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THE SUMMED PAPERFOLDING SEQUENCE

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Abstract

The sequence $a(1), a(2), a(3), \ldots$, labelled A088431 in the *Online Encyclopedia of Integer Sequences*, is defined by: $a(n)$ is half of the $(n+1)$th component, that is, the $(n+2)$th term, of the continued fraction expansion of

$$\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}.$$

Dimitri Hendriks has suggested that it is the sequence of run lengths of the paperfolding sequence, A014577. This paper proves several results for this summed paperfolding sequence and confirms Hendriks's conjecture.

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1. Introduction and preliminaries

According to Ben-Abraham *et al.* [4], the regular paperfolding sequence was discovered in the mid-1960s by three NASA physicists: John Heighway, Bruce Banks and William Hartner. Martin Gardner celebrated their work in *Scientific American* [7], after which Davis and Knuth [5] developed the mathematical underpinnings of paperfolding. Since that time, many papers have been written exploring the diverse features of this sequence, notably Dekking *et al.* [6], Allouche and Shallit [1], Mendès France and van der Poorten [9], and Mendès France and Shallit [8].

We start with two alternative definitions of sequence A088431, both found in the Online Encyclopedia of Integer Sequences [13].

DEFINITION 1.1 (A088431 continued fraction definition [13]). The sequence $A = a(1)a(2)a(3)\ldots$ is defined by: $a(n)$ is half of the $(n+1)$th component, that is, the $(n+2)$th term, of the continued fraction expansion of

$$\sum_{k=0}^{\infty} \frac{1}{2^{2^k}}.$$

TABLE 1. Initial terms of the sequence A088431.

$a(n)$	Formula	Value
$a(1)$	$a(1) = 2$	2
$a(2)$	1, since undefined	1
$a(3)$	$a(a(1) + 1) = a(2 + 1)$	2
$a(4)$	$a(a(1) + a(2) + 1) = a(2 + 1 + 1)$	2
$a(5)$	3, since undefined	3
$a(6)$	$a(a(1) + a(2) + a(3) + 1) = a(2 + 1 + 2 + 1)$	2
$a(7)$	1, since undefined	1

DEFINITION 1.2 (A088431 alternative definition [13]). The sequence $A = a(1)a(2)a(3)\ldots$ is given by the following rule: let $i = 1, 2, 3, \ldots$ and $a(1) = 2$. Then

$$a(a(1) + a(2) + a(3) + \cdots + a(n) + 1) = 2$$

and the ith undefined term of A is the ith term of the sequence

$$1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, \ldots.$$

EXAMPLE 1.3. Based on Definition 1.2, we give the first few terms of the sequence A088431 in Table 1.

Dimitri Hendriks in [13] has suggested that sequence A088431 appears to be the sequence of run lengths of the regular paperfolding sequence A014577. We prove several results concerning this summed paperfolding sequence and confirm Hendriks's conjecture.

In what follows, for simplicity and where no ambiguity exists, we remove the commas from sequences. For example, for a sequence only having values 1, 2 or 3, the sequence 2, 2, 1, 3 becomes 2213.

Davis and Knuth [5] prove the following result which we adopt as a definition for the paperfolding sequence. The notation is taken from Bates *et al.* [2].

DEFINITION 1.4 (Paperfolding sequence). Let S_n be the paperfolding sequence of length $2^n - 1$ and let $\overline{S_n^R}$ be S_n in reverse order with 0 becoming 1 and 1 becoming 0. Then, $S_1 = 1, S_{n+1} = S_n 1 \overline{S_n^R}$ and $\overline{S_{n+1}^R} = S_n 0\, \overline{S_n^R}$.

Bates *et al.* [2, 3] identify the following results for S_n.

THEOREM 1.5 (Expression for S_n). *For $n > 0$ and $h, k \geq 0$, $S_n = f_1 f_2 \ldots f_{2^n-1}$, where*

$$f_i = \begin{cases} 1 & \text{if } i = 2^k(4h + 1), \\ 0 & \text{if } i = 2^k(4h + 3). \end{cases}$$

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THEOREM 1.6 (Paperfolding runs). *The paperfolding sequence, S_n, contains only runs of single, double or triple entries of* 0; *or single, double or triple entries of* 1. *In particular, for $n \geq 4$, S_n contains:*

(i) 2^{n-4} *instances of the triple* 111, *and* $2^{n-4} - 1$ *instances of the triple* 000;
(ii) 2^{n-3} *instances of the double* 11, *and* $2^{n-3} + 1$ *instances of the double* 00;
(iii) 2^{n-4} *instances each of the single* 1, *and the single* 0.

THEOREM 1.7 (Interleaved paperfolding). *Let $S = f_1 f_2 f_3 \dots$ be the paperfolding sequence of infinite length. Then*

$$S = S_3\ f_1\ \overline{S_3^R}\ f_2\ S_3\ f_3\ \overline{S_3^R}\ f_4\ S_3\ f_5\ \overline{S_3^R}\ \dots.$$

That is, S is formed from the alternating interleave of two subsequences, S_3 and $\overline{S_3^R}$, with itself.

We consider the *runs* of identical terms in S_n or S, that is, the sizes of the sequence of successive digits 1 and 0 of S_n or S. We begin with a definition of these runs.

DEFINITION 1.8 (Summed paperfolding sequence). For $S_n = f_1 f_2 \dots f_{2^n-1}$:

- the summed paperfolding sequence, G_n, is the sequence of the sizes of successive digits 1 and 0 of S_n;
- the summed paperfolding sequence of infinite length is $G = \lim_{n\to\infty} G_n$ and is designated as $G = g(1)g(2)g(3)\dots$

We show in Theorem 1.10 that G_n has length 2^{n-1}.

EXAMPLE 1.9. $S_4 = 110110011100100$, $G_4 = 21223212$ and $G = 21223212\dots.$

The key results in this paper are:

- Theorem 2.1, a closed-form expression for G analogous to the expression for S_n (and by extension S) given at Theorem 1.5;
- Theorem 2.2 identifying the main internal relationships within G;
- Theorem 4.1 (Main Result) confirming Hendriks's conjecture [13]. That is, the sequence $A = a(1)a(2)a(3)\dots$ of A088431 is exactly the sequence $G = g(1)g(2)g(3)\dots$ of Definition 1.8.

THEOREM 1.10 (Length of G_n). $|G_n| = 2^{n-1}$, *where $|G_n|$ is the length of G_n.*

PROOF. For $n < 4$, our result is true. From Theorem 1.6, for $n \geq 4$, there are 2^{n-4} 111s, $2^{n-4} - 1$ 000s, 2^{n-3} 11s, $2^{n-3} + 1$ 00s, 2^{n-4} 1s and 2^{n-4} 0s. Hence,

$$|G_n| = 2^{n-4} + 2^{n-4} - 1 + 2^{n-3} + 2^{n-3} + 1 + 2^{n-4} + 2^{n-4} = 2^{n-1}. \qquad \square$$

Note that from Definition 1.4, since $S_{n+1} = S_n 1 \overline{S_n^R}$, the initial $|G_n|$ terms of G_{n+1} will be G_n. Since by Theorem 1.10, $|G_n| = 2^{n-1}$, we can write

$$G_n = g(1)g(2)\dots g(2^{n-1} - 1)g(2^{n-1}).$$

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2. Basic facts concerning G

We now develop an analogous expression for G to that given at Theorem 1.5 for S_n (and, by extension, S).

THEOREM 2.1 (Expression for G). *For $h, k, p \geq 0$, $G = g(1)g(2)g(3)\ldots$, where*

$$g(n) = \begin{cases} 1 \text{ if (i) } n = 8k+2, \text{ or} \\ \quad \text{if (ii) } n = 8k+7, \\ 2 \text{ if (iii) } n = 8k+1 \text{ and } k = 2^p(4h+3), \text{ or} \\ \quad \text{if (iv) } n = 8k+3, \text{ or} \\ \quad \text{if (v) } n = 8k+4 \text{ and } k \text{ is even, or} \\ \quad \text{if (vi) } n = 8k+5 \text{ and } k \text{ is odd, or} \\ \quad \text{if (vii) } n = 8k+6, \text{ or} \\ \quad \text{if (viii) } n = 8k+8 \text{ and } k+1 = 2^p(4h+1). \\ 3 \text{ if (ix) } n = 8k+1 \text{ and } k = 2^p(4h+1), \text{ or} \\ \quad \text{if (x) } n = 8k+4 \text{ and } k \text{ is odd, or} \\ \quad \text{if (xi) } n = 8k+5 \text{ and } k \text{ is even, or} \\ \quad \text{if (xii) } n = 8k+8 \text{ and } k+1 = 2^p(4h+3). \end{cases}$$

PROOF. From Theorem 1.6, S only contains singles, doubles and triples. Thus, the only possible terms in G are 1, 2 and 3. From Theorem 1.7, each S_3 and $\overline{S_3^R}$ starts with 11 and ends with 00 and the component $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ is followed by $S_3\ f_{i+2}\ \overline{S_3^R}\ f_{i+3}$, which is followed by $S_3\ f_{i+4}\ \overline{S_3^R}\ f_{i+5}$, and so on, indefinitely. Let the 0th component be $S_3\ f_1\ \overline{S_3^R}\ f_2$. Then the kth component is $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ where i is odd and $k = (i-1)/2$. Thus, the translation from S to G of the kth component, $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$, can be represented by eight terms, $g(8k+1)$ to $g(8k+8)$, where $k = (i-1)/2$ with two possible configurations:

- $f_i = 0:\ S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ becomes (2 or 3)123221(2 or 3); or
- $f_i = 1:\ S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$ becomes (2 or 3)122321(2 or 3),

where bracketed entries are determined by the values of f_{i-1} and f_{i+1}.

We consider each of $g(n) = 1$, 3 and 2 separately.

(1) $g(n) = 1$. In the 8-term translations above, we have $g(8k+2)$ and $g(8k+7)$ *always* taking the value 1, irrespective of the values of f_{i-1}, f_i or f_{i+1}, and there are *no other* values of 1 in this 8-term translation. Thus, $g(n) = 1$ if $n = 8k+2$ or $8k+7$.

(2) $g(n) = 3$. Consider the component $S_3\ f_i\ \overline{S_3^R}\ f_{i+1}$.

(i) If $f_i = 0$, then $g(4i) = g(8k+4)$ where from Theorem 1.5, $i = 2^p(4h+3)$. Since i is odd, $p = 0$. Thus, $8k+4 = 4(4h+3)$ and so $k = 2h+1$ which is odd. It follows that $g(8k+4) = 3$ for k odd.

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(ii) If $f_{i+1} = 0$, then $g(4(i+1)) = g(8k+8)$ where from Theorem 1.5, $i+1 = 2^m(4h+3)$. Since $i+1$ is even, $m > 0$. Thus, $8k+8 = 4(i+1) = 2^{m+2}(4h+3)$. That is, $k+1 = 2^{m-1}(4h+3) = 2^p(4h+3)$ for $p = m-1$, that is, for $p \geq 0$. It follows that $g(8k+8) = 3$ for $k+1 = 2^p(4h+3)$ where $p \geq 0$.

(iii) If $f_i = 1$, then $g(4i+1) = g(8k+5)$ where from Theorem 1.5, $i = 2^p(4h+1)$. Since i is odd, $p = 0$. Thus, $8k+5 = 4(4h+1)+1$ and so $k = 2h$. It follows that $g(8k+5) = 3$ for k even.

(iv) If $f_{i-1} = 1$, then for $i \geq 3$, that is, $k \geq 1$, $g(4(i-1)+1) = g(4i-3) = g(8k+1)$ where from Theorem 1.5 $i-1 = 2^m(4h+1)$. Since $i-1$ is even, $m > 0$. Thus, $8k+1 = 2^{m+2}(4h+1)+1$, that is, $k = 2^{m-1}(4h+1)$. It follows that $g(8k+1) = 3$ if $k = 2^p(4h+1)$ for $p = m-1 \geq 0$.

(3) $g(n) = 2$. Since the only possible terms in G are 1, 2 and 3, all terms not part of conditions (i) and (ii) must be terms having value 2. The result follows. □

The following theorem identifies important internal relationships within G.

THEOREM 2.2 (Relationships in G). *Let $G = g(1)g(2)g(3)\ldots$ be the summed paper-folding sequence of infinite length, then:*

(a) $g(2) = 1,\ g(2^n) = 2$ *for* $n > 1$;
(b) $g(3) = 2,\ g(2^n+1) = 3$ *for* $n > 1$;
(c) $g(2^n - i) = g(i+1)$ *for* $0 \leq i < 2^{n-1} - 1$;
(d) $g(2^n + i) = g(i)$ *for* $1 < i < 2^{n-1}$ *or* $2^{n-1} + 1 < i < 2^n - 1$;
(e) $g(6) = 2,\ g(2^n + 2^{n-1}) = 3$ *for* $n > 2$;
(f) $g(7) = 1,\ g(2^n + 2^{n-1} + 1) = 2$ *for* $n > 2$;
(g) $g(2^n + 2^m) = g(2^m) = 2$ *for* $n > m+1 > 2$;
(h) $g(2^n + 2^m + 2^r) = g(2^m + 2^r)$ *for* $n > m+1 > r+1$;
(i) $g(2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}) = g(2^{k_2} + 2^{k_3} + \cdots + 2^{k_r})$ *for* $k_1 > k_2 > \cdots > k_r$ *and* $r > 2$;
(j) $g(2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}) = g(2^{k_{r-2}} + 2^{k_{r-1}} + 2^{k_r})$ *for* $k_1 > k_2 > \cdots > k_r$ *and* $r > 2$.

PROOF. The assertions follow from Theorem 2.1, with its subcases denoted by items (i) to (xii).

(a) $g(2) = 1$ by item (i); $g(4) = 2$ by item (v); $g(2^n) = 2$ for $n > 2$ by item (viii).
(b) $g(3) = 2$ by item (iv); $g(5) = 3$ by item (xi); $g(2^n+1) = 3$ for $n > 2$ by item (ix).
(c) The first $2^{n-1} - 1$ elements of G_{n+1} are the sums of runs of 1 and 0, and the last $2^{n-1} - 1$ elements are the same sums, but of 0 and 1 and in reverse order. So, $g(2^n - i) = g(i+1)$ for $0 \leq i < 2^{n-1} - 1$.
(d) As $2^n + i = 2^{n+1} - (2^n - i)$ if $1 < i < 2^{n-1}$, by part (c) applied twice,

$$g(2^n + i) = g(2^n - (i-1)) = g(i).$$

If $2^{n-1} + 1 < i < 2^n - 1$, then $i = 2^{n-1} + 2^{n-2} + \cdots + 2^r + j$, where either $r = n-1$ and $1 < j < 2^{r-1} - 1$ or $r < n-1$ and $0 \leq j < 2^{r-1} - 1$. In both cases, by part (c),

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$$g(2^n + i) = g(2^{n+1} - (2^r - j)) = g(2^r - (j - 1)) = g(2^n - (2^r - j)) = g(i).$$

(e) $g(6) = 2$ by item (vii); $g(12) = 3$ by item (x); $g(2^n + 2^{n-1}) = g(3.2^{n-1}) = 3$ for $n > 3$ and $2^{n-1} + 1 < i < 2^n - 1$ by item (xii).

(f) $g(7) = 1$ by item (ii), $g(13) = 2$ by item (vi); $g(2^n + 2^{n-1} + 1) = g(3.2^{n-1} + 1) = 2$ for $n > 3$ by item (iii).

(g) For $p = n - m \geq 2$, $g(2^n + 2^m) = g(2^m(2^p + 1)) = 2$ by item (viii) and $g(2^m) = 2$ if $m = 0, 1$ and $g(2^m) = 2$ if $m \geq 3$ by item (viii). Thus, if $n > m + 1 > 2$, $g(2^n + 2^m) = g(2^m) = 2$, by parts (d) and (a).

(h) For $n > m + 1$ or $r > 0$, by item (viii),

$$g(2^m + 2^r) = g(2^r(2^{m-r} + 1)) = g(2^r(4h_0 + 1)) = 2.$$

For $n > m + 1 > r + 1$, by part (d) and item (viii),

$$\begin{aligned} g(2^n + 2^m + 2^r) &= g(2^r(2^{n-r} + (4h_0 + 1))) = g(2^r(4h_1 + (4h_0 + 1))) \\ &= g(2^r(4(h_1 + h_0) + 1)) = 2. \end{aligned}$$

(i) For $r > 2$ and $k_1 > k_2 > \cdots > k_r$, by part (d),

$$g(2^{k_1} + 2^{k_2} + \cdots + 2^{k_r}) = g(2^{k_2} + 2^{k_3} + \cdots + 2^{k_r}).$$

(j) By repeated use of part (i) of this proof. □

Note that for $n > 1$, $g(2^n) = 2$ and, for $n > 2$, $g(2^n) + 1 = 3$. This follows from observing that the sequence prior to $g(2^{n-1})$ is mirrored to give the sequence after $g(2^{n-1} + 1)$, reflected around 2, 3 in each case.

3. The expression $h(n)$

The following definition is important in developing our main result.

DEFINITION 3.1 (Expression for $h(n)$). For $n > 0$,

$$h(n) = g(1) + g(2) + g(3) + \cdots + g(n).$$

THEOREM 3.2 (Relationships involving $h(n)$). *Assume* $n > 0$.

(a) *If* $n = 4q + 1$, *then* $h(n) = 2n$.
(b) *If* $n = 2^k(4q + 1)$ *and* $k > 0$, *then* $h(n) = 2n - 1$.
(c) *If* $n = 4q + 3$, *then* $h(n) = 2n - 1$.
(d) *If* $n = 2^k(4q + 3)$ *and* $k > 0$, *then* $h(n) = 2n$.

PROOF. The proof is by induction on n. For $n = 1, 5$ and 6, $h(n) = 2n$; and for $n = 2, 3$ and 4, $h(n) = 2n - 1$. So conditions (a) to (d) hold for these minimal values of n. Assume conditions (a) to (d) for values less than some n. Suppose $n = 2^k(4q + 1) > 3$ and let $q = \sum_{i=1}^{r} 2^{q_i}$. Then

$$n = 2^k + \sum_{i=1}^{r} 2^{q_i + k + 2}$$

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and

$$h(n) = \sum_{j=1}^{2^{q_1+k+2}} g(j) + \sum_{j=1}^{n-2^{q_1+k+2}} g(2^{q_1+k+2} + j).$$

If $q_2 < q_1 - 1$, then $n - 2^{q_1+k+2} < 2^{q_1+k+1}$, so by Theorem 2.2(b) and (d),

$$g(2^{q_1+k+2} + 1) = 3 = g(1) + 1$$

and

$$h(n) = \sum_{j=1}^{2^{q_1+k+2}} g(j) + 1 + \sum_{j=1}^{n-2^{q_1+k+2}} g(j) = h(2^{q_1+k+2}) + 1 + h(n - 2^{q_1+k+2}).$$

So by the induction hypothesis, if $k > 0$,

$$h(n) = 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} - 1 = 2n - 1.$$

If $q_2 = q_1 - 1$, by Theorem 2.2(e), (b) and (f),

$$g(2^{q_1+k+2} + 2^{q_2+k+2}) = 3 = g(2^{q_2+k+2}) + 1$$
$$g(2^{q_1+k+2} + 2^{q_2+k+2} + 1) = 2 = g(2^{q_2+k+2} + 1) - 1.$$

So by Theorem 2.2(*d*), for the other values of j,

$$h(n) = \sum_{j=1}^{2^{q_1+k+2}} g(j) + 1 + \sum_{j=1}^{n-2^{q_1+k+2}} g(j) = 2n - 1$$

as before, so we have condition (b). If, instead, $k = 0$, the induction hypothesis gives

$$h(n) = 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} = 2n,$$

then we have condition (a). If $n = 2^k(4q + 3)$, the working is as above, except that the induction hypothesis gives, for $k > 0$,

$$h(n) = 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} = 2n,$$

and for $k = 0$,

$$h(n) = 2^{q_1+k+3} - 1 + 1 + 2n - 2^{q_1+k+3} - 1 = 2n - 1,$$

so we have conditions (d) and (c). □

THEOREM 3.3 (Limits on $h(n)$). *If* $h(n) + 1 \not\equiv 2, 7 \pmod 8$, *then* $g(h(n) + 1) = 2$ *and for no other values of* m *is* $g(m) = 2$.

PROOF. If $n = 4q + 1$, by Theorem 3.2(a), $h(n) + 1 = 2n + 1 = 8q + 3$. So by Theorem 2.1(iv), we have the result.

If $n = 2^k(4q + 1)$ and $k > 0$, by Theorem 3.2(b), $h(n) + 1 = 2n = 2^{k+1}(4q + 1)$ and we have the result by Theorem 2.1(viii) if $k > 1$, and by Theorem 2.1(v) if $k = 1$.

If $n = 4q + 3$, by Theorem 3.2(c), $h(n) + 1 = 2n = 8q + 6$. So by Theorem 2.1(vii), we have the result.

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If $n = 2^k(4q + 3)$, by Theorem 3.2(d), $h(n) + 1 = 2n + 1 = 2^{k+3}q + 2^{k+2} + 2^{k+1} + 1$. So by Theorem 2.2(j), $g(h(n) + 1) = g(2^{k+2} + 2^{k+1} + 1)$ and the result follows from Theorem 2.2(f).

By Theorem 3.2, the value of $g(m)$ when $m = 8q + 7$ or $8q + 2$ is 1, and for $m = 8q + 1, 2^{k+1}(4q + 3)$ with $k > 0$ or $2^{k+1}(4q + 1) + 1$ with $k > 0$, $g(m) = 3$.

Summarising, $g(m) = 2$, where $m = 8q + 3$, $8q + 6$, $2^{k+1}(4q + 1)$ with $k > 0$ and $2^{k+1}(4q + 3) + 1$ with $k > 0$. So all the cases when $g(m) = 2$ are obtained when $m = h(n) + 1$. □

4. Confirmation of Hendriks's conjecture

We are now able to state our main result, namely that the sequence $A = a(1)a(2)a(3)\ldots$ of A088431 is exactly the sequence $G = g(1)g(2)g(3)\ldots$.

THEOREM 4.1 (Confirmation of Hendriks's conjecture). *The sequences A and G are the same, that is, $a(n) = g(n)$.*

PROOF. By Theorem 2.2(c), $g(2^n - i) = g(i)$ for $0 \le i < 2^{n-1} - 1$, and

$$g(2^n - 2^{n-1} + 1) = g(2^{n-1} - 1) = 3 = g(2^{n-1}) + 1.$$

Let G_n^{R+1} denote the reverse of G_n with a 1 added to the new first term. Then

$$G_{n+1} = G_n G_n^{R+1}.$$

Since

$$G_5 = 2122321231232212,$$

G_5 has the subsequences: $321, 123, 1223, 3221, 212, 312, 232$ and 231. Thus, 321 and 213 will appear in G_5^{R+1} and so in G_6. For $n > 3$,

$$g(2^{n-1}) = g(2^{n-1} + 2) = 2 \quad \text{and} \quad g(2^{n-1} + 1) = 3,$$

so the middle sequence of any G_n will be 232. No new sequence of this kind can be generated in any G_n or G_n^{R+1}. Leaving out all the 2s in G, every 1 is followed by a 3 and every 3 is followed by a 1, giving $1, 3, 1, 3, \ldots$. By Theorem 3.3, $g(n)$ has the defining properties of $a(n)$ given at Definition 1.2. □

The referee advised us of the existence of the free software *Walnut,* which is described by Shallit [12]. *Walnut* can decide first-order predicates about automatic sequences, including the paperfolding sequence. The referee confirmed Theorem 4.1 using *Walnut* code. *Walnut* is authored by Hamoon Mousavi, and has been used extensively in confirming features of sequences found in the Online Encyclopedia of Integer Sequences (OEIS).

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5. Summed paperfolding and continued fractions

There is an interesting connection between continued fractions and summed paperfolding. Shallit [11] identifies the continued fraction expansion of $\sum_{k=0}^{u+1} 1/m^{2^k}$ for m an integer greater than 1, once the continued fraction expansion of $\sum_{k=0}^{u} 1/m^{2^k}$ is known. Thus, if

$$\sum_{k=0}^{u} \frac{1}{m^{2^k}} = [a_0; a_1, \ldots, a_n],$$

then

$$\sum_{k=0}^{u+1} \frac{1}{m^{2^k}} = [a_0; a_1, \ldots, a_{n-1}, a_n + 1, a_n - 1, a_n - 1, a_{n-1}, a_{n-2}, \ldots, a_2, a_1].$$

He acknowledges that for $m = 2$, which is the focus of our attention, the expansion contains a number of terms having value 0. This is not problematic as the following equality allows for the conversion of such continued fractions:

$$[a_0; a_1, \ldots, a_i, 0, a_{i+1}, a_{i+2}, \ldots, a_n] = [a_0; a_1, \ldots, a_i + a_{i+1}, a_{i+2}, \ldots, a_n].$$

He also shows that $\sum_{k=0}^{\infty} 1/m^{2^k}$ for $m \geq 2$ is irrational and proves some interesting results relating to the partial denominators of its continued fraction.

Pethö [10] generalises the method found by Shallit [11] to develop a continued fraction expansion for Fredholm numbers of the second kind. The number $\sum_{k=0}^{\infty} 1/2^{2^k}$, upon which the summed paperfolding sequence is based, is an example of a Fredholm number of the second kind.

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