Canad. Math. Bull. Vol. **21** (3), 1978

ORDER CHARACTERIZATION OF THE COMPLEX FIELD

BY
LINO GUTIERREZ NOVOA

1. **Introduction.** It is well known that the real number field can be characterized as an ordered field satisfied the "least upper bound" property.

Using the idea of n-ordered set, introduced in [3], and generalizing the notion of l.u.b. in a suitable way, it is possible to give a similar categorical definition of the complex field.

With these extended meanings, the main theorem of this paper (Theorem 7 in the text) is stated almost identically to the one for the real field. Any directly two-ordered field, in which the "supremum property" holds, is isomorphic to the complex field.

2. **Two-ordered sets.** (For the basic results and details on n-ordered sets the reader is referred to [3]. For an application of the idea of n-order to Geometry see [4].)

A two-order on a set S is a function ϕ defined on the classes of equivalent three-permutations of S, whose range is the set $\{-1, 0, 1\}$ and which satisfies 0_1 and 0_2. Two permutations are equivalent if they consist of the same elements and are of the same type (both odd or both even). The order is trivial if ϕ is identically 0. We write: $\phi(a, b, c) = \langle a, b, c\rangle$ for short. Note that a two-order is similar to the usual notion of orientation of the plane, but is not necessarily transitive in the sense that (xbc), (axc) may have the same orientation and (abc) the opposite one.

We state 0_1 and 0_2 (small Latin letter represent any elements of S):

$$0_1\colon\ \langle a, b, c\rangle = \langle b, c, a\rangle = -\langle b, a, c\rangle$$

$$\begin{aligned}0_2\colon\ &\text{If } \langle m, b, c\rangle\langle a, n, p\rangle, \langle n, b, c\rangle\langle m, a, p\rangle \quad \text{and}\\ &\langle p, b, c\rangle\langle m, n, a\rangle \quad \text{are non-negative, then}\\ &\langle a, b, c\rangle\langle m, n, p\rangle \geq 0.\end{aligned}$$

Let S be a two-ordered set. We give some definitions:

Definition 1. A triple (a, b, c) is *singular* if $\langle a, b, c\rangle = 0$. A pair (a, b) is singular if for any $x \in S$, $\langle a, b, x\rangle = 0$. An element (a) is singular if for any $x \in S$, (a, x) is singular.

Received by the editors August 3, 1977.

DEFINITION 2. If (a, b) is not singular, the *line* (a, b) is the set

$$L(a, b)=\{x; \langle a, b, x\rangle=0\}.$$

Any subset of a line is called a *linear set.*

The following theorem and its corollary are immediate consequences of the axioms of order and we omit the proofs.

THEOREM 0. *In any two-ordered set, if* $\langle a, b, x_i\rangle=0$ *for* $i=1,2,3$, *then either* (a, b) *or* (x_1, x_2, x_3) *is singular.*

COROLLARY 0. *If* $c\in L(a, b)$ *and* (a, c) *is not singular, then* $b\in L(a, c)$.

DEFINITION 3. Let $A\subset S$. We say that (h, k) is a *bound* for A if $\langle h, k, x\rangle\geq 0$ for every $x\in A$.

DEFINITION 4. We say that the pair (a, b) *separates* (c, d) if $\langle a, b, c\rangle\neq\langle a, b, d\rangle$.
It was proved in [3] that every line has two natural one-orders induced by the two-order of S. More precisely, if $e\notin L$, the function $\langle a, b\rangle=\langle a, b, e\rangle$ defined on $L\times L$ is a one-order which satisfies:

$$0_1'\colon\ \langle x, y\rangle=-\langle y, x\rangle$$

$$0_2'\colon\ \left.\begin{array}{l}\langle m, y\rangle\langle x, n\rangle\geq 0\\ \langle m, y\rangle\langle x, n\rangle\geq 0\end{array}\right\}\Rightarrow\langle x, y\rangle\langle m, n\rangle\geq 0,$$

These relative one-orders are independent of the choice of e (see [3], p. 1340) and are not necessarily transitive. Nevertheless, we shall use the notation $a<b$ meaning $\langle a, b\rangle=1$.

DEFINITION 5. If A is a subset of the one-ordered line L, we say that $s\in L$ is a *supremum* of A if:

(1) $s\geq a$ for any $a\in A$
(2) every bound (h, k) of A is also bound of $\{s\}$.

Notice that the above definition differs slightly from the usual one of least upper bound in a partially ordered set. (See [1], p. 16.)

DEFINITION 6. We say that (h, k) is an *upper bound* for the one-ordered linear set $A\subset L(a, b)$ where $a<b$ if the three following conditions hold:

(1) (h, k) is a bound for A.
(2) (a, b) separates (h, k).
(3) Either $\langle a, b, k\rangle=1$ or $\langle a, b, h\rangle=-1$ or both.

Lower bound is defined similarly, interchanging h and k in (3).

DEFINITION 7. The two-ordered set S is said to have the *supremum property* if every one-ordered linear set $A \subset L$ which has an upper bound, has a supremum $s \in L$.

DEFINITION 8. An *order-isomorphism* λ between two two-ordered sets (S_1, ϕ_1) and (S_2, ϕ_2) is a bijective function $\lambda : S_1 \to S_2$ which either preserves or reverses the order. That is, for every triple (a_1, a_2, a_3), $a_i \in S$, we have:

$$\phi_1(a_1, a_2, a_3) = k_\lambda \phi_2(\lambda(a_1), \lambda(a_2), \lambda(a_3))$$

where k_λ is either 1 or -1.

If $k_\lambda = 1$, we say λ is *direct*; if $k_\lambda = -1$, λ is *opposite*.

3. **Two-ordered fields.**

DEFINITION 9. Let F be a (commutative) field which is also a non-trivially two-ordered set. F is called a *two-ordered field* if the mappings $\lambda_h : x \to x + h$ and $\mu_k : x \to kx$, $k \neq 0$, are order-isomorphisms. If all of them are direct, F is a *directly two-ordered field.*

It is easy to show that there are no directly one-ordered fields. On the other hand, the complex field C with the two-order defined by:

$$\langle a, b, c\rangle = \operatorname{sign}\left(\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ \bar{a} & \bar{b} & \bar{c} \end{vmatrix} i\right), \qquad (\bar{a} = \text{conjugate of } a),$$

is a directly two-ordered field as can be readily seen.

Moreover, that the above property characterizes the complex field, when the supremum property is also present, will be the main result of this paper: (Theorem 7). The proof of this theorem will be a consequence of several preparatory results. In what follows F will denote a directly two-ordered field.

THEOREM 1. *If (a, b) is a singular pair of F, then $a = b$.*

Proof. Since F is not trivially ordered, there is some triple such that $\langle x, y, z\rangle \neq 0$. This implies $x \neq y$. Let $a \neq b$. Now, (a, b) being singular, $\langle a, b, u\rangle = 0$ for any $u \in F$. Taking $u = (z - x)(b - a)(y - x)^{-1} + a$, we have

$$\langle a, b, u\rangle = \langle x, y, z\rangle$$

because of Definition 9. This is a contradiction.

COROLLARY 2. *If $a \neq b$, then $L(a, b)$ is well-defined. In particular, we call* $R = L(0, 1)$.

Now we prove that R is a subfield of F.

4. **The subfield R.**

THEOREM 2. *If $a, b \in R$ and $b \neq 0$, then $a - b \in R$ and $ab^{-1} \in R$.*

Proof. We have $\langle 0, 1, a\rangle = \langle 0, 1, b\rangle = \langle 0, a, b\rangle = 0$ by Corollary 0. Now,

$$0 = \langle 0, a, b\rangle = \langle 0, -a, -b\rangle \qquad \text{(By definition 9).}$$

Thus, if $a \neq 0$, $a - b \in L(0, a) = R$. If $a = 0$, we use $\langle 0, 1, -1\rangle = -\langle 0, -1, 1\rangle = -\langle 0, 1, -1\rangle$ (by 0_1 and Definition 9 respectively) to show that $\langle 0, 1, -1\rangle = 0$ and therefore $\langle 0, b, -b\rangle = 0$, so that

$$a - b = -b \in L(0, b) = R.$$

For the second part, $0 = \langle 0, a, b\rangle = \langle 0, ab^{-1}, 1\rangle$. Hence, $ab^{-1} \in R$.

COROLLARY 3. *R is a subfield of F.*

Let $(R, <)$ denote the field R with the relative one-order for which $0 < 1$. We leave the proof that the order $<$ is transitive and that R is an ordered field in the usual sense to the reader.

THEOREM 3. *If F is a directly two-ordered field with the supremum property, then R is isomorphic to the real number field.*

Proof. One shows easily that the supremum property implies the existence of an l.u.b. for any subset of R bounded (in the usual sense) from above. It is well known (see [2], p. 95) that this property characterizes the real number field among the ordered fields.

From now on, F will denote a directly two-ordered field with the supremum property, and because of Theorem 3, we identify R with the real numbers.

THEOREM 4. *There is no upper bound in F for the set N of natural numbers.*

Proof. Otherwise, N would have a supremum $s \in R$, $s \geq n$, for $n \in N$, and we know this is not true.

The proof of the following two corollaries are left to the reader.

COROLLARY 4. *There is no lower bound in F for the set N' of negative integers.*

COROLLARY 5. *If $(0, 1)$ separates (s, y), then there are elements $a, b \in R$ such that (s, y) separates (a, b).*

5. **The complex field.** Our final goal is to show that F is isomorphic to the field C of complex numbers. The next theorem is basic for that purpose.

THEOREM 5. *If $(0, 1)$ separates (x, y), some $r \in R$ satisfies*

$$\langle x, y, r\rangle = 0.$$

Proof. If either x or y is in R, the theorem is obvious. So we assume, for

https://doi.org/10.4153/CMB-1978-054-6 Published online by Cambridge University Press

instance, $\langle 0, 1, x\rangle = -\langle 0, 1, y\rangle = 1$. Hence, we may consider that the order of R is induced by x.

Call

$$A = \{u;\, u \in R, \langle u, y, x\rangle \geq 0\}$$

and

$$A' = \{v;\, v \in R, \langle v, y, x\rangle \leq 0\}.$$

By Corollary 5, neither A nor A' is empty. Now (y, x) is an upper bound for A. Similarly, (x, y) is an upper bound for A' in the opposite order of R (the one induced by y). It follows from the supremum property that there are elements $z_1, z_2 \in R$ such that:

$$z_1 \geq a \quad \text{for} \quad a \in A \quad \text{and} \quad \langle y, x, z_1\rangle \geq 0$$

and

$$z_2 \leq a' \quad \text{for} \quad a' \in A' \quad \text{and} \quad \langle x, y, z_2\rangle \leq 0.$$

But R being dense and $A \cup A' = R$, it follows that

$$z_1 = z_2 = r \quad \text{and} \quad \langle x, y, r\rangle = 0.$$

THEOREM 6. *Every element $z \in F$ is a root of some quadratic equation with real coefficients.*

Proof. We assume that both z and $z + z^{-1}$ are not in R. Otherwise, the statement is obvious. Since $\langle 0, 1, z\rangle = -\langle 0, 1, z^{-1}\rangle$, one of the two pairs $(z + z^{-1}, z)$ or $(z + z^{-1}, z^{-1})$ is separated by $(0, 1)$. In any case, let $r \in R$ be the element furnished by Theorem 5.

In the first case we have:

$$\langle 0, 1, z(r - z)\rangle = \langle 0, z^{-1}, r - z\rangle = \langle z, z + z^{-1}, r\rangle = 0,$$

and, therefore, $s = z(r - z) \in R$, so that

$$z^2 - rz + s = 0.$$

In the second case:

$$\langle 0, 1, z^{-1}(r - z^{-1})\rangle = \langle 0, z, r - z^{-1}\rangle = \langle z^{-1}, z + z^{-1}, r\rangle = 0,$$

and

$$s = z^{-1}(r - z^{-1}) \in R.$$

Hence,

$$sz^2 - rz + 1 = 0.$$

We have therefore completed the proof of: *Theorem* 7. Any directly two-ordered field with the supremum property is isomorphic to the complex field.

References

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University of Alabama