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ON A GENERALIZATION OF ONE DIMENSIONAL RANDOM WALK WITH A PARTIALLY REFLECTING BARRIER

BY
B. R. HANDA

1. **Introduction and summary.** Consider a one-dimensional random walk model where a particle starting at the origin at any instant either takes a jump through a unit distance to the right with probability p_1, or stays at the same position with probability p_0, or else takes a jump through either of $1, 2, \ldots, \mu$ units of distance to the left with probabilities $p_{-1}, p_{-2}, \ldots, p_{-\mu}$ respectively. Assume $\sum_{\nu=-\mu}^{1} p_\nu = 1$. Furthermore a partially reflecting barrier which behaves in the following manner is placed at the position $m_1(\geq 0)$. As soon as the particle reaches the barrier it is reflected to either of the positions $m_1-1, \ldots, m_1-\mu$ with probabilities $q_{-1}, \ldots, q_{-\mu}$ or else gets absorbed at the position m_1 with probability $q_0 = (1-\sum_{\nu=1}^{\mu} q_{-\nu})$. This paper mainly attempts the problem of obtaining the exact expressions for the probabilities of the events

(A) of being at the position m at the Nth step
(B) of being at the barrier at the Nth step and getting absorbed in the next step.

A graphical representation of the above random walk introduced in the next section together with certain combinatorial arguments shall be used in the proofs. In §3 letting $\mu=1$ we derive the probabilities for the events (A) and (B). However, for the case of general μ we obtain in §4, the probability of event (B) only. The expression for the probability of (A) in general case is very cumbersome. The last section uses the result of §4 to solve a ballot problem with $(\mu+1)$ candidates, which shall be formulated in that section.

2. **Graphical representation.** We would call the line segment between two lattice points in the plane on which no other lattice point lie as a step. If a step is parallel to the line $x=ty$, where $t \geq 0$ is an integer, we say that the step is of the type S_t, whereas by a step of the type S we mean a step in the horizontal direction. With any realization of the generalized random walk we associate a minimal lattice path with diagonal steps (called a random walk graph) in the following manner.

The minimal lattice path starts at the origin and at any instant proceeds either by a step of the type S with probability p_1 or a step of any of the types S_t, $t=0, 1, \ldots, \mu$ with probabilities $p_{t-\mu}$, $t=0, 1, \ldots, \mu$ respectively. The line $x=m_1+\mu y$ acts as a

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barrier. Consequently the minimal lattice path reaching this line is reflected in such a way that the next step is either of the types S_t, $t=0, 1, \ldots, \mu-1$ (referred to as the reflections of the types D_t) with probabilities $q_{t-\mu}$, $t=0, 1, \ldots, \mu-1$, or else it terminates there with probability q_0.

The minimal lattice path that ends on the line $x=\mu y+t$ must do so in $t+\sum_{i=0}^{\mu}(\mu+1-i)r_i$ steps, where r_i is the number of steps of the types S_i, $i=0, 1, \ldots, \mu$ and remaining $t+\sum_{i=0}^{\mu}(\mu-i)r_i$ steps are of the type S. Therefore, corresponding to the events (A) and (B) of §1, we consider the following events in the terminology of minimal lattice paths

(A_1) A minimal lattice path having exactly r_i steps of the type S_i ($i=0, 1, \ldots, \mu$) ends on the line $x=m+\mu y$.

(B_1) A minimal lattice path having exactly r_i steps of the type S_i($i=0, 1, \ldots, \mu$) ends on the barrier $x=m_1+\mu y$ and the path terminates in the next step.

In what follows we use the symbol $\binom{x}{j_1, \ldots, j_k}$ to denote the multinomial coefficient $x(x-1)\ldots(x-\sum_{i=1}^{k}j_i+1)/\prod_{i=1}^{k}j_i!$ We state below a theorem from [4] in slightly different notations which shall be needed in the subsequent sections.

THEOREM. *For nonnegative integral values of* α, β *and* $0 \le t \le \beta$ *let* $N(\alpha, \beta; r_0, \ldots, r_t)$ *represent the number of paths from* $(0, 0)$ *to* $(\alpha+\beta n, n)$ *where* $n=\sum_{i=0}^{t} r_i$ *never touching the line* $x=\beta y$ *except possibly for the end points and having* r_i *steps of the type* $S_i(i=0, 1, \ldots, t)$. *Then*

$$(1) \qquad N(\alpha, \beta; r_0, \ldots, r_t) = \begin{cases} \dfrac{1}{\sum\limits_{i=0}^{t}(\beta+1-i)r_i-1}\dbinom{\sum\limits_{i=0}^{t}(\beta+1-i)r_i-1}{r_0, \ldots, r_t}, & \alpha = 0 \\ \dfrac{\alpha}{\alpha+\sum\limits_{i=0}^{t}(\beta+1-i)r_i}\dbinom{\alpha+\sum\limits_{i=0}^{t}(\beta+1-i)r_i}{r_0, \ldots, r_t}, & \alpha > 0 \end{cases}$$

3. **The simple case** ($\mu=1$). Here the paths involve only three types of steps, viz., horizontal, along the line $x=y$ (referred to as the diagonal step) and vertical. Also there is only one type of reflection at the barrier $x=y+m_1$, viz., of the type D_0 which is in the vertically upward direction.

Denote by $P_n(m; r_0, r_1)$ the probability that a path with r_0 vertical steps and r_1 diagonal steps ends in the point $(m+r_0+r_1, r_0+r_1)$ on the line $x=m+y$ and has exactly n reflections at the line $x=m_1+y$. Obviously $P_n(m; r_0, r_1)$ is nonzero for $m \le m_1$. Also since

$$(2) \qquad P_n(m_1; r_0, r_1) = P_n(m_1-1; r_0, r_1)p_1$$

we shall obtain the expression for $P_n(m; r_0, r_1)$, when $m<m_1$ only.

It might be possible to use the reflection principle in the lines of Lehner [2] in order to obtain the required expression, but we propose to use a combinatorial method involving convolution identities proved in [3]. The later technique is

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equally elegant as the first one but is more powerful in the sense that it is capable of being extended to the case of general μ, to be dealt with in the next section.

The expression for $P_n(m; r_0, r_1)$ is stated in the following theorem.

THEOREM 1. *For* $m<m_1$:

$$
(3)\qquad \begin{cases} P_0(m; r_0, r_1) = \left[\binom{m+2r_0+r_1}{r_0, r_1} - \binom{m+2r_0+r_1}{r_0+m-m_1, r_1}\right] p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0}, \\ P_n(m; r_0, r_1) = \dfrac{2m_1-m+n-1}{m+2r_0+r_1-(n-1)} \binom{m+2r_0+r_1-(n-1)}{r_0+m-m_1-(n-1), r_1} \left(\dfrac{q_{-1}}{p_{-1}}\right)^n \\ \qquad \times p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0}, \quad \textit{for } 1 \le n \le r_0+m-m_1+1. \end{cases}
$$

Proof. For $n=0$:

$$
(4)\qquad P_0(m; r_0, r_1) = N_0(m; r_0, r_1) p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0},
$$

where $N_0(m; r_0, r_1)$ denotes the number of paths from (0, 0) to $(m+r_0+r_1, r_0+r_1)$ having exactly r_0 vertical and r_1 diagonal steps and never touching or crossing the line $x=m_1+y$. Subtracting from the total number of paths from (0, 0) to

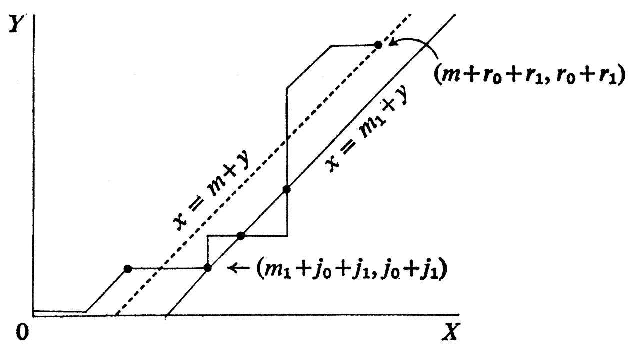

Figure 1.

$(m+r_0+r_1, r_0+r_1)$, those paths that actually touch or cross the line $x=m_1+y$ (see Fig. 1) we have, by using (1),

$$
(5)\qquad \begin{aligned} N_0(m; r_0, r_1) &= \binom{m+2r_0+r_1}{r_0, r_1} \\ &\quad - \sum_{j_0=0}^{r_0-(m_1-m)} \sum_{j_1=0}^{r_1} \left\{ N(m, 1; j_0, j_1) \binom{m-m_1+2(r_0-j_0)+(r_1-j_1)}{r_0-j_0, r_1-j_1} \right\}. \end{aligned}
$$

The sum on the right can be written as

$$
\sum_{j_0=0}^{r_0-(m_1-m)} \sum_{j_1=0}^{r_1} \frac{m_1}{m_1+2j_0+j_1} \binom{m_1+2j_0+j_1}{j_0, j_1} \binom{m_1+2(r_0-j_0)+(r_1-j_1)}{r_0-j_0, r_1-j_1}
$$

which, with the help of convolution (10) in [3], simplifies to

$$
\binom{m+2r_0+r_1}{r_0+m-m_1, r_1}.
$$

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Inserting the value of $N_0(m; r_0, r_1)$ from (5) in (4) we have proved the result for $n=0$.

For $n \geq 1$ we prove the result by induction. When $n=1$ we divide the path into two segments, the first segment containing j_0 vertical and j_1 diagonal steps ends in the point $(m_1+j_0+j_1, j_0+j_1)$ on the line $x=m_1+y$ where the reflection takes place. Then with the help of (1), we can write the expression for the paths from (0, 0) to $(m_1+r_0+r_1, r_0+r_1)$ with one reflection as

$$\sum_{j_0=0}^{r_0+m-m_1} \sum_{j_1=0}^{r_1} N(m_1, 1; j_0, j_1) N(m_1-m, 1; r_0-m_1+m-j_0, r_1-j_1)$$

which, by using the convolution (9) in [3], simplifies to

$$\frac{2m_1-m}{m+2r_0+r_1}\binom{m+2r_0+r_1}{r_0-m_1+m, r_1}.$$

Since each such path has the same probability

$$p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0-1} q_{-1} = p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0}\left(\frac{q_{-1}}{p_{-1}}\right)$$

the theorem is proved for $n=1$. Assume (3) to be true up to $n=r$. For $n=r+1$, we consider a similar division of the path as for the case $n=1$, but here the point $(m_1+j_0+j_1, j_0+j_1)$ is the point of $(r+1)$th reflection. Thus

$$\begin{aligned} P_{r+1}(m; r_0, r_1) = & \sum_{j_0=r}^{r_0+m-m_1} \sum_{j_1=0}^{r_1} P_r(m_1-1; j_0, j_1) . p_1 \\ & \times q_{-1} N(m_1-m, 1; r_0-m_1+m-j_0, r_1-j_1) \\ & \times p_1^{m-m_1+r_0-j_0} p_0^{r_1-j_1} p_{-1}^{r_0-j_0-1} \end{aligned}$$

Routine calculations reduce the right-hand member to the right member of (3) with $n=r+1$. This completes the proof.

Denoting by $P(A_1; \mu, m)$ and $P(B_1; \mu)$ the probabilities of the events A_1 and B_1 respectively defined in §2, one immediately proves the following corollaries.

COROLLARY 1. *For* $m<m_1$

$$\begin{aligned} P(A_1; 1, m) = & \Bigg[\binom{m+2r_0+r_1}{r_0, r_1} + \binom{m+2r_0+r_1}{r_0+m-m_1, r_1}\Bigg\{1+\left(\frac{q_{-1}}{p_{-1}}-2\right) \\ & \times {}_2F_1\left(1, -(r_0+m-m_1); -(m+2r_0+r_1); \frac{q_{-1}}{p_{-1}}\right)\Bigg\} \\ & -\frac{q_{-1}}{p_{-1}}\binom{m+2r_0+r_1-1}{r_0+m-m_1, r_1-1} \\ & \times {}_2F_1\left(1, -(r_0+m-m_1); -(m+2r_0+r_1-1); \frac{q_{-1}}{p_{-1}}\right)\Bigg] \\ & \times p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0} \end{aligned} \tag{6}$$

where ${}_2F_1$ *is the well known hypergeometric function.*

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Proof. From Theorem 1

$$
\begin{aligned}
P(A_1; 1, m) = \Bigg[& \binom{m+2r_0+r_1}{r_0, r_1} - \binom{m+2r_0+r_1}{r_0+m-m_1, r_1} \\
& + \sum_{n=1}^{r_0+m-m_1+1} \frac{2m_1-m+n-1}{m+2r_0+r_1-(n-1)} \binom{m+2r_0+r_1-(n-1)}{r_0+m-m_1-(n-1), r_1} \\
& \times \left(\frac{q_{-1}}{p_{-1}}\right)^n \Bigg] p_1^{m+r_0} p_0^{r_1} p_{-1}^{r_0}.
\end{aligned} \tag{7}
$$

Though it might be possible to express (7) as (6) by modifying the steps used for obtaining equations (29) through (37) in [2], a much simpler method would be to use the identity

$$
\begin{aligned}
& \frac{2m_1-m+n}{m+2r_0+r_1-n} \binom{m+2r_0+r_1-n}{r_0+m-m_1-n, r_1} \\
& \quad = \binom{m+2r_0+r_1-n}{r_0+m-m_1-n, r_1} - 2\binom{m+2r_0+r_1-n-1}{r_0+m-m_1-n-1, r_1} - \binom{m+2r_0+r_1-n-1}{r_0+m-m_1-n, r_1-1}
\end{aligned}
$$

along with some rearrangement of the terms, in the summation on the right of (7).

COROLLARY 2.

$$
\begin{aligned}
P(B_1; 1) = \Bigg[& \binom{m+2r_0+r_1}{r_0, r_1} \Bigg\{ \frac{2p_{-1}}{q_{-1}} + \left(1 - \frac{2p_{-1}}{q_{-1}}\right) \\
& \times {}_2F_1\left(1, -r_0; -(m_1+2r_0+r_1); \frac{q_{-1}}{p_{-1}}\right) \Bigg\} \\
& - \binom{m_1+2r_0+r_1-1}{r_0, r_1-1} {}_2F_1\left(1, -r_0; -(m_1+2r_0+r_1-1); \frac{q_{-1}}{p_{-1}}\right) \Bigg] \\
& \times q_0 p_1^{m_1+r_0} p_0^{r_1} p_{-1}^{r_0}.
\end{aligned} \tag{8}
$$

Proof. Observe that

$$
\begin{aligned}
P(B_1; 1) &= P(A_1; 1, m_1-1) p_1 . q_0 \\
&= \sum_{n=0}^{r_0} \frac{m_1+n}{m_1+2r_0+r_1-n} \binom{m_1+2r_0+r_1-n}{r_0-n, r_1} p_1^{m_1+r_0} p_0^{r_1} p_{-1}^{r_0} q_0 \quad \text{(from (7))}
\end{aligned}
$$

which, when expressed in terms of hypergeometric function, is as in (8).

4. **The general case.** Denoting by $P(m; r_0, \ldots, r_\mu; k_0, \ldots, k_{\mu-1})$ the probability that the minimal lattice path having r_i steps of the type $S_i (i=0, 1, \ldots, \mu)$ ends on the line $x = m + \mu y$ and has exactly k_i reflections of the type D_i $(i=0, \ldots, \mu-1)$, we prove the following theorem.

THEOREM 2.

$$
\begin{aligned}
& P(m_1; r_0, \ldots, r_\mu; k_0, \ldots, k_{\mu-1}) \\
& = \binom{k_0+\cdots+k_{\mu-1}}{k_0, \ldots, k_{\mu-1}} \frac{m_1 + \sum_{i=0}^{\mu-1} (\mu-i)k_i}{m_1 - \sum_{i=0}^{\mu-1} k_i + \sum_{i=0}^{\mu} (\mu+1-i)r_i} \binom{m_1 - \sum_{i=0}^{\mu-1} k_i + \sum_{i=0}^{\mu} (\mu+1-i)r_i}{r_0-k_0, \ldots, r_{\mu-1}-k_{\mu-1}, r_\mu} \\
& \quad \times \prod_{i=0}^{\mu-1} \left(\frac{q_{i-\mu}}{p_{i-\mu}}\right)^{k_i} \prod_{i=0}^{\mu} p_{i-\mu}^{r_i} p_1^{m_1 + \sum_{i=0}^{\mu} (\mu-i) r_i} q_0
\end{aligned} \tag{8}
$$

Proof. The proof is by induction which uses the following equations

$$P(m_1; r_0, \ldots, r_\mu; 1, 0, \ldots, 0) = \sum_{j_\mu=0}^{r_\mu} \cdots \sum_{j_1=0}^{r_1} \sum_{j_0=0}^{r_0-1} N(m_1, \mu; j_0, \ldots, j_\mu) \prod_{i=0}^{\mu} p_{i-\mu}^{j_i} \cdot \times p_1^{m_1+\sum_{i=0}^{\mu}(\mu-1)j_i} q_{-\mu} N(\mu, \mu; r_0-j_0-1, r_1-j_1, \ldots, r_\mu-j_\mu) \times \prod_{i=0}^{\mu} p_{i-\mu}^{r_i-j_i} \frac{1}{p_{-\mu}} p_1^{\sum_{i=0}^{\mu}(\mu-i)(r_i-j_i)} q_0,$$

and for $j=0, 1, \ldots, \mu-1$

$$P(m_1; r_0, \ldots, r_\mu; k_0, \ldots, k_j+1, \ldots, k_{\mu-1}) = \sum_{l=0}^{\mu-1} \sum_{j_\mu=0}^{r_\mu} \cdots \sum_{j_j=k_j+1}^{r_j} \cdots \sum_{j_l=k_l-1}^{r_l-1} \cdots \sum_{j_0=k_0}^{r_0} \times P(m_1; j_0, \ldots, j_\mu; k_0, \ldots, k_{j+1}, \ldots, k_l-1, \ldots, k_{\mu-1}) \times q_{l-\mu} N(\mu-l, \mu; r_0-j_0, \ldots, r_l-1-j_l, \ldots, r_\mu-j_\mu) \times \prod_{i=0}^{\mu} p_{i-\mu}^{r_i-j_i} \frac{1}{p_{l-\mu}} p_1^{\sum_{i=0}^{\mu}(\mu-i)(r_i-j_i)} q_0$$

[when $l=j$, the limits of summation over j_l is modified as from k_0 to r_0-1 and the first term under summation becomes $P(m_1; j_0, \ldots, j_\mu; k_0, \ldots, k_{\mu-1})$]. The rest of the proof involves the usual simplifications.

The following corollaries from Theorem 2 are immediate.

COROLLARY 3.

(9) $$P(B_1; \mu) = \sum_{n=0}^{r_0+\cdots+r_{\mu-1}} \sum^{n} P(m_1; r_0, \ldots, r_\mu; k_0, \ldots, k_{\mu-1})$$

where $\sum^n$ is the sum over all k_i's subject to the restrictions

$$\left\{\sum_{i=0}^{\mu-1} k_i = n, \quad 0 \le k_i \le r_i, \quad i = 0, 1, \ldots, \mu-1\right\}$$

COROLLARY 4. *When $r_1 = \cdots = r_\mu = 0$, i.e. the random walk has two moves, viz. 1 or $-\mu$ at any stage, we have*

$$P(B_1; \mu) = \left[\sum_{n=0}^{r_0} \frac{m_1+n\mu}{m_1-n+(\mu+1)r_0} \binom{m_1-n+(\mu+1)r_0}{r_0-n} \left(\frac{q_{-\mu}}{p_{-\mu}}\right)^n\right] \times p_1^{m_1+\mu r_0} p_{-\mu}^{r_0} q_0$$

(10) $$= \binom{m_1+(\mu+1)r_0}{r_0} \left[\frac{(\mu+1)p_{-\mu}}{q_{-\mu}} + \left(1-\frac{(\mu+1)p_{-\mu}}{q_{-\mu}}\right) \times {}_2F_1\left(1_0-r_0; -(m_1+(\mu+1)r_0); \frac{q_{-\mu}}{p_{-\mu}}\right)\right] \times p_1^{m_1+\mu r_0} p_{-\mu}^{r_0} q_0.$$

A particular case of (10) when $\mu=1$ is the result (41) in [2].

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5. **An application to ballot problem.** Let us consider a ballot with $(\mu+1)$ candidates $A_0, \ldots, A_{\mu-1}$ and A having $r_0, \ldots, r_{\mu-1}$ and r votes respectively. Represent by $A_i(k)$ (by $A(k)$) the number of votes for the candidate A_i (for A), $i=0, 1, \ldots, \mu-1$ when the first k votes have been counted. $1 \le k \le \sum_{i=0}^{\mu-1} r_i + r$. The probability of the event

$$E_{n,\mu}:(A(k) \geq \sum_{i=0}^{\mu-1} (\mu-i)A_i(k), \tag{11}$$

$k=1, \ldots, \sum_{i=0}^{\mu-1} r_i+r$ and the equality sign holds for exactly n number of k's), $0 \le n \le \sum_{i=0}^{\mu-1} r_i$ will be obtained in this section.

Assume that all possible counting records are equally likely, each having the same probability

$$1 \Big/ \binom{r+\sum_{i=0}^{\mu-1} r_i}{r_0, \ldots, r_{\mu-1}}.$$

Any counting record can be represented by a minimal lattice path from (0, 0) to $(r+\sum_{i=0}^{\mu-1} ir_i, \sum_{i=0}^{\mu-1} r_i)$ such that the ith step $(i=1, \ldots, r+\sum_{i=0}^{\mu-1} r_i)$ is of the type S_t $(t=0, \ldots, \mu-1)$ or S, according as the ith vote is cast for the candidate A_t or A. Favourable counting records to the event $E_{n,\mu}$ are those whose corresponding minimal lattice paths lie below the line $x=\mu y$ and touch this line exactly at n points. Clearly $P(E_{n,\mu})$ is nonzero when $r \ge \sum_{i=0}^{\mu-1} (\mu-i)r_i$. Reviewing Theorem 2, by taking $r_\mu=0$ and $m_1=r-\sum_{i=0}^{\mu-1} (\mu-i)r_i$ a moment's reflection, yields the number of favourable paths to be

$$\sum^n \binom{k_0+\cdots+k_{\mu-1}}{k_0, \ldots, k_{\mu-1}} \frac{r-\sum_{i=0}^{\mu-1} (r_i-k_i)(\mu-i)}{r+\sum_{i=0}^{\mu-1} (r_i-k_i)} \binom{r+\sum_{i=0}^{\mu-1} (r_i-k_i)}{r_0-k_0, \ldots, r_{\mu-1}-k_{\mu-1}} \tag{12}$$

where $\sum^n$ denotes as before the sum over all k_i's subject to restrictions $\{\sum_{i=0}^{\mu-1} k_i=n$, $0 \le k_i \le r_i$, $i=0, \ldots, \mu-1\}$. Then by multiplying (12) by

$$1 \Big/ \binom{r+\sum_{i=0}^{\mu-1} r_i}{r_0, \ldots, r_{\mu-1}}$$

one gets $P(E_{n,\mu})$.

Observe that $P(E_{0,1})$ and $\sum_{n=0}^{r_0} P(E_{n,1})$ are the strict sense and weak sense probabilities (see [1]) of the classical ballot problem. Whereas if we set $r_1=\cdots=r_{\mu-1}=0$ in $P(E_{n,\mu})$ we can get probabilities corresponding to the Babier generalization of classical ballot problem.

We conclude with the following identity.

$$
\begin{aligned}
(13)\qquad & N\left(r-\sum_{i=0}^{\mu-1}(\mu-i)r_i+1,\mu;r_0,\ldots,r_{\mu-1}\right)\\
&\quad=\sum_{n=0}^{r_0+\cdots+r_{\mu-1}}\sum^{n}\binom{k_0+\cdots+k_{\mu-1}}{k_0,\ldots,k_{\mu-1}}\\
&\quad\times\frac{r-\sum_{i=0}^{\mu-1}(\mu-i)(r_i-k_i)}{r+\sum_{i=0}^{\mu-1}(r_i-k_i)}\binom{r+\sum_{i=0}^{\mu-1}(r_i-k_i)}{r_0-k_0,\ldots,r_{\mu-1}-k_{\mu-1}}.
\end{aligned}
$$

Note that the left-hand side uses (1) and represents the number of paths from (0, 0) to $(r+\sum_{i=0}^{\mu-1} ir_i, \sum_{i=0}^{\mu-1} r_i)$, never crossing the line $x=\mu y$.

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INDIAN INSTITUTE OF TECHNOLOGY,
HAUZ KHAS, NEW DELHI, INDIA