot be successful. Stellingwerf (1976) has pointed out that an outward integration might work, and has presented a solution to a simple model problem. Realistic stellar envelopes can be computed in this way only if the convection zone is thin; otherwise a more stable numerical procedure must be adopted.

Attempts to include p_t in realistic stellar envelopes have been made by Henyey, Vardya and Bodenheimer (1965) and by Travis and Matsushima (1971). In both cases the structure equations were simplified in a manner tantamount to ignoring the left hand side of equation (9.3), thereby reducing the order of the differential system and removing the singularities. Henyey et al. anticipated that this approximation was not serious. Unpublished computations by Baker, Gough and Stellingwerf of RR Lyrae envelopes with shallow convection zones using

the full system of equations revealed that at least in those stars the effect of p_t is not profound. Its inclusion smooths out the region near the top of the convection zone, so that dp_t/dr remains smaller in magnitude than dp/dr , and has little influence on the remainder of the convection zone.

10. REFINEMENTS AND GENERALIZATIONS

The discussion in §6 demonstrated how F_c and p_t depend on the growth rate σ of convective eddies. This dependence was emphasized by Spiegel (1963), who also showed how the expressions are modified when viscosity is considered.

The averaging procedure used to derive (6.11) and (6.14) does not depend on the precise nature of the turbulent flow. The description of the breaking up of eddies is not refined enough to distinguish between the different circumstances to which the theory might be applied. Detailed descriptions of the dynamics is confined to eddy growth, and is contained in the expression for σ . It is to this that refinements and generalizations are most easily made.

Transport by small-scale turbulence

As an illustration, an attempt will be made to incorporate into the dynamics the exchange of heat and momentum by smaller scale turbulence that was ignored in §6. It will be assumed that turbulence on a scale smaller than the heat carrying eddies is isotropic, so the transport might be roughly represented in terms of a scalar eddy diffusivity

$$\nu_e = \left(\overline{\underset{\sim}{u}' \cdot \underset{\sim}{u}'}\right)^{1/2} \Big/ k' , \tag{10.1}$$

where $\underset{\sim}{u}'$ is a characteristic velocity and k' a characteristic wavenumber of the background turbulence. This diffusity will be taken to be the same for both momentum and heat. Its value is related to the velocity and length scales of the major eddies, whose disruption seeds the small scale motion, and may be rewritten

$$\begin{aligned} \nu_e &= \epsilon \left(\overline{\underset{\sim}{u} \cdot \underset{\sim}{u}}\right)^{1/2} k^{-1} = \epsilon k^{-1} \left(\Phi \overline{w^2}\right)^{1/2} \\ &= \epsilon k^{-1} \left(\Phi p_t/\rho\right)^{1/2} , \end{aligned} \tag{10.2}$$

where ϵ is of order unity and depends on the spectrum of the turbulence. It is likely that ϵ is only weakly dependent on the amplitude of the convection and can probably be safely assumed constant. This expression can now be incorporated into the expression for the growth rate of a disturbance in a viscous conducting

fluid (e.g. Spiegel 1963):

$$\sigma = \tfrac{1}{2}(\tilde{\kappa} + \tilde{\nu})\, k^2 \left\{ \left[1 + 4\left(\mu^2 \Phi^{-1} k^{-4} - \tilde{\kappa}\,\tilde{\nu}\right)(\tilde{\kappa} + \tilde{\nu})^{-2}\right]^{1/2} - 1 \right\}, \qquad (10.3)$$

where $\tilde{\kappa}$ and $\tilde{\nu}$ are the effective thermal diffusivity and kinematic viscosity:

$$\tilde{\kappa} = \nu_e + K/\rho c_\tau \;, \qquad \tilde{\nu} = \nu_e . \qquad (10.4)$$

Equations (10.2) - (10.4) define a growth rate σ which can be substituted into
(6.11) and (6.14) to obtain equations for F_c and p_t . The prescription is
algebraically more complicated than the previous formulation which led to (4.10)
and (4.11), though its effect can be approximated by simply multiplying the value
of σ obtained previously by the factor $\left\{ 1 + \tfrac{1}{4}\pi \epsilon \Phi(\Phi - 1)^{-1/2}\right\}^{-1}$.
It is perhaps not surprising that the modifications to the results hardly change
the functional dependence of F_c and p_t on S, because the two extreme approaches
discussed in §4 led to the same formulae. The new results may be no better than
(4.10) and (4.11), because the attempted improvement to the representation of the
physics may be insignificant compared with the errors that remain. It should be
noted, however, that the modifications cannot simply be absorbed into the
definition of ℓ .

The small scale turbulence not only influences the dynamics of the larger
eddies but also contributes directly to the fluxes. The heat flux can be
accounted for by replacing K by $\rho c_p \tilde{\kappa}$ in the equation for the radiative flux.
The relevant Reynolds stress component must be augmented by $\tfrac{1}{3}\rho \overline{u'.u'}$ which can
be written as $\tilde{\epsilon}\, p_t$, where $\tilde{\epsilon}$ is yet another undetermined parameter of order
unity that depends on the spectrum of the turbulence.

Other refinements can be included, such as a representation of entrainment
and erosion of eddies, or the generation of waves. The former has been considered
by Ulrich (1970a), who used the meteorologists' model of convection based on
rising thermals. D.W. Moore and Spiegel (unpublished) considered the influence of
acoustic generation by convective eddies, and found that this noticeably reduces the
turbulent velocities when the Mach number is of order unity. Generation of gravity
waves with wavelengths comparable with ℓ , which occurs at the boundaries of
convection zones, probably requires a nonlocal theory for an adequate description.
Further refinements are discussed by Spiegel (1971).

Convection in slowly rotating stars

Aside from suggesting improvements to the standard theory, this approach can
be used to formulate mixing-length theories for more general circumstances.
Rotation or a magnetic field, for example, can easily be incorporated into the

stability analysis that determines σ . If the convection zone is rotating, the maximally contributing eddies are rolls aligned with the horizontal component of the rotation rate $\underset{\sim}{\Omega}$ (e.g. Chandrasekhar 1961). Their growth rate is determined by

$$(\sigma + 2\kappa)\left\{\sigma^2 - \mu^2 + 4\Phi^{-1}(\Phi-1)\Omega^2\right\} + 2\kappa\mu^2 = 0, \tag{10.5}$$

where Ω is the vertical component of $\underset{\sim}{\Omega}$. Only if Ω is small might one reasonably hope to obtain meaningful results by just using this growth rate in the normal mixing-length formulae, since the effect of the rotation on eddy disruption has been ignored. In that event the solution to (10.5) can be approximated by

$$\sigma \simeq \left\{\nu^2 - 4\Phi^{-1}(\Phi-1)(\nu+\kappa)(\nu-\kappa)^{-1}\Omega^2\right\}^{1/2} - \kappa, \tag{10.6}$$

and F_c is given in terms of it by (6.11). Note that Ω measures the local rotation in the vicinity of the eddy, and should therefore be interpreted not as the *angular velocity* but as *half the vorticity of the mean flow*.

It is more difficult to calculate the Reynolds stress. The rotation introduces a degree of order to the turbulence that destroys the axisymmetry of the stress tensor and rotates its principal axes. Provided $\Omega^2 \ll \mu^2$ the effect is small and for the purposes of computing the hydrostatic structure of the star can no doubt be safely ignored. Equation (6.14) can be used for p_t with σ determined by (10.6). But this approximation is not good enough for the horizontal components of the mean momentum equation, since the relatively small off-diagonal terms in the stress tensor generated by the rotation are important for determining the angular momentum transport by the turbulence. It is straightforward to construct a Reynolds stress tensor from the eigenfunctions of linear stability theory, but in the absence of experimental tests it would be most unwise to rely on it.

Influence of a magnetic field

A magnetic field $\underset{\sim}{B}$ can be treated similarly, provided its turbulent distortion may be considered random and does not lead to organized concentrations such as sunspots. Once again the turbulent motion is most efficient as rolls, aligned with the horizontal field, and the growth rate is determined by the equation obtained from (10.5) or (10.6) by replacing $4\Phi^{-1}(\Phi-1)\Omega^2$ by $\pi^2 B^2/\ell^2$, where B is the vertical component of $\underset{\sim}{B}$. The caveats concerning the Reynolds stress mentioned in connection with rotation apply here too.

Nuclear reactions and composition gradients

The interaction between nuclear reactions and convection is of particular interest when reaction timescales are comparable with σ^{-1}. This can be the case in late stages of stellar evolution. The fluctuations in energy generation rate induced by the convection influence the eddy dynamics through modifications in both the temperature and chemical composition of fluid elements. The convection influences the nuclear reactions not only via F_c and P_t, but also by transporting the products of the reactions.

The mixing-length theory can be generalized as before. Variations x_i' in the abundances x_i of elements i must now be taken into account when calculating both the buoyancy and, of course, the energy generation rate $\varepsilon(p, T, x_i)$ per unit mass that must be introduced into equation (3.7). The amplitudes W and Θ are now determined by

$$\Phi \frac{\partial W}{\partial t} = g(\delta T^{-1} \Theta - \rho_i X_i), \tag{10.7}$$

$$\frac{\partial \Theta}{\partial t} - \beta W = -(2\varkappa - \varepsilon_T) \Theta + \varepsilon_i X_i, \tag{10.8}$$

where X_i is the amplitude of x_i', defined in a way analogous to W and Θ in (6.1) and (6.2), and

$$\rho_i \equiv \left(\frac{\partial \ln \rho}{\partial x_i}\right)_{p,T}, \qquad \varepsilon_i \equiv \frac{1}{c_p}\left(\frac{\partial \varepsilon}{\partial x_i}\right)_{p,T}, \qquad \varepsilon_T \equiv \frac{1}{c_p}\left(\frac{\partial \varepsilon}{\partial T}\right)_{p,x_i}.$$

The summation convention is being used. The abundances are determined by

$$\frac{D x_i}{D t} = R_i(p, T, x_j), \tag{10.9}$$

where R_i measures the rate of production of i. The linearized fluctuation equation derived from (10.9) can be combined with (10.7) and (10.8) to yield the following equation for σ:

$$\Psi \sigma^2 + \left\{ \Psi(2\varkappa - \varepsilon_T) - S_{ij}\left(g\Phi^{-1}\rho_i \frac{dx_j}{dz} + \varepsilon_i R_{jT}\right) \right\} \sigma$$
$$- \mu^2 + g\Phi^{-1} S_{ij}\left\{ \delta T^{-1}\varepsilon_i - (2\varkappa - \varepsilon_T)\rho_i \right\} \frac{dx_j}{dz} = 0, \tag{10.10}$$

where

$$R_{ij} \equiv \left(\frac{\partial R_i}{\partial x_j}\right)_{p,T} \quad , \qquad R_{iT} \equiv \left(\frac{\partial R_i}{\partial T}\right)_{p,x_j} \quad , \qquad \Psi \equiv 1 + \frac{T}{\delta} f_i S_{ij} R_{jT}$$

and S_{ij} is defined by

$$\left(\sigma \delta_{ij} - R_{ij} - T R_{iT} f_j/\delta\right) S_{jk} = \delta_{ik} \ . \tag{10.11}$$

where δ_{ij} is the Kronecker delta.

Overbars on mean quantities have been omitted as usual. Once again F_i and p_t are given in terms of σ by (6.11) and (6.14). In addition one can easily derive for the flux of x_i, which is in the z direction,

$$F_i = \overline{w x_i'} = f^{-1} S_{ij} \left(\frac{\sigma \Phi T}{g \delta} R_{jT} - \frac{dx_j}{dz}\right) p_t \ . \tag{10.12}$$

The first term represents the transport that arises solely because ᵪchemical elements are created or destroyed at different rates in upward and downward moving fluid. The second term represents turbulent mixing, though that too is influenced by the reactions and is not a simple scalar diffusion.

In the special case when there are no reactions $S_{ij} = \sigma^{-1} \delta_{ij}$,

$$\sigma^3 + 2\kappa\sigma^2 - \left(\mu^2 + g\Phi^{-1} f_i \frac{dx_i}{dz}\right)\sigma - 2\kappa g \Phi^{-1} f_i \frac{dx_i}{dz} = 0 \tag{10.13}$$

and

$$F_i = -(\sigma f)^{-1} p_t \frac{dx_i}{dz} \ , \tag{10.14}$$

which leads to a simple diffusion equation governing the mean abundance x_i.

Convection in pulsating envelopes

Application of the mixing-length formalism to stellar pulsation is somewhat more complicated than the examples considered above, because now the time dependence of the coefficients in the fluctuation equations must be taken into account. Additional assumptions must also be made. Only a few brief remarks are made here, since detailed discussion of this problem is to be found in Unno's

contribution to this volume.

The case of radial pulsations is the simplest to discuss, provided attention is restricted to fundamental and low overtone modes that vary on a length scale greater than ℓ . If Lagrangian co-ordinates defined in terms of the mean flow are used to describe the pulsations, but locally defined Eulerian co-ordinates for the convection, the equations governing the convective fluctuations are rather similar to those used for a static atmosphere, though additional terms must be added to account for the mean dilation and q must be modified because the co-ordinate frame is no longer inertial. A mixing-length theory can therefore be developed by following one of the procedures outlined in §§ 4-6.

Unno (1967) formulated a theory by generalizing the model that assumes continuous turbulent exchange of momentum and heat between a convective element and its surroundings. Though the general growth of convective fluctuations during the lifetime of an eddy is ignored in this approach, acceleration of a convective element and modulations in its temperature induced by the pulsations are taken into account. An eddy is presumed to maintain its identity, deforming instantaneously with the mean environment. The theory now requires one to recognize that the lifetime of an eddy is finite, for a turbulent eddy retains some memory of the conditions at the time of its formation. Thus much of the apparent simplicity enjoyed by this model when applied to a stationary stellar envelope is lost. Alternatively, the discussion of § 6 can be adapted for a pulsating star by introducing the appropriate time dependence into the equations of motion (Gough 1977).

These approaches each require an explicit statement about how the initial state of a convective element depends on conditions at the time of creation. Since the mixing length, which determines both the destruction rate and the initial dimensions of elements, is assumed to depend only on the mean (horizontally averaged) state and not on convective fluctuations, it is perhaps most natural, and certainly simplest, to assume that it has the same functional form as for a stationary envelope (and thus does not depend explicitly on time derivatives) and to make a similar assumption about all other aspects of creation. It must be realized that this is yet another unverified assumption of the theory. It may not be a good approximation, for although it is the mean stratification of the convection zone that controls which eddies grow most rapidly, the level of turbulence at the instant of creation presumably does have some influence on the perturbations out of which those eddies grew.

A similar objection may be levelled at the assumption that the mixing length, when it determines eddy annihilation, depends only on the mean environment. If breakup is determined by shear within the eddy, perhaps the current eddy dimensions provide a more appropriate length scale. These depend on the history of the eddy and not just on instantaneous conditions. Likewise turbulent drag

and heat exchange depends on the current eddy size, and also on the intensity of the small scale turbulence which may not vary in phase with the larger eddies.

The different versions of mixing-length theory yield different formulae for F_c and p_t when applied to radially pulsating stars. This emphasizes the uncertainties in the assumptions. The differences offer some hope of choosing between them by observation.

The range of possible assumptions widens further when nonradial pulsations are considered. One prescription has been offered by Gabriel, Scuflaire, Noels & Boury (1974) who generalized Unno's approach in a natural way. Amongst the approximations is the neglect of anisotropy in the turbulent flow, which circumvents the complicated problem of determining how the changing shear associated with the pulsations modifies the convective velocity field. Anisotropy appears to have a more complicated influence on the pulsational perturbations of the heat flux and Reynolds stress in this case than it does for radial pulsations, so the assumption may be critical. It would be useful to know how sensitive pulsations of stellar models are to changes in this and other assumptions in order to assess where effort to improve the theory might most profitably be directed.

One of the motivations for developing a convection theory in a time-dependent envelope is to study the pulsations of the cooler Cepheid and RR Lyrae variables. Unpublished computations by N.H. Baker and myself of the linear stability of such stars to radial pulsations, using a generalization of the formulation in §6, indicate that the modulation of F_c generally has a stabilizing influence on the pulsations, and is responsible for determining the red edge of the instability strip. The phase of the modulation of p_t is such as to drive the pulsations in some regions of the convection zone and damp them in others. The driving is greater in the cooler stars, and may be a significant factor in the excitation of the long period variables.

Comments

These examples illustrate how the basic ideas of mixing-length theory might be applied to a variety of situations. The generalizations all concentrate on describing the dynamics of the major eddies prior to breakup, and ignore the more difficult issues concerning creation and annihilation. To do more would require a more sophisticated study of the mechanisms of turbulence.

In particular, there is no prescription for determining the mixing length. One could choose the same value as one believes is applicable to ordinary convection. In that case the theory predicts, for example, that a vertical component of rotation or magnetic field reduces the heat flux. It appears however that there can be circumstances where rotation increases the heat

flux through a convecting fluid (Rossby 1969, Sommerville & Lipps 1973, Baker & Spiegel 1975), which shows that the mixing-length prescription hasn't even predicted the correct sign of the change. Perhaps the influence of a small composition gradient is more reliably described, because the perturbation is via a scalar rather than a vector field, and influences the dynamics only by modifying buoyancy. However this could be the case only when that modification is small, for we know from experimental studies of thermohaline convection that once composition gradients are sufficient to change the stability characteristics of the mean stratification the gross structure of the flow suffers a qualitative change (Turner 1973). Thus the theory is not immediately adaptable to semi-convection.

11. NONLOCAL THEORIES

One of the obvious inaccuracies in the theory developed above comes from assuming velocity and temperature fluctuations to depend on only local properties of the environment. This would be justified if ℓ were much less than all relevant scale heights, but the stellar calibration suggests that this is not the case. In particular β can vary on a scale much shorter than ℓ. Nonlocal treatments take some account of the finite extent of convective eddies, and lead to prescriptions for F_c and p_t that involve averages over distances of order ℓ. Thus sharp gradients in β no longer lead to rapid variations in the convective transports. Moreover, the treatments aim at representing overshoot into adjacent stable regions.

There are two nonlocal properties of eddies that can be represented in a straightforward way. One is that an eddy centred at z_o samples β over the range $(z_o - \tfrac{1}{2}\ell,\ z_o + \tfrac{1}{2}\ell)$; the other is that $F_c(z)$ and $p_t(z)$ are determined not only by eddies with $z_o = z$, but by all the eddies centred between $z_o - \tfrac{1}{2}\ell$ and $z_o + \tfrac{1}{2}\ell$. These can be taken into account within the framework of the Boussinesq approximation, which entails ignoring the variation of all other variables over the scale of an eddy.

Averaging over eddies

The only place β enters into the formulae (6.11) and (6.14) for F_c and p_t is in the growth rate σ. As was noticed by Spiegel (1963), the linearized equations of motion used to determine the eddy growth are the Euler equations of a variational equation for σ, whose solution is (6.6) with β replaced by

$$\langle \beta(z) \rangle \equiv \frac{\int \beta(z') \, w^2(z',z) \, dz'}{\int w^2(z',z) \, dz'} \quad ,$$

where $w(z,z_o)$ is the vertical component of the velocity of an eddy centred at z_o and the range of the integrals is the vertical extent of the eddy. With the introduction of (6.1) this becomes

$$\langle \beta \rangle = \frac{2}{\ell} \int_{z-\frac{1}{2}\ell}^{z+\frac{1}{2}\ell} \beta(z') \cos^2[\pi(z'-z)/\ell] \, dz'. \tag{11.1}$$

Taking account of contributions to F_c and p_t from eddies centred at different heights leads to similar averages: the assumptions (6.1) and (6.2) imply that both wT' and w^2 have a z dependence quadratic in $\cos[\pi(z-z_o)/\ell]$. Thus if $F_{co}(z)$ and $p_{to}(z)$ are defined as the right hand sides of (6.11) and (6.14) with β replaced by $\langle \beta \rangle$, the nonlocal formulae for the heat flux and Reynolds stress may be written

$$F_c = \frac{2}{\ell} \int_{z-\frac{1}{2}\ell}^{z+\frac{1}{2}\ell} F_{co}(z_o) \cos^2[\pi(z-z_o)/\ell] \, dz_o \quad , \tag{11.2}$$

$$p_t = \frac{2}{\ell} \int_{z-\frac{1}{2}\ell}^{z+\frac{1}{2}\ell} p_{to}(z_o) \cos^2[\pi(z-z_o)/\ell] \, dz_o \quad . \tag{11.3}$$

Use of these expressions converts the ordinary differential equations of stellar structure that obtain from local mixing-length theory into integro-differential equations.

The fluid element approach

The extent of the region over which F_{co} and p_{to} are averaged in (11.2) and (11.3) depends on the mixing length in its role of being a measure of the eddy size. The description of mixing-length theory in terms of rising and falling

fluid elements, however, averages over a mean-free-path, and so depends on ℓ as a measure of the annihilation rate. The analysis will now be repeated for this more commonly used picture, to illustrate how uncertain the fine details of the theory are. The arguments are similar to those used by Spiegel (1963).

It is a straightforward matter to repeat the analysis of §6 with the fluid particle picture in mind. The mathematical structure is almost the same, with the principal difference that the integrals in the equations leading to (6.9) and (6.11) are now to be considered as line integrals along fluid element trajectories. It is simplest to use the vertical displacement ξ of a parcel from its initial position to define an independent variable s according to

$$ds = d\xi/\ell . \tag{11.4}$$

The bottom and top of the convection zone are assumed to be at $s = 0$ and $s = s_1$. Then the contribution to F_c from rising elements is approximately

$$F_{c+}(\mu,s) = \frac{1}{2(\ln\lambda - \gamma)} \int_0^s \rho c_p \frac{\sigma^3 \Phi T \ell^2}{g\delta} (s - s_0) e^{-\frac{s-s_0}{\mu}} \, ds_0 , \tag{11.5}$$

and that from sinking elements is

$$F_{c-}(\mu,s) = \frac{1}{2(\ln\lambda - \gamma)} \int_s^{s_1} \rho c_p \frac{\sigma^3 \Phi T \ell^2}{g\delta} (s_0 - s) e^{-\frac{s_0-s}{\mu}} \, ds_0 . \tag{11.6}$$

In obtaining these equations the temperature fluctuation of an element was taken to be

$$T' = \frac{\sigma^2 \Phi T \ell}{g\delta} (s - s_0) ,$$

which was estimated by integrating W in (6.4) and using this and (6.5) to eliminate $t - t_0$ and Θ_0 in the expression for Θ. Once again σ is assumed to be defined in terms of an average $\langle\beta\rangle$ such as (11.1) to take account of both the finite size of fluid elements and the fact that they traverse a finite distance through their environment. Note that the creation rate n has been taken to be the same as in the local theory. It has been assumed that the

motion is not necessarily vertical, as does Spiegel, the cosine of the angle made by the velocity with the vertical being denoted by μ . Thus $F_{c\pm}\,d\mu$ is the flux due to elements moving in directions between μ and $\mu + d\mu$ when $\mu \gtrless 0$. The total flux is obtained by integrating over μ and yields

$$F_c(s) = \int_0^{s_1} F_{co}(s_0)\,|\,s_0 - s\,|\,E_2(|\,s_0 - s\,|)\,ds_0 \,,\qquad(11.7)$$

where E_2 is an exponential integral. The expression for p_t is similar. These averages are rather different from (11.2) and (11.3), the main weight coming from $|\,s - s_0\,| \simeq 0.6$ rather than being concentrated near zero. The value of λ defining the initial conditions is once again undetermined. If it is fixed by insisting that (11.7) approaches (4.10) in the limit $\ell \to 0$, one finds

$$\lambda \simeq e^{\,4/3\,-\,\gamma} \simeq 7\,.\qquad(11.8)$$

The averaging procedure in both this formulation and the eddy approach is rather crude, and depends in particular on an assumed structure for the velocity and temperature fluctuations based on local theory. Other versions of the theory that pay more explicit attention to the motion of elements have been formulated, notably by Faulkner, Griffiths and Hoyle (1965), Ulrich (1970a), Shaviv & Salpeter (1973) and Maeder (1975). Nordlund (1976) has recently studied a model based on rising and sinking columns. The differences in outcome between the various procedures appears to derive mainly from variances in the rather arbitrary choices of scaling factors.

Spiegel's theory

A major drawback to the methods described so far is that they require one to solve the equations of motion for the eddies. This becomes especially awkward when the theory is generalized for application to more complicated circumstances, such as pulsating stars. It may be possible to alleviate the difficulties by working within the framework suggested by Spiegel (1963) who started from an element conservation equation in phase space. Spiegel considered a plane parallel atmosphere and set

$$\mu\,\frac{dv\,\psi}{dz} + \frac{v\,\psi}{\ell} = \frac{Q}{\ell} \,,\qquad(11.9)$$

where ψ is the element distribution function and v is the magnitude of the velocity. The term $\partial(\dot{u}_i \psi) / \partial u_i$, which depends on the dynamics of elements and which would normally appear on the left hand side of a conservation equation, has been absorbed into the source function Q . This equation can be formally solved for ψ in terms of Q , as is sometimes done in radiative transfer theory, and the heat flux and Reynolds stress computed by averaging appropriate moments of ψ over μ . In particular, the heat flux is

$$F_c = \int_{-1}^{1} \mu \, v \, h' \psi \, d\mu = \int_{0}^{S_1} |h'| \, Q(s_o) \, E_2(|s_o - s|) \, ds_o , \qquad (11.10)$$

where h' is the specific enthalpy fluctuation in an element. Rather than discuss the element dynamics explicitly, Spiegel simply assumed that $|h'| \, Q(s_o)$ is independent of s and then chose it to make (11.10) reduce to (4.10) in the limit $\ell \to 0$. The result is

$$F_c(s) = \int_{0}^{S_1} F_{co}(s_o) \, E_2(|s_o - s|) \, ds_o , \qquad (11.11)$$

with λ given by (6.13). This result differs from (11.7) because of the assumption about the functional form of $|h'| Q$.

Approximations

Since integral equations are not readily incorporated into most stellar structure programmes it is tempting to approximate the equations for F_c and P_t with differential equations. Spiegel's approach now exhibits the advantage that one can immediately draw on the techniques of radiative transfer theory. In particular, Eddington's first approximation provides simple equations relating F_c and P_t to $\langle \beta \rangle$ that are no doubt accurate enough. To obtain the equation for F_c , for example, moment equations are first constructed by multiplying (11.9) by $h' \ell$ and by $\mu h' \ell$ and integrating with respect to μ , remembering that $h' \gtrless 0$ when $\mu \gtrless 0$. This gives

$$\frac{dF_c}{ds} - J = 0 , \qquad (11.12)$$

$$\frac{dK}{ds} - F_c = -F_{co} \quad , \tag{11.13}$$

where

$$J \equiv \int_{-1}^{1} \hbar' \psi \, d\mu \quad , \qquad K \equiv \int_{-1}^{1} \mu^2 \hbar' \psi \, d\mu .$$

Eddington's approximation is to take $\psi = \psi_+$ if $\mu > 0$, $\psi = \psi_-$ if $\mu < 0$, where ψ_+ and ψ_- are independent of μ. This implies $K = \frac{1}{3} J$, and hence

$$a^{-2} \frac{d^2 F_c}{ds^2} - F_c \simeq -F_{co} \quad , \tag{11.14}$$

where $a = \sqrt{3}$ (cf. Travis & Matsushima 1973). The equation for p_t is similar. But there remains the problem of finding an approximate equation determining $\langle \beta \rangle$. Guidance may be found by attempting to rederive an equation of the type (11.14) directly from the integral relation (11.11).

The approximation (11.14) is equivalent to replacing the kernel $\mathcal{K}(s_o - s) = E_2(|s_o - s|)$ in (11.11) by the simpler function

$$\mathcal{K}_o(s_o - s) = \frac{1}{2} b \exp\left(-b|s_o - s|\right) \tag{11.15}$$

with $b = a$. Equation (11.1) might therefore be approximated in a similar manner. But how does one best choose b? Equation (11.11) may be rewritten

$$F_c(s) = \int_{-\infty}^{\infty} \mathcal{K}_o(s_o - s) \, \mathcal{F}(s_o) \, ds_o + \int_{-\infty}^{\infty} \{\mathcal{K}(s_o - s) - \mathcal{K}_o(s_o - s)\} \, \mathcal{F}(s_o) \, ds_o$$

$$\equiv F_c^{(o)}(s) + F_c^{(1)}(s) \quad , \tag{11.16}$$

where

$$\mathcal{F}(s) = \begin{array}{ll} F_{co}(s) \quad , & 0 \leqslant s \leqslant s_1 \quad , \\ 0 \quad , & s < 0, \ s > s_1 \quad . \end{array} \tag{11.17}$$

The limits of integration have formally been written as $\pm\,\infty$, and are meant to denote positions well into the bounding stable regions where F_c is small. Obvious adjustments must be made when two convection zones are close together, or if the domain of integration includes the central regions of the star.

It is clear that b is best chosen in such a way as to minimize the magnitude of $F_c^{(1)}$. This problem is of a kind that has been encountered in radiative transfer theory (Monaghan 1970) and statistical mechanics (e.g. Barker & Henderson 1976) and its solution depends on the features of (11.11) one wishes to represent most accurately. Here, an approximation will be sought that roughly represents the solution when the scale of variation ℓ_F of $F_{c\,\bullet}$ is not a great deal less than ℓ ; to find a representation approximately valid for all scales ℓ_F would entail an analysis of the equations that determine $F_{c\bullet}$. Thus $\mathfrak{F}(s_\bullet)$ is replaced by its Taylor series about s, and the leading terms of the expansion of $F_c^{(1)}$ so generated are made to vanish. The first two terms are automatically zero, and the third vanishes provided $b = \sqrt{2}$. This result differs somewhat from the value obtained from the Eddington approximation, which is a representation that appears to be good at both extremes of ℓ_F , at least for radiative transfer. If equation (11.1) is treated similarly one obtains

$$b^{-2}\,\frac{d^2\langle\beta\rangle}{ds^2} - \langle\beta\rangle \;=\; -\beta \;, \tag{11.18}$$

where b, which is calculated as before but now with $\mathcal{H}(s) = 2\cos^2\pi s$, is given by

$$b \;=\; \left\{ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(s'\cos\pi s'\right)^2 ds' \right\}^{-\frac{1}{2}} \;\simeq\; 7\cdot 8 \;. \tag{11.19}$$

If equations (11.2) and (11.3) are used to determine F_c and p_t , this value must also replace a in (11.14) and the analagous equation for p_t .

The differential equations determining the mean structure of the star, with this approximation to nonlocal mixing-length theory, is of order five higher than when local theory is used. Computing time is therefore increased. However the singular points at the edges of the convection zone discussed in §9, and the numerical difficulties associated with them, are no longer present. Equation (11.14), its analogue for p_t and equation (11.18) should be solved subject to the boundary conditions $F_c \rightarrow 0$, $p_t \rightarrow 0$ and $\langle\beta\rangle \rightarrow \beta$ as $s \rightarrow \pm\,\infty$.

Comments

The factor of about 5 by which the values of b obtained from the two kernels differ emphasizes the different roles played by the mixing length. Fluid crossing the midplane of a convective eddy of diameter ℓ is likely to have risen vertically by perhaps about half the radius, which is only about $\frac{1}{4}$ the mean-free-path of a fluid element. The mean-free-path and the element or eddy diameter have been arbitrarily set equal, but perhaps a factor of order unity should have been introduced between them. Consequently the coefficients in (11.14) and (11.18) are parameters that, like the mixing length itself, are not determined by the theory but await calibration by comparing theoretical models with observation.

The most obvious testing ground is the top of the solar convection zone, where overshooting into the stable regions can be studied. However it is in the regions of overshoot that new uncertainties seem to enter. The integral equation (11.16) for F_c, for example, explicitly assumes that the stable regions provide no source of convective elements: $\mathcal{H} = 0$ outside the convection zone. The decceleration of elements in the stably stratified regions is represented by the averaging of β, but this does not adequately account for the possible oscillation of elements and the generation of waves. Negative values of $\mathcal{H}$ would be required, and Spiegel (1963) has suggested replacing σ in the formula (6.11) for F_{co} by its real part, presumably to account for the damping of those waves. However, that does not account for possible propagation of energy by the waves, and subsequent dissipation far from the site of generation. A more careful analysis of the coupling between the convection and the waves must be undertaken before one can have confidence in the procedure.

Calibration of nonlocal theories is at present in an unsatisfactory state. Attempts are made to construct model solar atmospheres and to compare overshoot velocities or limb darkening with observation, adjusting parameters where necessary (e.g. Ulrich 1970b; Travis & Matsushima 1973; Nordlund 1974), with some diversity in the conclusions. Indeed Spruit (1974) has fitted the limb darkening function using a local mixing-length theory. Moreover, though the models are constructed with an averaged F_c, β is not always averaged and p_t is ignored entirely. Ulrich (1976) has recently investigated the sensitivity of solar type model atmospheres to variations in the parameter b in the kernel (11.15), with $\langle \beta \rangle \equiv \beta$, and to the addition of a multiple of a delta function to that kernel.

The refinements and generalizations discussed in §10 can easily be incorporated into these nonlocal procedures. Nonlocal effects of small scale shear turbulence have been discussed by Kraichnan (1962), using rather different arguments which suggest that at very high values of S there is a qualitative change in the functional dependence of F_c on S. Scalo and Ulrich (1973) have incorporated nuclear reactions into a nonlocal theory.

12. COMPRESSIBLE CONVECTION

Compressibility plays two kinds of role. First it influences the structure of the convective flow discussed above, and secondly it introduces new phenomena that are not represented by the Boussinesq approximation.

Various studies of the structure of the linear eigenfunctions of convective motion in a compressible atmosphere have been made, though no attempt has been made to incorporate the results into a mixing-length theory. This is not surprising. One reason is that the mathematical difficulties are rather greater than for Boussinesq theories, but a more fundamental reason is that it is not at all clear how the mixing-length hypothesis should be interpreted in these circumstances. It should be recalled, however, that the assumption $\ell \propto H$ is based on arguments concerned with the structure of eddies in a compressible stratified medium, and that in some sense, therefore, compressibility is acknowledged. New phenomena that must be considered include the pressure fluctuations in the equation of state, which not only modify the structure of the eigenfunctions but also must be included in the formula for the heat flux. Viscous dissipation must be included in the mean energy equation. Unno assesses the importance of such mechanisms in his contribution to this volume.

13. CONCLUDING REMARKS

The mixing-length formulae derived in § 4 are based on very rough order-of-magnitude estimates. The physical arguments supporting them are based on imprecisely defined models. Moreover the observational evidence for the validity of the formula for the heat flux is very weak; the Reynolds stress is ignored in almost all stellar structure computations.

Even if it could be ascertained that the Boussinesq formulation outlined in this article is sound, there would still be the difficulty of extrapolating the theory to stellar conditions where compressibility is important. The major point at which compressible arguments are invoked is in the choice of the mixing length ℓ . It is commonly believed that effective heat carrying eddies cannot extend over much more than a scale height H of density or pressure, and accordingly ℓ is taken to be of order H . The solar calibration of the heat flux is not inconsistent with this assumption, though it is inconsistent with the conditions under which the Boussinesq approximation is justified. However, numerical computations of compressible convection that either solve the equations of fluid motion directly in two dimensions (Graham 1975) or three (Graham, these proceedings), or represent the solutions in the single-mode approximation (Toomre, Zahn, Latour & Spiegel 1976b; Van der Borght 1975) predict large eddies extending over the entire convection zone that show little tendency to break up into smaller scales. But perhaps the computations do not mimic solar conditions well enough, since they lack the thin transition zone at the top of the convective region in which the

temperature gradient is very strongly superadiabatic. The convection just beneath the photosphere is observed to have a characteristic length scale comparable with H and it is not unlikely that the vertical scale is similar. Whether in the region below, the dominant scale of motion is always of order H, or whether it is quite different, is hardly relevant for most purposes, because any plausible formula for F_c implies that beneath the first scale height the temperature gradient is very close to being adiabatic. Moreover the detailed structure of the transition zone doesn't influence the interior significantly, so any formula for F_c with an adjustable factor multiplying it can serve to construct models of the sun and solar type stars that have the correct luminosity and radius. Of course the motion in the transition zone is important for determining the photospheric velocity field, but here mixing-length theory is currently inadequate for making reliable predictions.

One must not conclude from these remarks that a good convection theory is unnecessary to stellar evolution theory for modelling solar type stars. On the contrary, though it is only the integrated properties of the transition zone that are required to determine the adiabat deep down, a theory is required for extrapolating from models of the sun to other solar type stars. And, of course, as soon as one wishes to discuss the structure of a stellar atmosphere, a knowledge of the subphotospheric velocity field is essential.

The structure of convective envelopes of red giants is more sensitive to ℓ , but calibration is difficult because there are other uncertainties in both theory and observation. The degree of overshooting and consequent material mixing at the edges of convective cores is also of interest, but difficult to assess observationally.

It is common practice to argue that because the mixing-length hypothesis, in whatever guise it is to be used, is so uncertain, it is hardly worth the trouble to calculate its consequences accurately. Indeed it is sometimes the case that so coarse a mesh is used for the numerical solution of the stellar structure equations that the solutions are not resolved in the convection zone, and that the differential equations are therefore not adequately represented by the finite difference equations. It is also common, once a formula for F_c has been decided upon, not to calibrate the mixing length, nor even to report precisely the formula that was used for F_c . Though it may be true that in our present state of knowledge there is little reason to prefer, say, a red giant model computed with a mixing-length formula that has been carefully calibrated on the main sequence to a model computed with a similar formula that has not, there would be greater hope of improving our understanding of stellar convection and its influence on stellar structure if investigations were more meticulously carried out and reported. The prospects of an imminent supersession of mixing-length theory by a theory that is demonstrably more reliable for describing stellar convection zones

is bleak. Therefore it seems worthwhile to invest some effort into trying to improve the theory we already have. Modern sophisticated mixing-length theories have achieved some measure of success in describing turbulent flows in the laboratory (e.g. Launder & Spalding, 1972), so there is some hope that the effort would not be in vain.

E.A. Spiegel and I have recently been attempting to consolidate the theory by synthesizing the ideas that have been severally used in the past. The approach is based on a two-fluid model, one so-called fluid being an assembly of thermals and the other being the background environment. Entrainment, erosion and turbulent exchange of energy and momentum are represented in the equations of motion, using laboratory calibrations where possible. The goal is to derive a set of equations determining the heat flux and the Reynolds stresses that would be applicable to a sufficiently wide variety of circumstances for a meaningful calibration to be possible. The success or failure will be reported elsewhere.

It has been the aim of this article to clarify the ideas and assumptions behind the simple mixing-length theories used in astrophysics, and so provide a basis for the necessary improvement and generalization to circumstances more complicated than those for which the theory was originally formulated. Some indication of how this might be achieved has been given. Other measures that may have to be taken include abandoning the idea that the flow can always be described adequately in terms of a single length scale ℓ . This may be necessary for a theory of semiconvection, for example. It must be realized, however, that many attempts to improve or generalize the theory involve additional physical mechanisms, and consequently the introduction of new parameters that must be determined by observation.

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