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ON THE CANONICAL MODULE OF A 0-DIMENSIONAL SCHEME

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ABSTRACT. The main topic of this paper is to give characterizations of geometric properties of 0-dimensional subschemes $X \subseteq \mathbb{P}^d$ in terms of the algebraic structure of the canonical module of their projective coordinate ring. We characterize Cayley-Bacharach, (higher order) uniform position, linearly and higher order general position properties, and derive inequalities for the Hilbert functions of such schemes. Finally we relate the structure of the canonical module to properties of the minimal free resolution of X.

Introduction. In the study of 0-dimensional schemes X embedded in some projective space $\mathbb{P}^d$ over an algebraically closed field k one oftentimes considers the homogeneous ideal I_X and the projective coordinate ring $R = k[X_0, \ldots, X_d]/I_X$ of the embedding $X \subseteq \mathbb{P}^d$. Here we want to pursue a "dual" point of view and try to characterize geometric properties of that embedding in terms of algebraic properties of the canonical module ω_R of R.

The canonical module can be described by $\omega_R \cong \underline{\mathrm{Hom}}_{k[x_0]}(R, k[x_0])(-1)$, where $\underline{\mathrm{Hom}}$ means graded homomorphisms and x_0 is the image in R of a linear form which does not pass through any point in the support of X. This module is a finitely generated graded R-module which starts in degree $-\sigma_X$, where $\sigma_X = \max\{n \in \mathbb{Z} : H_X(n) < \deg X\}$. The multiplication maps of this module will be used to describe geometrical properties of X like the Cayley-Bacharach property, uniform position property, general position property, *etc.* The starting point of our investigations is the following theorem whose reduced version was shown in [GKR] and whose nonreduced version can be found in [K1].

THEOREM 1. *The scheme X is locally Gorenstein and a Cayley-Bacharach scheme if and only if there exists an element $\varphi \in (\omega_R)_{-\sigma_X}$ which satisfies* $\mathrm{Ann}_R(\varphi) = (0)$.

Here we say that X is a *Cayley-Bacharach scheme*, if every hypersurface of degree σ_X which contains a subscheme of degree $\deg X - 1$ of X automatically contains X. In Section 2 we shall give a new proof of Theorem 1 which is based on a detailed study of R and ω_R in Section 1. It was already pointed out in [GKR] that structural results like Theorem 1 tend to have implications for the growth behaviour of the Hilbert function of X.

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COROLLARY. *If X is locally Gorenstein and a Cayley-Bacharach scheme, then $H_X(n) + H_X(\sigma_X - n) \leq \deg X$ for all $n \in \mathbb{Z}$.*

If we drop the assumption "X is locally Gorenstein" in Theorem 1, we obtain the following characterization.

THEOREM 2. *The scheme X is a Cayley-Bacharach scheme if and only if the multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ is nondegenerate.*

Equivalently, we could have defined Cayley-Bacharach schemes as the ones for which all subschemes $Y \subseteq X$ of degree $\deg X - 1$ have Hilbert function $H_Y(n) = \min\{H_X(n), \deg X - 1\}$. This approach is used in Section 3 where we discuss higher uniformities. We say that X is *n-uniform*, if every subscheme $Y \subseteq X$ with $\deg X - n \leq \deg Y \leq \deg X$ has the Hilbert function $H_Y(m) = \min\{H_X(m), \deg Y\}$ for $m \in \mathbb{Z}$. Then X is *in uniform position* if and only if X is $(\deg X - 1)$-uniform. We prove the following characterizations.

THEOREM 3. *Let $\Delta_X := \deg X - H_X(\sigma_X)$. Then X is Δ_X-uniform if and only if the multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ is biinjective.*

COROLLARY. *If X is Δ_X-uniform, then $H_X(n) + H_X(\sigma_X - n) \leq \deg X - \Delta_X + 1$ for all $n \in \{0, \ldots, \sigma_X\}$.*

THEOREM 4. *The scheme X is in uniform position if and only if for every $n \in \{0, \ldots, \sigma_X\}$ the multiplication map $R_n \otimes (\omega_R)_{-n} \longrightarrow (\omega_R)_0$ is biinjective.*

Here "biinjective" means that $r \in R_n$, $\varphi \in (\omega_R)_{-n}$, and $r \cdot \varphi = 0$ imply $r = 0$ or $\varphi = 0$. Of course, many intermediate uniformities can be characterized in an analogous way.

Recently another kind of uniformity has received some attention. We say that X is in *linearly general position*, if $\deg(X \cap L) \leq 1 + \dim L$ for every proper linear subspace $L \subset \mathbb{P}^d$. In Section 4 we characterize schemes in linearly general position as follows.

THEOREM 5. *A nondegenerate subscheme $X \subseteq \mathbb{P}^d$ is in linearly general position if and only if the multiplication map $R_1 \otimes (\omega_R)_{-1} \longrightarrow (\omega_R)_0$ is biinjective.*

COROLLARY. *If X is nondegenerate and in linearly general position, then we have $\Delta H_X(n) \geq d$ for all $n \in \{1, \ldots, \sigma_X\}$.*

In particular, $H_X(n) \geq \min\{1 + nd, \deg X\}$ for all $n \geq 0$.

The second inequality of this corollary has been obtained in [EH] with a different method. Theorem 5 is also generalized for schemes in quadratically or higher order general position. We go on to describe the relation of those notions with the classical terminology "X imposes independent conditions on forms of degree n".

Our last section deals with characterizations of properties of the minimal projective resolution of R as a $k[X_0, \ldots, X_d]$-module using the multiplication maps of ω_R. For example, we show the following theorem.

THEOREM 6. *If the projective resolution of X is almost linear, then the multiplication map $R_{\sigma_X-\alpha_X+2}\otimes(\omega_R)_{-\sigma_X}\longrightarrow(\omega_R)_{-\alpha_X+2}$ is injective.*

Moreover, we give characterizations of schemes with almost linear or higher order resolutions in terms of the algebraic structure of ω_R.

1. **The projective coordinate ring and its canonical module.** This section consists of a series of definitions and easy lemmas which will prove useful later on. The canonical module is defined and some of its most elementary properties are noted.

First of all we want to fix the notations which will be used throughout the paper. We work over an algebraically closed field k of arbitrary characteristic. By $\mathbb{P}^d$ we denote the d-dimensional projective space over k. We want to study 0-dimensional subschemes X of $\mathbb{P}^d$. Since we always consider X together with a fixed embedding $X\subseteq\mathbb{P}^d$, we shall say "X has property $\mathcal{P}$" when we really mean "the embedding $X\subseteq\mathbb{P}^d$ has property $\mathcal{P}$".

The coordinates $\{X_0,\ldots,X_d\}$ of $\mathbb{P}^d$ are always chosen such that no point of the support of X lies on the hyperplane $\mathcal{V}(X_0)$. By I_X we denote the homogeneous saturated ideal of X in $k[X_0,\ldots,X_d]$. The projective (or homogeneous) coordinate ring of X is then given by $R:=k[X_0,\ldots,X_d]/I_X$. It is a standard graded k-algebra $R=\bigoplus_{n\geq 0}R_n$, *i.e.* we have $R_0=k$, $\dim_k R_1$ is finite, and $R=k[R_1]$. We let $\mathfrak{m}:=\bigoplus_{n>0}R_n$ be the homogeneous maximal ideal of R. The image of X_i in R is denoted by x_i for $i=0,\ldots,d$. By the choice of coordinates, $x_0\in R_1$ is not a zero divisor of R. Hence R is a 1-dimensional Cohen-Macaulay ring and $\bar{R}:=R/(x_0)$ is a 0-dimensional ring.

The Hilbert function of X is denoted by $H_X\colon\mathbb{Z}\longrightarrow\mathbb{N}$ $(n\longmapsto\dim_k R_n)$. We have $H_X(n)=0$ for $n<0$, $H_X(0)=1$, and $H_X(n)=\deg X$ for $n\gg 0$. The invariant

$$\sigma_X:=\max\{n\in\mathbb{Z}:H_X(n)<\deg X\}$$

will play an important rôle throughout this paper. By $\Delta H_X\colon\mathbb{Z}\longrightarrow\mathbb{N}$ $\big(n\longmapsto H_X(n)-H_X(n-1)\big)$ we denote the first difference function of H_X. Because of the exact sequence of graded R-modules

$$0\longrightarrow R(-1)\xrightarrow{x_0}R\longrightarrow\bar{R}\longrightarrow 0$$

we have $\Delta H_X(n)=\dim_k\bar{R}_n$ for all $n\in\mathbb{Z}$. Here $R(-1)$ denotes the shift of R, *i.e.* the graded R-module with $R(-1)_n=R_{n-1}$ for all $n\in\mathbb{Z}$. Since $\bar{R}$ is also a standard graded k-algebra, we have $\Delta H_X(n)\neq 0$ if and only if $n\in\{0,\ldots,\sigma_X+1\}$, and therefore $H_X(0)<\cdots<H_X(\sigma_X+1)=\deg X$. The number $\Delta_X:=H_X(\sigma_X+1)-H_X(\sigma_X)$ denotes the last nonzero difference of H_X. It is clear that $R_n=(x_0)_n$ for every $n>\sigma_X+1$, hence $R_n=x_0^{n-\sigma_X-1}R_{\sigma_X+1}$ for every $n\geq\sigma_X+1$.

Now we shall examine subschemes $Y\subseteq X$ of degree $\deg X-1$. Let $I_{Y/X}$ be the ideal of Y in R, and let $\alpha_{Y/X}:=\min\{n\in\mathbb{N}:(I_{Y/X})_n\neq 0\}$ be its initial degree. Then $I_{Y/X}$ is a saturated ideal of R and $\alpha_{Y/X}$ is well-defined, because $\dim_k(R/I_{Y/X})_n=H_Y(n)=\deg Y<\deg X=\dim_k R_n$ for $n\gg 0$.

The projective coordinate ring of Y is $S=R/I_{Y/X}$. As $Y\subseteq X$, the element x_0 is not a zero divisor of S. Consequently, $\bar{S}:=S/(x_0)$ has Hilbert function $\dim_k\bar{S}_n=\Delta H_Y(n)=$

$H_Y(n) - H_Y(n-1)$ for $n \in \mathbb{Z}$. From $Y \subseteq X$ and $\deg Y = \deg X - 1$ we get $\Delta H_Y(n) \leq \Delta H_X(n)$ for $n \in \mathbb{Z}$ and $\sum_{n\in\mathbb{Z}} \Delta H_Y(n) = \sum_{n\in\mathbb{Z}} \Delta H_X(n) - 1$. By definition of $\alpha_{Y/X}$, we have $\Delta H_Y(\alpha_{Y/X}) < \Delta H_X(\alpha_{Y/X})$.

Altogether, the Hilbert function of Y is given by

$$H_Y(n) = \begin{cases} H_X(n) & \text{for } n < \alpha_{Y/X}, \\ H_X(n) - 1 & \text{for } n \geq \alpha_{Y/X}. \end{cases}$$

We shall call $\alpha_{Y/X}$ the degree of Y in X. From the above discussion it follows that $\alpha_{Y/X} \leq \sigma_X + 1$.

A nonzero element $f_Y^* \in (I_{Y/X})_{\alpha_{Y/X}}$ is called a *minimal separator* of Y. Since x_0 is not a zero divisor on R, the element $x_0^n f_Y^*$ is a k-basis of $(I_{Y/X})_{\alpha_{Y/X}+n}$ for each $n \geq 0$. A nonzero element $f_Y \in k \cdot x_0^{\sigma_X+1-\alpha_{Y/X}} f_Y^*$ is called a *separator* of Y. For any $r \in R_n$, $n \geq 0$, we get $rf_Y \in (I_{Y/X})_{\sigma_X+1+n}$, hence $rf_Y = \lambda x_0^n f_Y$ for some $\lambda \in k$.

Next we want to derive a local description of separators which will be useful later on. The coordinate ring Γ of X in the affine space $\mathbb{A}^d \cong D_+(X_0)$ is $\Gamma = R/(x_0 - 1)$. The canonical epimorphism $R \twoheadrightarrow \Gamma$ is given by dehomogenization. If we equip Γ with the ascending filtration $\mathcal{F}$ induced by the degree filtration of $k[X_1,\ldots,X_d]$ via $\Gamma \cong k[X_1,\ldots,X_d]/\bar{I}_X$, where $\bar{I}_X = I_X/I_X \cap (X_0 - 1)$, we can form the homogenization $f^* = x_0^{\operatorname{ord}_{\mathcal{F}} f} f(x_1/x_0,\ldots,x_d/x_0)$ of any element $f \in \Gamma$. For details on those procedures the reader may consult [KK].

LEMMA 1.1. *If we restrict dehomogenization to elements of degree $\sigma_X + 1$, we obtain an isomorphism $R_{\sigma_X+1} \xrightarrow{\sim} \Gamma$.*

PROOF. Because of $\dim_k R_{\sigma_X+1} = \deg X = \dim_k \Gamma$, it suffices to show that every element of Γ is the dehomogenization of an element of R_{σ_X+1}. For $f \in \Gamma$ we have $\operatorname{ord}_{\mathcal{F}} f \leq \sigma_X + 1$, because no element of $\operatorname{gr}_{\mathcal{F}}(\Gamma) \cong \bar{R}$ has a degree larger than that. Now $x_0^{\sigma_X+1-\operatorname{ord}_{\mathcal{F}} f} f^* \in R_{\sigma_X+1}$ has dehomogenization f. ∎

Combining the isomorphism of the lemma with the canonical isomorphism $\Gamma \xrightarrow{\sim} \prod_{P\in X} \mathcal{O}_{X,P}$, we obtain an isomorphism $\imath\colon R_{\sigma_X+1} \xrightarrow{\sim} \prod_{P\in X} \mathcal{O}_{X,P}$ which maps each homogeneous element $r \in R_{\sigma_X+1}$ to the tuple $(r_P)_{P\in X}$ of its germs at the points of X. In particular, we have $\imath(x_0^{\sigma_X+1}) = (1)_{P\in X}$.

The ideal of Y in $\prod_{P\in X} \mathcal{O}_{X,P}$ is of the form $k \cdot (0,\ldots,0,s_P,0,\ldots,0)$, where $P \in X$ and $s_P \in \mathfrak{S}(\mathcal{O}_{X,P})$ is an element of the socle $\mathfrak{S}(\mathcal{O}_{X,P}) = \{r_P \in \mathcal{O}_{X,P} : \mathfrak{m}_{X,P} \cdot r_P = 0\}$ of $\mathcal{O}_{X,P}$. Clearly, $\imath(f_Y) = (0,\ldots,0,s_P,0,\ldots,0)$ for some separator $f_Y \in R_{\sigma_X+1}$ of Y, and the image of any other separator of Y is a nonzero scalar multiple of this element. In particular, for two subschemes $Y, Y' \subseteq X$ with $\deg Y = \deg Y' = \deg X - 1$ we obtain

$$f_Y \cdot f_{Y'} \in \begin{cases} k \cdot x_0^{\sigma_X+1} f_Y & \text{if } \dim_k \mathcal{O}_{X,P} = 1 \text{ and } Y = Y' = X \setminus \{P\}, \\ (0) & \text{otherwise.} \end{cases}$$

When dealing with ideals of R defining subschemes of X, it is necessary to keep the following lemma in mind.

LEMMA 1.2. *If $J \subseteq R$ is a homogeneous ideal, then its saturation $J^{\text{sat}} := \{r \in R : \mathfrak{m}^n r \subseteq J \text{ for some } n \geq 0\}$ is already given by*

$$J^{\text{sat}} = \{r \in R : x_0^n r \in J \text{ for some } n \geq 0\}.$$

In particular, J is saturated if and only if $x_0 r \in J$ implies $r \in J$.

PROOF. As noted earlier, $R_n = x_0^{n-\sigma_X-1} R_{\sigma_X+1}$ for $n \geq \sigma_X + 1$. Thus $x_0^n r \in J$ implies $\mathfrak{m}^{\sigma_X+1+n} r = \mathfrak{m}^{\sigma_X+1} x_0^n r \subseteq J$, and the claims of the lemma follow. ∎

In order to define the canonical module of R, we need some basic properties of the category of graded R-modules. Its homomorphisms are homogeneous R-linear maps $\varphi\colon M \longrightarrow N$. We also let $\underline{\operatorname{Hom}}_R(M,N)$ be the graded R-module whose homogeneous components are the sets of homogeneous R-linear maps $\varphi\colon M \longrightarrow N(n)$ for $n \in \mathbb{Z}$. The functor $H^0_{\mathfrak{m}}(M) := \{x \in M : \mathfrak{m}^n x = 0 \text{ for some } n \geq 0\}$ is a left-exact covariant functor on that category. We can form its right derived functors $H^i_{\mathfrak{m}}(-)$, $i \geq 0$. The modules $H^i_{\mathfrak{m}}(M)$ are graded R-modules whose underlying R-modules agree with the usual local cohomology modules. We equip k with the trivial grading and let $M^* := \underline{\operatorname{Hom}}_k(M,k)$ for every graded R-module M.

DEFINITION. The graded R-module $\omega_R := H^1_{\mathfrak{m}}(R)^*$ is called the *canonical module* of R.

The following properties of ω_R are proven in [GW].

LEMMA 1.3. *a) The graded R-module ω_R is finitely generated.*

b) There is a canonical isomorphism of graded R-modules

$$\omega_R \cong \underline{\operatorname{Hom}}_{k[x_0]}(R, k[x_0])(-1).$$

In particular, x_0 is not a zero divisor on ω_R.

c) There is an exact sequence of graded R-modules

$$0 \longrightarrow R \longrightarrow \Gamma_* \mathcal{O}_X \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow 0.$$

In particular, the Hilbert function of ω_R satisfies

$$H_{\omega_R}(n) = \deg X - H_X(-n) \quad \textit{for all } n \in \mathbb{Z}.$$

d) If $Y \subseteq X$ is a subscheme and $S = R/I_{Y/X}$ its projective coordinate ring, then there is a canonical isomorphism of graded R-modules

$$\omega_S \cong \{\varphi \in \omega_R : I_{Y/X} \cdot \varphi = 0\}.$$

Notice that Lemma 1.3.c implies $-\sigma_X = \min\{n \in \mathbb{Z} : (\omega_R)_n \neq 0\}$ and

$$H_{\omega_R}(-\sigma_X) = \Delta_X < \cdots < H_{\omega_R}(0) = \deg X - 1 < H_{\omega_R}(1) = \deg X.$$

From now on, we shall always use Lemma 1.3.b to think of elements of ω_R as $k[x_0]$-linear forms on R, and we shall use 1.3.d to identify ω_S with a submodule of ω_R. Comparing Hilbert functions then yields $\sigma_Y \leq \sigma_X$ for every subscheme $Y \subseteq X$.

Our final three lemmas of this section will help us deal with those linear forms.

LEMMA 1.4. *For $\varphi \in \omega_R$, the following conditions are equivalent.*

a) $\varphi = 0$

b) $\varphi\,|_{R_{\sigma_X+1}}: R_{\sigma_X+1} \longrightarrow k[x_0]$ *is the zero linear transformation.*

PROOF. It suffices to show that b) implies a). If $r \in R_n$, $n \leq \sigma_X$, then $0 = \varphi(x_0^{\sigma_X+1-n}r) = x_0^{\sigma_X+1-n}\varphi(r)$, so $\varphi(r) = 0$. And if $r \in R_n$, $n \geq \sigma_X + 1$, then we can write $r = x_0^{n-\sigma_X-1}r'$ with $r' \in R_{\sigma_X+1}$, and $\varphi(r) = x_0^{n-\sigma_X-1}\varphi(r') = 0$ again. ∎

LEMMA 1.5. *There is a* 1–1 *correspondence between elements φ of $(\omega_R)_{-\sigma_X}$ and k-linear maps $\bar\varphi: R_{\sigma_X+1} \longrightarrow k$ with $\bar\varphi(x_0R_{\sigma_X}) = 0$.*

PROOF. If $\varphi \in (\omega_R)_{-\sigma_X} \cong \mathrm{Hom}_{k[x_0]}(R, k[x_0])_{-\sigma_X-1}$, then its restriction $\bar\varphi := \varphi\,|_{R_{\sigma_X+1}}: R_{\sigma_X+1} \longrightarrow k$ vanishes on $x_0R_{\sigma_X}$, because we have $\varphi(x_0R_{\sigma_X}) = x_0\varphi(R_{\sigma_X}) \subseteq x_0k[x_0]_{-1} = (0)$.

Conversely, if $\bar\varphi: R_{\sigma_X+1} \longrightarrow k$ is a k-linear map, we define a homogeneous k-linear map $\varphi: R \longrightarrow k[x_0]$ of degree $-\sigma_X - 1$ as follows:

1) For $r \in R_n$, $n \leq \sigma_X$, we let $\varphi(r) := 0$.

2) For $r \in R_n$, $n \geq \sigma_X + 1$, we write $r = x_0^{n-\sigma_X-1}r'$ with $r' \in R_{\sigma_X+1}$ and let $\varphi(r) := x_0^{n-\sigma_X-1}\bar\varphi(r')$.

Now $\bar\varphi(x_0R_{\sigma_X}) = 0$ is exactly the right condition to make φ even $k[x_0]$-linear, so that φ defines an element of $(\omega_R)_{-\sigma_X}$. Obviously those two constructions are inverses of each other and define the desired bijection. ∎

LEMMA 1.6. *For $\varphi \in \omega_R$ and $Y \subseteq X$ with* $\deg Y = \deg X - 1$, *the following conditions are equivalent.*

a) $f_Y \cdot \varphi = 0$

b) $\varphi(f_Y) = 0$.

PROOF. If $\varphi(f_Y) = 0$ and $r \in R_n$, $n \geq 0$, then $f_Y\varphi(r) = \varphi(rf_Y) \in k \cdot x_0^n\varphi(f_Y) = (0)$, because $rf_Y \in k \cdot x_0^n f_Y$, as noted earlier. ∎

2. **Cayley-Bacharach schemes.** In this section we want to study the canonical module of 0-dimensional schemes having the Cayley-Bacharach property with respect to hypersurfaces of degree σ_X, the maximum possible degree (*cf.* [K2]). In particular, Theorems 1 and 2 of the introduction will follow from Theorems 2.4 and 2.6, respectively.

DEFINITION. A 0-dimensional scheme $X \subseteq \mathbb{P}^d$ is called a *Cayley-Bacharach scheme*, if every hypersurface of degree σ_X which contains a subscheme $Y \subseteq X$ of degree $\deg Y = \deg X - 1$ automatically contains X.

PROPOSITION 2.1. *The following conditions are equivalent.*

a) X is a Cayley-Bacharach scheme.

b) If $Y \subseteq X$ and $\deg Y = \deg X - 1$, *then* $\alpha_{Y/X} = \sigma_X + 1$.

c) Each subscheme $Y \subseteq X$ of degree $\deg X - 1$ *has Hilbert function* $H_Y(n) = \min\{H_X(n), \deg X - 1\}$ *for all* $n \in \mathbb{Z}$.

PROOF. a) $\Rightarrow$ b): By definition, $\alpha_{Y/X}$ is the least degree of a hypersurface containing Y, but not X. Our assumption implies $\alpha_{Y/X} \geq \sigma_X + 1$. The other inequality holds always.

b) $\Rightarrow$ c): This follows from the description of H_Y in Section 1.

c) $\Rightarrow$ a): Since $H_X(\sigma_X) \leq \deg X - 1$, we have $H_Y(\sigma_X) = H_X(\sigma_X)$, and this means that every hypersurface of degree σ_X which contains Y also contains X. ∎

In order to be able to characterize Cayley-Bacharach schemes in terms of their canonical modules, we need two lemmas.

LEMMA 2.2. *a) Let $n \geq 0$ and $r \in R_n \setminus \{0\}$. Then there exists an element $r' \in R_{\sigma_X+1}$, a subscheme $Y \subseteq X$ with* $\deg Y = \deg X - 1$, *and a separator $f_Y \in R_{\sigma_X+1}$ of Y such that $rr' = x_0^n f_Y$.*

b) If $Y \subseteq X$ is a subscheme and $\deg Y \leq n \leq \deg X$, *then there exists a subscheme Z of X such that* $\deg Z = n$ *and $Y \subseteq Z \subseteq X$.*

PROOF. a): Since $r \neq 0$, there is a point $P \in X$ such that $(r_Q)_{Q\in X} := \imath(x_0^{\sigma_X+1-n}r)$ satisfies $r_P \neq 0$. Then we can find an element $r'_P \in \mathcal{O}_{X,P}$ such that $s_P := r_P r'_P \in \mathfrak{S}(\mathcal{O}_{X,P})$ is in the socle of $\mathcal{O}_{X,P}$. Now we use $r' := \imath^{-1}\big((0,\ldots,0,r'_P,0,\ldots,0)\big)$ and $f_Y := \imath^{-1}\big((0,\ldots,0,s_P,0\ldots,0)\big)$ and obtain the desired equality $rr' = x_0^n f_Y$.

b): By induction, it suffices to do the case $n = \deg X - 1$. Let $I_{Y/X} \subseteq R$ be the ideal of Y, and let $r \in (I_{Y/X})_m \setminus \{0\}$ for some $m \geq 0$. Using a) we find a subscheme $Z \subseteq X$ with $\deg Z = \deg X - 1$ and a separator $f_Z \in R_{\sigma_X+1}$ of Z such that $rr' = x_0^m f_Z$ for some $r' \in R_{\sigma_X+1}$. Hence $I_{Z/X} = (f_Z)^{\mathrm{sat}} \subseteq (r)^{\mathrm{sat}} \subseteq I_{Y/X}$, and thus $Y \subseteq Z \subseteq X$. ∎

LEMMA 2.3. *A homogeneous element $\varphi \in \omega_R$ satisfies* $\mathrm{Ann}_R(\varphi) = (0)$ *if and only if $\varphi(f_Y) \neq 0$ for any separator of a subscheme $Y \subseteq X$ of degree* $\deg Y = \deg X - 1$.

PROOF. If $\mathrm{Ann}_R(\varphi) = (0)$, then $f_Y \cdot \varphi \neq 0$, and thus $\varphi(f_Y) \neq 0$ by Lemma 1.6. Conversely, if $r \cdot \varphi = 0$ for some $r \in R_n \setminus \{0\}$, $n \geq 0$, then we can use Lemma 2.2.a to conclude that $x_0^n f_Y \varphi = 0$ for some separator f_Y of a subscheme $Y \subseteq X$ of degree $\deg X - 1$. By Lemma 1.3, we then have $f_Y \cdot \varphi = 0$ and $\varphi(f_Y) = 0$, a contradiction. ∎

THEOREM 2.4. *The following conditions are equivalent.*

a) X is a Cayley-Bacharach scheme and locally Gorenstein.

b) There exists an element $\varphi \in (\omega_R)_{-\sigma_X}$ such that $\varphi(f_Y) \neq 0$ for all separators $f_Y \in R_{\sigma_X+1}$ of subschemes $Y \subseteq X$ of degree $\deg Y = \deg X - 1$.

c) There exists an element $\varphi \in (\omega_R)_{-\sigma_X}$ such that $\mathrm{Ann}_R(\varphi) = (0)$.

d) A generic element $\varphi \in (\omega_R)_{-\sigma_X}$ satisfies $\mathrm{Ann}_R(\varphi) = (0)$.

PROOF. a) $\Rightarrow$ d): Since X is locally Gorenstein, there is for each point $P \in X$ a unique subscheme $Y \subseteq X$ with $\deg Y = \deg X - 1$ corresponding to a socle element of $\mathcal{O}_{X,P}$. Since X is a Cayley-Bacharach scheme, a separator $f_Y \in R_{\sigma_X+1}$ satisfies $f_Y \notin x_0 R_{\sigma_X}$. Thus a generic k-linear map $\bar{\varphi}\colon R_{\sigma_X+1} \longrightarrow k$ with $\bar{\varphi}(x_0 R_{\sigma_X}) = 0$ satisfies $\bar{\varphi}(f_Y) \neq 0$ for those finitely many subschemes $Y \subseteq X$. Now Lemma 1.5 and Lemma 2.3 together imply the claim.

"d) $\Rightarrow$ c)" is clear and "b)$\Leftrightarrow$c)" follows from Lemma 2.3.

b) $\Rightarrow$ a): From $R(\sigma_X) \cong R\varphi \subseteq \omega_R$ and $\dim_k R_n = \deg X = \dim_k(\omega_R)_n$ for $n \gg 0$ we conclude that $\omega_{X,P} \cong \mathcal{O}_{X,P}$ for all $P \in X$, *i.e.* that X is locally Gorenstein. Since $\varphi(f_Y) \neq 0$ and $\varphi(x_0 R_{\sigma_X}) \subseteq k[x_0]_{-1} = (0)$, we must have $f_Y \notin x_0 R_{\sigma_X}$, and therefore $\alpha_{Y/X} = \sigma_X + 1$, for all subschemes $Y \subseteq X$ of degree $\deg X - 1$. Hence X is a Cayley-Bacharach scheme by Proposition 2.1. ∎

The corollary of Theorem 1 stated in the introduction follows now by simply comparing Hilbert functions for the inclusion $R(\sigma_X) \cong R\varphi \subseteq \omega_R$. If we have equality here, ω_R is a graded free R-module of rank one. It is well-known that this is the case if and only if R is a Gorenstein ring or, in other words, if X is arithmetically Gorenstein.

COROLLARY 2.5. *a) If X is a Cayley-Bacharach scheme and $\Delta_X = 1$, then X is locally Gorenstein.*

b) A 0-dimensional scheme X is arithmetically Gorenstein if and only if X is a Cayley-Bacharach scheme and $H_X(n) + H_X(\sigma_X - n) = \deg X$ for all $n \in \mathbb{Z}$.

PROOF. In view of what we explained above, it only remains to show "a)". Suppose there is a point $P \in X$ such that $\dim_k \mathfrak{S}(\mathcal{O}_{X,P}) \geq 2$. Let $f_Y, f_{Y'} \in R_{\sigma_X+1}$ be separators of subschemes $Y, Y' \subseteq X$ with $\deg Y = \deg Y' = \deg X - 1$ such that $\imath(f_Y)$ and $\imath(f_{Y'})$ are k-linearly independent elements of $\mathfrak{S}(\mathcal{O}_{X,P})$. Since $\Delta_X = \dim_k \bar{R}_{\sigma_X+1} = 1$, there are scalars $\lambda, \lambda' \in k$ such that $\lambda f_Y + \lambda' f_{Y'} \in x_0 R_{\sigma_X}$. But then the subscheme $Y'' = \mathcal{V}(\lambda f_Y + \lambda' f_{Y'}) \subseteq X$ satisfies $\deg Y'' = \deg X - 1$ and $\alpha_{Y''/X} \leq \sigma_X$, a contradiction. ∎

EXAMPLE. Let $X \subseteq \mathbb{P}^2$ be concentrated at two points P_1 and P_2 with $\mathcal{O}_{X,P_i} \cong \mathcal{O}_{\mathbb{P}^2,P_i} / \mathfrak{m}^2_{\mathbb{P}^2,P_i}$ for $i = 1, 2$. Then X is not locally Gorenstein, and from its Hilbert function H_X: $13566\cdots$ we read off $\Delta_X = 1$. Therefore we know immediately that X is not a Cayley-Bacharach scheme.

Our next theorem contains Theorem 2 of the introduction.

THEOREM 2.6. *The following conditions are equivalent.*

a) X is a Cayley-Bacharach scheme.

b) The multiplication map $R_{\sigma_X+1} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_1$ is nondegenerate.

c) The multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ is nondegenerate.

d) For all $m, n \geq 0$, the multiplication map $R_m \otimes (\omega_R)_{-\sigma_X+n} \longrightarrow (\omega_R)_{-\sigma_X+m+n}$ is nondegenerate.

PROOF. If $\varphi \in (\omega_R)_{-\sigma_X+n}$ and $R_m\varphi = 0$ for some $m, n \geq 0$, then $x_0^m \varphi = 0$, and therefore $\varphi = 0$. Thus the multiplication maps appearing in the theorem are always nondegenerate in the second argument.

a) $\Rightarrow$ b): Suppose that $r \in R_{\sigma_X+1} \setminus \{0\}$ and $r(\omega_R)_{-\sigma_X} = 0$. Then Lemma 2.2.a shows that $f_Y(\omega_R)_{-\sigma_X} = 0$ for some separator $f_Y \in R_{\sigma_X+1}$ of a subscheme $Y \subseteq X$ of degree $\deg X - 1$. Let $\bar{f}_Y$ be the image of f_Y in $\bar{R}_{\sigma_X+1}$. Since X is a Cayley-Bacharach scheme, we have $\bar{f}_Y \neq 0$. Choose a complement V of $k \cdot \bar{f}_Y$ in $\bar{R}_{\sigma_X+1}$, and let $\pi\colon \bar{R}_{\sigma_X+1} \longrightarrow k$ be the projection to $\bar{f}_Y$ along V. Then π lifts to a k-linear map $\bar{\varphi}\colon R_{\sigma_X+1} \longrightarrow k$ with $\bar{\varphi}(x_0 R_{\sigma_X}) = 0$

and $\bar{\varphi}(f_Y) = 1$. Using Lemma 1.5 we obtain an element $\varphi \in (\omega_R)_{-\sigma_X}$ such that $\varphi(f_Y) = 1$. In view of Lemma 1.6 this is a contradiction.

b) $\Rightarrow$ d): Let $r \in R_m$ such that $r(\omega_R)_{-\sigma_X+n} = 0$. In particular, this implies $rx_0^n(\omega_R)_{-\sigma_X} = 0$, and therefore $r(\omega_R)_{-\sigma_X} = 0$. If $m \leq \sigma_X+1$, we conclude from $rx_0^{\sigma_X+1-m} \in R_{\sigma_X+1}$ and $rx_0^{\sigma_X+1-m}(\omega_R)_{-\sigma_X} = 0$ that $r = 0$. If $m > \sigma_X + 1$, we write $r = x_0^{m-\sigma_X-1}r'$ with $r' \in R_{\sigma_X+1}$ and conclude from $r'(\omega_R)_{-\sigma_X} = 0$ that $r = 0$.

Since "d) $\Rightarrow$ c)" is clear, it remains to show "c) $\Rightarrow$ a)". Suppose that X is not a Cayley-Bacharach scheme, *i.e.* that there is a subscheme $Y \subseteq X$ of degree $\deg X - 1$ such that $f_Y \in x_0R_{\sigma_X}$ for some separator f_Y of Y. Write $f_Y = x_0g_Y$ with $g_Y \in R_{\sigma_X}$. For $\varphi \in (\omega_R)_{-\sigma_X}$ we have $\varphi(f_Y) = x_0\varphi(g_Y) \in x_0k[x_0]_{-1} = (0)$, hence $f_Y\varphi = 0$ by Lemma 1.6. Thus $x_0g_Y(\omega_R)_{-\sigma_X} = f_Y(\omega_R)_{-\sigma_X} = 0$, implying $g_Y(\omega_R)_{-\sigma_X} = 0$, a contradiction. ∎

COROLLARY 2.7. *If X is a Cayley-Bacharach scheme, then*

$$H_X(n) \leq \Delta_X\big(\deg X - H_X(\sigma_X - n)\big) \quad \textit{for all } n \geq 0.$$

PROOF. Since $R_n \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+n}$ is nondegenerate for every $n \geq 0$, we get injections $R_n \hookrightarrow \operatorname{Hom}_k\big((\omega_R)_{-\sigma_X}, (\omega_R)_{-\sigma_X+n}\big)$. Now compare dimensions. ∎

Of course, if $\Delta_X = 1$, the inequalities of Corollary 2.7 and the corollary of Theorem 1 are equivalent. We do not know whether the stronger inequalities of the latter corollary hold for arbitrary Cayley-Bacharach schemes. The following example shows that, at any rate, Theorems 2.4 and 2.6 are not equivalent, if $\Delta_X > 1$.

EXAMPLE. Let $X \subseteq \mathbb{P}^2$ be the subscheme defined by $I_X = (X_1^2, X_1X_2, X_2^2)$. It is concentrated at the point $P = (1 : 0 : 0)$ and has Hilbert function H_X: $133\cdots$ and $\sigma_X = 0$. Since $\mathcal{O}_{X,P} \cong \mathcal{O}_{\mathbb{P}^2,P}/\mathfrak{m}^2_{\mathbb{P}^2,P}$, we see that X is not locally Gorenstein. In particular, X has infinitely many subschemes of degree two. But each subscheme $Y \subseteq X$ of degree two has Hilbert function H_Y: $122\cdots$, so X is a Cayley-Bacharach scheme.

Let us also check what happens in the canonical module in this example. For an element $\varphi \in (\omega_R)_0$, let $a := \varphi(x_1)$ and $b := \varphi(x_2)$. We claim that $(bx_1 - ax_2)\varphi = 0$. This follows from Lemma 1.4, since $\big((bx_1 - ax_2)\varphi\big)(x_0) = x_0b\varphi(x_1) - x_0a\varphi(x_2) = 0$, and $\big((bx_1 - ax_2)\varphi\big)(x_i) = b\varphi(x_1x_i) - a\varphi(x_2x_i) = (b - a)\varphi(0) = 0$ for $i = 1, 2$. Hence no element $\varphi \in (\omega_R)_0$ has $\operatorname{Ann}_R(\varphi) = 0$, while the multiplication map $R_0 \otimes (\omega_R)_0 \longrightarrow (\omega_R)_0$ is clearly nondegenerate.

The final proposition of this section applies Theorem 2.6 to give a generalization of the well-known fact that if one removes a point from a 0-dimensional reduced complete intersection, one is still left with a Cayley-Bacharach scheme.

PROPOSITION 2.8. *Suppose that X is a Cayley-Bacharach scheme with $\Delta_X = 1$, and $Y \subseteq X$ is a subscheme of degree $\deg Y = \deg X - 1$. Then Y is a Cayley-Bacharach scheme.*

In particular, if X is arithmetically Gorenstein, every subscheme of degree $\deg X - 1$ *is a Cayley-Bacharach scheme.*

PROOF. Let $f_Y \in R_{\sigma_X+1}$ be a separator of Y. As explained in Section 1, the projective coordinate ring of Y is $S = R/(f_Y)$, and its canonical module is given by $\omega_S \cong \{\psi \in \omega_R : f_Y\psi = 0\}$. Since $\Delta_X = 1$, the Hilbert function of Y is $H_Y(n) = H_X(n)$ for $n \leq \sigma_X$ and $H_Y(n) = \deg Y$ for $n \geq \sigma_X$. Thus $\sigma_Y = \sigma_X - 1$ and $S_{\sigma_Y} \cong R_{\sigma_X-1}$.

Suppose that $r \in R_{\sigma_X-1}$ annihilates $(\omega_S)_{-\sigma_X+1}$. By Corollary 2.5.a and Theorem 2.4, there is an element $\varphi \in (\omega_R)_{-\sigma_X}$ such that $\mathrm{Ann}_R(\varphi) = (0)$. So, $(\omega_S)_{-\sigma_X+1}$ contains $\{\ell\varphi : \ell \in R_1, f_Y\ell\varphi = 0\}$, and therefore $r\ell = 0$ for all $\ell \in R_1$ such that $f_Y\ell = 0$.

We let $\imath(x_0^2 r) =: (r_Q)_{Q\in X}$, $\imath(x_0^{\sigma_X}\ell) =: (\ell_Q)_{Q\in X}$, and $\imath(f_Y) =: (0,\ldots,0,s_P,0,\ldots,0)$ with $P \in X$, $s_P \in \mathfrak{S}(\mathcal{O}_{X,P})$. If $\ell \in R_1$ defines a hyperplane $\mathcal{V}(\ell)$ which passes through P, but through no other point of X, we have $f_Y\ell = \imath^{-1}\big((0,\ldots,0,s_P\ell_P,0,\ldots,0)\big) = 0$. Thus $r\ell = 0$ and $r_P\ell_P = 0$ for such ℓ. Since the various germs ℓ_P generate the maximal ideal of $\mathcal{O}_{X,P}$, it follows that r_P is an element of the socle of $\mathcal{O}_{X,P}$. Then the hypothesis that X is locally Gorenstein implies that r_P is a multiple of s_P.

On the other hand, for $Q \in X$, $Q \neq P$, the germ ℓ_Q is a unit of $\mathcal{O}_{X,Q}$, and $r\ell = 0$ implies $r_Q = 0$. Altogether this shows that $x_0^2 r$ is a scalar multiple of f_Y. But $f_Y \notin x_0 R_{\sigma_X}$, because X is a Cayley-Bacharach scheme, so the only possibility left is $r = 0$. Hence the multiplication $S_{\sigma_Y} \otimes (\omega_S)_{-\sigma_Y} \longrightarrow (\omega_S)_0$ is nondegenerate, and therefore Y is a Cayley-Bacharach scheme by Theorem 2.6. ∎

EXAMPLE. If $X \subseteq \mathbb{P}^2$ is reduced and concentrated at the points $(1:0:0)$, $(1:1:0)$, $(1:2:0)$, $(1:0:1)$, $(1:1:1)$, $(1:0:2)$, $(1:2:2)$, then its Hilbert function is $H_X : 13677\cdots$, and X is a Cayley-Bacharach scheme because of [GKR], 4.9. From the proposition it follows that any subset of six points of X is a Cayley-Bacharach scheme, too.

3. **Higher uniformities.** In Proposition 2.1 we have characterized Cayley-Bacharach schemes X as those 0-dimensional schemes for which every subscheme Y of degree $\deg X - 1$ has the same Hilbert function $H_Y(n) = \min\{H_X(n), \deg X - 1\}$ for all $n \in \mathbb{Z}$. This can be interpreted as a weak uniformity of X and invites the following generalization.

DEFINITION. Let $n \geq 1$. We say that X is *n-uniform*, if every subscheme $Y \subseteq X$ of degree $\deg X - n \leq \deg Y \leq \deg X$ has Hilbert function $H_Y(m) = \min\{H_X(m), \deg Y\}$ for all $m \in \mathbb{Z}$.

We say that X is *in uniform position*, if X is $(\deg X - 1)$-uniform.

EXAMPLES. a) By definition, X is 1-uniform if and only if X is a Cayley-Bacharach scheme.

b) Proposition 2.8 says that every Cayley-Bacharach scheme X with $\Delta_X = 1$ is 2-uniform. In particular, every arithmetically Gorenstein scheme is 2-uniform.

c) J. Harris has shown in [H] that if char $k = 0$ and X is the general hyperplane section of an integral curve $C \subseteq \mathbb{P}^{d+1}$, then X is in uniform position. For most cases, this has been extended to char $k > 0$ by J. Rathmann in [R].

d) The 0-dimensional reduced complete intersection scheme $X = \{(1:0:0),(1:1:0),(1:2:0),(1:0:1),(1:1:1),(1:2:1)\} \subseteq \mathbb{P}^2$ is 2-uniform by Proposition 2.8, but not 3-uniform, because $Y := \{(1:0:0),(1:1:0),(1:2:0)\}$ has $H_Y(1) = 2 < 3 = \min\{H_X(1), \deg Y\}$. Therefore Proposition 2.8 cannot be improved for complete intersections.

The next theorem contains Theorem 3 of the introduction. In the sequel we shall say that a k-linear map $\mu\colon U \otimes V \longrightarrow W$ of finite dimensional k-vector spaces is *biinjective*, if $\mu(u \otimes v) = 0$ implies $u = 0$ or $v = 0$ for all $u \in U$, $v \in V$.

THEOREM 3.1. *The following conditions are equivalent.*

a) X is Δ_X-uniform.

b) The multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ is biinjective.

c) For each $n \in \{0, \ldots, \sigma_X\}$, the multiplication map $R_n \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+n}$ is biinjective.

PROOF. a) $\Rightarrow$ b): Let $r \in R_{\sigma_X} \setminus \{0\}$, and let $Y := \mathcal{V}(r) \subseteq X$. Then $r \in (I_{Y/X})_{\sigma_X}$ implies $H_Y(\sigma_X) < H_X(\sigma_X)$. Since X is Δ_X-uniform, this yields $\deg Y < H_X(\sigma_X) = \deg X - \Delta_X$. By applying Lemma 2.2.b, we find a subscheme $Y' \subseteq X$ such that $\deg Y' = \deg X - \Delta_X$ and $Y \subseteq Y'$. By assumption, $\sigma_Y \leq \sigma_{Y'} = \sigma_X - 1$. If $S := R/I_{Y/X}$, it follows that no nonzero element of $(\omega_R)_{-\sigma_X}$ lies in $\omega_S = \{\varphi \in \omega_R : I_{Y/X} \cdot \varphi = 0\}$. By Lemma 1.2, this means that no element of $(\omega_R)_{-\sigma_X} \setminus \{0\}$ is annihilated by r.

b) $\Rightarrow$ c): Let $\varphi \in (\omega_R)_{-\sigma_X}$ and $r \in R_n$, $n \leq \sigma_X$, such that $r\varphi = 0$. Then $rx_0^{\sigma_X - n} \in R_{\sigma_X}$ and $rx_0^{\sigma_X - n}\varphi = 0$ imply $rx_0^{\sigma_X - n} = 0$ or $\varphi = 0$. Hence we have $r = 0$ or $\varphi = 0$.

c) $\Rightarrow$ a): Let $Y \subseteq X$ be a subscheme of degree $\deg X - \Delta_X \leq \deg Y \leq \deg X$. It suffices to show $H_Y(\sigma_X) = H_X(\sigma_X)$, because $H_Y(n) = \deg Y$ for $n \geq \sigma_X + 1$ follows already from $\sigma_Y \leq \sigma_X$. Suppose that $H_Y(\sigma_X) < H_X(\sigma_X) \leq \deg Y$. Then there exists a nonzero element $r \in (I_{Y/X})_{\sigma_X}$, and we can also conclude that $\sigma_Y = \sigma_X$. Therefore there is a nonzero element $\varphi \in (\omega_R)_{-\sigma_X}$ which is annihilated by $I_{Y/X}$. In particular, $r\varphi = 0$, a contradiction. ∎

REMARK. It is useful to be able to check computationally whether the multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ is biinjective. Otherwise we would have to compute the Hilbert functions of *all* subschemes $Y \subseteq X$ of degree $\deg Y \geq \deg X - \Delta_X$ in order to check Δ_X-uniformity. We start by computing a minimal homogeneous presentation

$$\bigoplus_{i=1}^{\beta_{d-1}} R(\alpha_{d-1\,i} - d - 1) \xrightarrow{\delta} \bigoplus_{i=1}^{\beta_d} R(\alpha_{di} - d - 1) \xrightarrow{\epsilon} \omega_R \longrightarrow 0$$

as explained in Section 5. Let $e_1, \ldots, e_\Delta \in \bigoplus_{i=1}^{\beta_d} R(\alpha_{di} - d - 1)$ be those standard basis vectors which have degree $-\sigma_X$. Their images $\varphi_i := \epsilon(e_i)$ form a k-basis of $(\omega_R)_{-\sigma_X}$. Compute monomials (or polynomials) $m_1, \ldots, m_{s-\Delta} \in k[X_0, \ldots, X_d]_{\sigma_X}$ whose images in R_{σ_X} form a k-basis of that vector space ($s = \deg X$).

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Using elimination, we can find the variety $V \subseteq \mathbb{A}_k^s$ of all solutions $(a_1, \ldots, a_\Delta, b_1, \ldots, b_{s-\Delta}) \in k^s$ of

$$(a_1e_1 + \cdots + a_\Delta e_\Delta)(b_1m_1 + \cdots + b_{s-\Delta}m_{s-\Delta}) \in \operatorname{im}\delta.$$

Let $L \subseteq \mathbb{A}_k^{s-\Delta}$ be the linear subspace of all solutions $(b_1, \ldots, b_{s-\Delta}) \in k^{s-\Delta}$ of $b_1m_1 + \cdots + b_{s-\Delta}m_{s-\Delta} \in (I_X)_{\sigma_X}$. Now check whether V equals $\{0\} \times \mathbb{A}_k^{s-\Delta} \cup \mathbb{A}_k^\Delta \times L$ in $\mathbb{A}_k^s \cong \mathbb{A}_k^\Delta \times \mathbb{A}_k^{s-\Delta}$.

More generally, it is clear that the described method can be applied to check biinjectivity of any bilinear map of vector spaces whose matrix we know. Another method can be derived from [E], Lemma 1.1.

Like Theorem 2.4, Theorem 3.1 also has implications for the growth behaviour of H_X. The key here is the following lemma.

BIINJECTIVE MAP LEMMA. *Let $\mu\colon U \otimes V \longrightarrow W$ be a k-linear, biinjective map of finite dimensional k-vector spaces. Then*

$$\dim_k W \geq \dim_k U + \dim_k V - 1.$$

For a nice, elementary proof of this lemma see [G]. In the situation of Theorem 3.1 we can apply it and obtain the corollary stated in the introduction. This corollary was obtained earlier in [HE] under the stronger hypothesis that X is reduced and in uniform position.

Next we want to characterize 0-dimensional schemes with even higher uniformities.

THEOREM 3.2. *Let $i \in \{0, \ldots, \sigma_X\}$. The following conditions are equivalent.*

a) X is $\big(\deg X - H_X(i)\big)$-uniform.

b) For each $n \in \{i, \ldots, \sigma_X\}$, the multiplication map $R_n \otimes (\omega_R)_{-n} \longrightarrow (\omega_R)_0$ is biinjective.

c) If $n \in \{0, \ldots, \sigma_X - i\}$ and $m \in \{0, \ldots, \sigma_X - n\}$, the multiplication map $R_m \otimes (\omega_R)_{-\sigma_X+n} \longrightarrow (\omega_R)_{-\sigma_X+m+n}$ is biinjective.

PROOF. a) $\Rightarrow$ b): We proceed by downward induction on i. The case $i = \sigma_X$ is handled by Theorem 3.1. By induction hypothesis, we only have to show that $R_i \otimes (\omega_R)_{-i} \longrightarrow (\omega_R)_0$ is biinjective. Suppose $r \in R_i \setminus \{0\}$ and $\varphi \in (\omega_R)_{-i}$ are such that $r\varphi = 0$. Let $Y := \mathcal{V}(r) \subseteq X$. Then the ideal $I_{Y/X}$ of Y in R satisfies $r \in (I_{Y/X})_i \neq (0)$. Since X is $\big(\deg X - H_X(i)\big)$-uniform, this implies $\deg Y < H_X(i)$. Using Lemma 2.2.b we find a subscheme $Y' \subseteq X$ such that $\deg Y' = H_X(i)$ and $Y \subseteq Y'$. By assumption, $\sigma_Y \leq \sigma_{Y'} = i - 1$. Let $S := R/I_{Y/X}$. From $r\varphi = 0$ and Lemma 1.2 we conclude $I_{Y/X} \cdot \varphi = (r)^{\mathrm{sat}}\varphi = 0$. Hence $\varphi \in (\omega_S)_{-i}$. Now $-i < -\sigma_Y$ yields $\varphi = 0$, as was to be shown.

b) $\Rightarrow$ c): Suppose that $r \in R_m$ and $\varphi \in (\omega_R)_{-\sigma_X+n}$ satisfy $r\varphi = 0$. Then $rx_0^{\sigma_X-n-m} \in R_{\sigma_X-n}$ and $rx_0^{\sigma_X-n-m}\varphi = 0$ imply $rx_0^{\sigma_X-n-m} = 0$ or $\varphi = 0$ by b). Hence $r = 0$ or $\varphi = 0$.

Since "c) $\Rightarrow$ b)" is clear, it remains to show "b) $\Rightarrow$ a)". Again we proceed by downward induction on i, the case $i = \sigma_X$ being the corresponding statement in 3.1. Suppose that $Y \subseteq X$ is a subscheme of degree $H_X(i) \leq \deg Y \leq \deg X$. If $\deg Y \geq H_X(i+1)$, we get the claim from the induction hypothesis. Thus we can assume that $H_X(i) \leq \deg Y < H_X(i+1)$.

We find a subscheme $Y' \subseteq X$ such that $\deg Y' = H_X(i+1)$ and $Y \subseteq Y'$. From the induction hypothesis we know $H_{Y'}$. Hence $\sigma_Y \leq \sigma_{Y'} = i$, and therefore we have $H_Y(n) = \deg Y$ for $n \geq i+1$. It remains to show $H_Y(i) = H_X(i)$. Suppose $H_Y(i) < H_X(i)$. Then there is an element $r \in (I_{Y/X})_i \setminus \{0\}$, and $H_Y(i) < \deg Y$ implies $\sigma_Y \leq i$. Thus we also find an element $\varphi \in (\omega_S)_{-i} \setminus \{0\}$. Now $r\varphi = 0$ contradicts b). ∎

Notice that in order to obtain Theorem 4 of the introduction, we only have to apply Theorem 3.2 with $i = 0$.

EXAMPLE. Consider the 0-dimensional reduced subscheme $X \subseteq \mathbb{P}^2$ concentrated at the eleven points $P_1 = (1:2:1)$, $P_2 = (1:1:2)$, $P_3 = (1:2:-1)$, $P_4 = (1:1:-2)$, $P_5 = (1:-2:1)$, $P_6 = (1:-2:-1)$, $P_7 = (1:3:2)$, $P_8 = (1:3:3)$, $P_9 = (1:4:0)$, $P_{10} = (1:5:0)$, and $P_{11} = (1:0:6)$. Its Hilbert function is H_X: 1 3 6 10 11 11 $\cdots$, so $\Delta_X = 1$, and using the procedure given in [GKR] it is easy to check that X is a Cayley-Bacharach scheme.

Since $Y_1 := \{P_1, \ldots, P_6\}$ has Hilbert function H_{Y_1}: 1 3 5 6 6 $\cdots$ and $Y_2 := \{P_1, P_2, P_8, \ldots, P_{11}\}$ has Hilbert function H_{Y_2}: 1 3 6 6 $\cdots$, we see that X is not 5-uniform, where $5 = \deg X - H_X(\sigma_X - 1)$. Therefore there exist nonzero elements $\psi \in (\omega_R)_{-2}$ and $r \in R_2$ such that $r\psi = 0$. After calculating a presentation of ω_R as in Section 5, it is a straightforward exercise to compute those elements explicitly.

COROLLARY 3.3. *If X is locally Gorenstein and in uniform position, the multiplication maps $R_m \otimes R_n \longrightarrow R_{m+n}$ are biinjective for all $m, n \geq 0$ such that $m+n \leq \sigma_X$.*

PROOF. Since X is a locally Gorenstein Cayley-Bacharach scheme, we find an element $\varphi \in (\omega_R)_{-\sigma_X}$ such that $\mathrm{Ann}_R(\varphi) = 0$. Now restrict the multiplication maps from part c) of the theorem to $R\varphi \subseteq \omega_R$ and identify $R(\sigma_X) \cong R\varphi$. ∎

Of course, unlike Theorem 4, the above corollary is not a characterization of 0-dimensional schemes in uniform position, since the inclusion $R\varphi \subseteq \omega_R$ is a strict one as long as X is not arithmetically Gorenstein.

Finally, we also want to give a characterization of n-uniform schemes for those values of n which lie between two consecutive values of $\deg X - H_X(i)$. Here we restrict ourselves to the case of reduced schemes. Also, we only formulate the case $n \in \{1, \ldots, \Delta_X\}$ explicitly, and leave appropriate generalizations to the reader.

In the case of reduced 0-dimensional subschemes $X \subseteq \mathbb{P}^d$ we shall use the following notations. We let $s := \deg X$ and write $X = \{P_1, \ldots, P_s\}$. For each $i \in \{1, \ldots, s\}$, we let $f_i \in R_{\sigma_X+1}$ be a separator corresponding to the subscheme $X \setminus \{P_i\} \subseteq X$. As shown in [GKR], the set $\{f_1, \ldots, f_s\}$ is a k-basis of R_{σ_X+1}. Therefore the images $\{\bar{f}_1, \ldots, \bar{f}_s\}$ in $\bar{R}_{\sigma_X+1}$ are a set of generators of that vector space. The relations among those images determine the structure of the multiplication map $R_{\sigma_X} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_0$ in a natural way.

PROPOSITION 3.4. *Let $n \in \{1, \ldots, \Delta_X\}$. The following conditions are equivalent.*

a) X is n-uniform.

b) Every subset of n elements from $\{\bar{f}_1, \ldots, \bar{f}_s\}$ is linearly independent in $\bar{R}_{\sigma_X+1}$.

PROOF. a) $\Rightarrow$ b): Let $\nu_1, \ldots, \nu_n \in \{1, \ldots, s\}$ be pairwise distinct elements, and let $Y := X \setminus \{P_{\nu_1}, \ldots, P_{\nu_n}\}$. Then $H_Y(i) = H_X(i)$ for $i \leq \sigma_X$ and $H_Y(i) = s - n$ for

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$i \geq \sigma_X + 1$. Also note that $(f_{\nu_1}, \ldots, f_{\nu_n}) \subseteq I_{Y/X}$. If we compare Hilbert functions, we see that in fact we must have $I_{Y/X} = (f_{\nu_1}, \ldots, f_{\nu_n})$. Thus $(f_{\nu_1}, \ldots, f_{\nu_n})$ is a saturated ideal of R, *i.e.* $\lambda_1 f_{\nu_1} + \cdots + \lambda_n f_{\nu_n} \in x_0 R_{\sigma_X}$ with $\lambda_i \in k$ implies $\lambda_1 = \cdots = \lambda_n = 0$. This is clearly equivalent to what we claimed in b).

b) $\Rightarrow$ a): Let $m \leq n$, let $\nu_1, \ldots, \nu_m \in \{1, \ldots, s\}$ be pairwise distinct, and let $Y := X \setminus \{P_{\nu_1}, \ldots, P_{\nu_m}\}$. We have to show $H_Y(i) = \min\{H_X(i), s-m\}$ for all $i \in \mathbb{Z}$. Since the ideal $(f_{\nu_1}, \ldots, f_{\nu_m})$ defines Y in X scheme-theoretically, and since $R/(f_{\nu_1}, \ldots, f_{\nu_m})$ has the correct Hilbert function, the claim is equivalent to showing that the ideal $(f_{\nu_1}, \ldots, f_{\nu_m})$ is saturated.

Let $i \geq 0$ and $r \in R_i$ such that $x_0 r \in (f_{\nu_1}, \ldots, f_{\nu_m})$. We consider three cases.

1) If $i \geq \sigma_X + 1$, we can write $x_0 r = \lambda_1 x_0^{i-\sigma_X} f_{\nu_1} + \cdots + \lambda_m x_0^{i-\sigma_X} f_{\nu_m}$ with $\lambda_j \in k$, and we get $r = \lambda_1 x_0^{i-\sigma_X-1} f_{\nu_1} + \cdots + \lambda_m x_0^{i-\sigma_X-1} f_{\nu_m} \in (f_{\nu_1}, \ldots, f_{\nu_m})$.

2) If $i = \sigma_X$, we can write $x_0 r = \lambda_1 f_{\nu_1} + \cdots + \lambda_m f_{\nu_m}$ with $\lambda_j \in k$, and using b) we get $\lambda_1 = \cdots = \lambda_m = 0$, and thus $r = 0$.

3) If $i < \sigma_X$, we have $x_0 r \in (f_{\nu_1}, \ldots, f_{\nu_m})_{i+1} = (0)$, hence $r = 0$.

In any case, $x_0 r \in (f_{\nu_1}, \ldots, f_{\nu_m})$ implies $r \in (f_{\nu_1}, \ldots, f_{\nu_m})$. By Lemma 1.2, this yields the desired conclusion. ∎

EXAMPLE. Suppose $\operatorname{char} k = p > 0$ and $X \subseteq \mathbb{P}^d$ is a reduced 0-dimensional subscheme consisting of $s := \deg X \geq p + 2$ $\mathbb{F}_p$-rational points and having $\Delta_X = 2$. Then X is not 2-uniform.

Suppose that X was 2-uniform. Since X is defined over $\mathbb{F}_p$, its separators f_i and their residue classes $\bar{f}_i$ are elements of $\mathbb{F}_p[x_0, \ldots, x_d]$ resp. $\mathbb{F}_p[x_1, \ldots, x_d]$ for $i = 1, \ldots, s$. By the proposition, we have $\bar{f}_i \neq 0$ and $\bar{f}_i \notin k \cdot \bar{f}_j$ for all $i, j = 1, \ldots, s$ such that $i \neq j$. W.l.o.g. let $\{\bar{f}_1, \bar{f}_2\}$ be a k-basis of $\bar{R}_{\sigma_X+1}$. For $i = 1, \ldots, s$ write $\bar{f}_i = \lambda_i \bar{f}_1 + \mu_i \bar{f}_2$ with $\lambda_i, \mu_i \in \mathbb{F}_p$. Then $\{(\lambda_1 : \mu_1), \ldots, (\lambda_s : \mu_s)\}$ is a set of $s \geq p + 2$ distinct points in $\mathbb{P}^1_{\mathbb{F}_p}$, a contradiction.

4. **Schemes in general position.**

In this section let X again be an arbitrary 0-dimensional subscheme of $\mathbb{P}^d$. We want to study 0-dimensional schemes which exhibit the following kind of uniformity.

DEFINITION. We say that X is in *linearly general position*, if $\deg(L \cap X) \leq 1 + \dim L$ for every proper linear subspace $L \subsetneq \mathbb{P}^d$.

It is useful to rephrase this condition in terms of Hilbert functions of various subschemes of X.

PROPOSITION 4.1. *The following conditions are equivalent.*

a) X is in linearly general position.

b) If $Y \subseteq X$ is a subscheme of degree $\deg Y \leq d + 1$, *then* $H_Y(n) = \deg Y$ *for all* $n \geq 1$.

c) Each subscheme $Y \subseteq X$ of degree $\deg Y \leq d + 1$ *satisfies* $\sigma_Y \leq 0$.

d) *If* $\deg X \geq d+1$, *then* $H_Y(1) = d+1$ *for every subscheme* $Y \subseteq X$ *of degree* $\deg Y = d+1$, *and if* $1 \leq \deg X \leq d$, *then* $H_X(1) = \deg X$.

PROOF. a) $\Rightarrow$ b): Let $\langle X\rangle$ be the linear span of X, *i.e.* the linear subspace $\langle X\rangle := \mathcal{V}\big((I_X)_1\big) \subseteq \mathbb{P}^d$. Then we have $\dim\langle X\rangle = d - \dim_k(I_X)_1 = H_X(1) - 1$. If X does not span $\mathbb{P}^d$, *i.e.* if $\dim\langle X\rangle < d$, we can use $L = \langle X\rangle$ in the definition, and we obtain $\deg X \leq H_X(1) \leq d$. Therefore $H_X(n) = \deg X$ for all $n \geq 1$. Clearly, this must then also be true for all subschemes of X.

In case $\dim\langle X\rangle \geq d$, we have $H_X(1) = d+1 \leq \deg X$. Choose a subscheme $Z \subseteq X$ such that $\deg Z = d+1$ and $Y \subseteq Z$. It suffices to show $H_Z(1) = d+1$. Then $\sigma_Y \leq \sigma_Z$ yields the claim. Suppose there is a hyperplane $L \subseteq \mathbb{P}^d$ such that $Z \subseteq L$. Then $X \underset{\neq}{\subset} L$ implies $L \cap X \underset{\neq}{\subset} X$. Thus $d+1 = \deg Z \leq \deg(L\cap X) \leq 1 + \dim L = d$, a contradiction. Therefore Z is not contained in any hyperplane, *i.e.* $H_Z(1) = d+1$.

"b)$\Leftrightarrow$c)" is clear by definition of σ_Y, and "c) $\Rightarrow$ d)" is also clear.

d) $\Rightarrow$ a): First we consider the case $\deg X \geq d+1$. Let $Z = L \cap X$ for some proper linear subspace $L \underset{\neq}{\subset} \mathbb{P}^d$. If $\deg Z \geq d+1$, we find a subscheme $Y \subseteq Z$ of degree $\deg Y = d+1$. Then $H_Y(1) = d+1$ follows from our assumption, but contradicts $Y \subseteq Z \subsetneq X$. Hence we must have $\deg Z \leq d$. Then we find a subscheme $Y \subseteq X$ such that $\deg Y = d+1$ and $Z \subseteq Y$. By assumption we have $\sigma_Y = 0$, hence $\sigma_Z \leq 0$. Altogether we obtain

$$\deg(L\cap X) = \deg Z = H_Z(1) \leq H_L(1) = 1 + \dim L.$$

Finally, we consider the case $1 \leq \deg X \leq d$. By assumption we have $\sigma_X \leq 0$. Therefore $\sigma_{L\cap X} \leq 0$ for every proper linear subspace $L \underset{\neq}{\subset} \mathbb{P}^d$. Thus we find $\deg(L\cap X) = H_{L\cap X}(1) \leq H_L(1) = 1 + \dim L$, as desired. ∎

EXAMPLE. Schemes in linearly general position are, for instance, obtained naturally by taking general hyperplane sections of nondegenerate integral curves $C \subseteq \mathbb{P}^{d+1}$ which are not strange. This is the content of the General Position Lemma shown in [R].

A 0-dimensional scheme $X \subseteq \mathbb{P}^d$ is called *nondegenerate*, if it is not contained in any hyperplane. Our next theorem characterizes nondegenerate schemes in linearly general position and contains Theorem 5 of the introduction. The easy task of formulating a similar theorem also in the degenerate case is left to the reader.

Notice that if X is in linearly general position and $\deg X \geq d+1$, then X is automatically nondegenerate. Conversely, if X is nondegenerate, we obviously must have $\deg X \geq d+1$.

THEOREM 4.2. *Let* $X \subseteq \mathbb{P}^d$ *be a nondegenerate* 0*-dimensional subscheme. The following conditions are equivalent.*

a) *X is in linearly general position.*

b) *The multiplication map* $R_1 \otimes (\omega_R)_{-1} \longrightarrow (\omega_R)_0$ *is biinjective.*

c) *For each* $n \in \{1,\ldots,\sigma_X\}$, *the multiplication map* $R_1 \otimes (\omega_R)_{-n} \longrightarrow (\omega_R)_{-n+1}$ *is biinjective.*

PROOF. a) $\Rightarrow$ b): Let $\ell \in R_1 \setminus \{0\}$ and $\varphi \in (\omega_R)_{-1}$ such that $\ell\varphi = 0$. Since X is in linearly general position, we have $\deg \mathcal{V}(\ell) \leq d$. Using Lemma 2.2.b we find a subscheme $Y \subseteq X$ such that $\deg Y = d+1$ and $\mathcal{V}(\ell) \subseteq Y$. Then $I_{Y/X}$, the ideal of Y in R, is contained in $(\ell)^{\mathrm{sat}}$, the ideal of $\mathcal{V}(\ell)$ in R. By Lemma 1.2, this implies that for every $r \in I_{Y/X}$ there exists a number $n \geq 0$ such that $x_0^n r \in (\ell)$. In particular, we have $x_0^n r\varphi = 0$, and therefore $r\varphi = 0$. Let $S := R/I_{Y/X}$. By assumption and Proposition 4.1, we have $H_Y(1) = d+1$. Thus $H_{\omega_S}(-1) = \deg Y - H_Y(1) = 0$. It follows that $\varphi \in \{\psi \in (\omega_R)_{-1} : I_{Y/X} \cdot \psi = 0\} = (\omega_S)_{-1} = (0)$, hence $\varphi = 0$, as was to be shown.

b) $\Rightarrow$ c): If $\ell \in R_1 \setminus \{0\}$ and $\varphi \in (\omega_R)_{-n}$ are such that $\ell\varphi = 0$, then $\ell x_0^{n-1}\varphi = 0$ implies $x_0^{n-1}\varphi = 0$ by b), and hence $\varphi = 0$.

c) $\Rightarrow$ a): Suppose that X is not in linearly general position. By Proposition 4.1, there is a subscheme $Y \subseteq X$ such that $\deg Y = d+1$ and $H_Y(1) \leq d$. Since X is nondegenerate, we find an element $\ell \in R_1 \setminus \{0\}$ such that $\ell \in I_{Y/X}$. Let $S := R/I_{Y/X}$. Since $H_{\omega_S}(-1) = \deg Y - H_Y(1) \geq 1$, we find an element $\varphi \in (\omega_S)_{-1} \setminus \{0\}$. But then $I_{Y/X} \cdot \varphi = 0$ implies $\ell\varphi = 0$, a contradiction. ∎

As usual, also this theorem has implications for the Hilbert function of schemes in linearly general position. In fact, an application of the Biinjective Map Lemma to b) yields the corollary stated in the introduction. The second statement of that corollary follows by simply adding up the inequalities obtained before.

EXAMPLE. If X is a nondegenerate 0-dimensional subscheme of $\mathbb{P}^3$ and $\Delta H_X(\sigma_X) = 2$, *i.e.* if H_X is of the form H_X: $1\ 3\ \cdots s-\Delta_X-2, s-\Delta_X, s, s\cdots$, where $s := \deg X$, then X is not in linearly general position. This follows from the aforementioned corollary, since $\Delta H_X(\sigma_X) = 2 < 3 = \Delta H_X(1)$.

Our next corollary is an immediate consequence of Theorem 3.2 and Theorem 4.2.

COROLLARY 4.3. *If X is nondegenerate and in uniform position, then X is in linearly general position.*

In view of Proposition 4.1.b, we find it natural to extend the concept of linearly general position as follows.

DEFINITION. Let $i \geq 1$. We say that X is in *i-th-order general position*, if every subscheme $Y \subseteq X$ of degree $\deg Y \leq H_{\mathbb{P}^d}(i)$ satisfies $H_Y(n) = \min\{H_{\mathbb{P}^d}(n), \deg Y\}$ for all $n \geq 0$.

In other words, the Hilbert function of each subscheme $Y \subseteq X$ of degree $\deg Y \leq H_{\mathbb{P}^d}(i)$ agrees with $H_{\mathbb{P}^d}$ as long as possible, and then immediately attains its maximum value. By Proposition 4.1.b, X is in 1-st-order general position if and only if X is in linearly general position. Also, X is in (σ_X+1)-th-order general position if and only if X is in uniform position and in generic position (*i.e.* $H_X(n) = \min\{H_{\mathbb{P}^d}(n), \deg X\}$ for all $n \geq 0$).

By now, it should be clear to the reader how i-th-order general position is reflected by the structure of the canonical module. Again we restrict ourselves to the case of sufficiently many points and leave the degenerate cases as an exercise.

THEOREM 4.4. *Let $i \geq 0$ and let $\alpha_X := \min\{n \in \mathbb{N} : (I_X)_n \neq 0\} \geq i+1$. The following conditions are equivalent.*

a) *X is in i-th-order general position.*
b) *For each $n \in \{1, \ldots, i\}$, the multiplication map $R_n \otimes (\omega_R)_{-n} \longrightarrow (\omega_R)_0$ is biinjective.*
c) *For all $m \in \{1, \ldots, i\}$ and $n \in \{0, \ldots, \sigma_X - m\}$, the multiplication map $R_m \otimes (\omega_R)_{-\sigma_X+n} \longrightarrow (\omega_R)_{-\sigma_X+m+n}$ is biinjective.*

PROOF. a) $\Rightarrow$ b): We proceed by induction on i. The case $i = 1$ is contained in Theorem 4.2. Using the induction hypothesis, we only have to show that $R_i \otimes (\omega_R)_{-i} \longrightarrow (\omega_R)_0$ is biinjective. Let $r \in R_i \setminus \{0\}$ and $\varphi \in (\omega_R)_{-i}$ such that $r\varphi = 0$. Consider the subscheme $Z := \mathcal{V}(r) \subseteq X$. Since $r \in (I_{Z/X})_i$ and X is in i-th-order general position, we have $\deg Z < H_{\mathbb{P}^d}(i)$. Choose a subscheme $Y \subseteq X$ such that $\deg Y = H_{\mathbb{P}^d}(i)$ and $Z \subseteq Y$. Then $I_{Y/X}$ is contained in $I_{Z/X} = (r)^{\mathrm{sat}}$. Thus, for every $s \in I_{Y/X}$, there exists $n \geq 0$ such that $x_0^n s \in (r)$. This implies $I_{Y/X} \cdot \varphi = 0$. Let $S = R/I_{Y/X}$. From $H_{\omega_S}(-i) = 0$ and $\varphi \in (\omega_S)_{-i}$ we conclude $\varphi = 0$.

"b) $\Rightarrow$ c)" is standard by now, and "c) $\Rightarrow$ b)" is clear, so we still have to prove "b) $\Rightarrow$ a)". Again we proceed by induction on i, the case $i = 1$ being provided by 4.2. By induction hypothesis, each subscheme $Y \subseteq X$ of degree $\deg Y \leq H_{\mathbb{P}^d}(i-1)$ has the desired Hilbert function. So, let $Y \subseteq X$ be a subscheme of degree $H_{\mathbb{P}^d}(i-1) < \deg Y \leq H_{\mathbb{P}^d}(i)$. By choosing a subscheme of degree $H_{\mathbb{P}^d}(i-1)$ of Y and applying the induction hypothesis, we see that $H_Y(i-1) = H_{\mathbb{P}^d}(i-1)$. Therefore it only remains to show $\sigma_Y = i - 1$.

Find a subscheme $Z \subseteq X$ such that $\deg Z = H_{\mathbb{P}^d}(i)$ and $Y \subseteq Z$. Since $\deg Y > H_{\mathbb{P}^d}(i-1)$ implies $i - 1 \leq \sigma_Y \leq \sigma_Z$, it suffices to show $H_Z(i) = H_{\mathbb{P}^d}(i)$. Suppose there is a hypersurface of degree i containing Z. The image $r \in R_i$ of its equation in R does not vanish because of $\alpha_X \geq i + 1$. Let $S := R/I_{Z/X}$. Since $H_{\omega_S}(-i) \geq 1$, we find a nonzero element $\varphi \in (\omega_S)_{-i}$. Then $I_{Z/X} \cdot \varphi = 0$ implies $r\varphi = 0$, a contradiction. ∎

REMARK. More generally, if we drop the assumption $\alpha_X \geq i + 1$ in Theorem 4.4, condition 4.4.b is equivalent to the statement "every subscheme $Y \subseteq X$ of degree $\deg Y \leq H_X(i)$ has Hilbert function $H_Y(n) = \min\{H_X(n), \deg Y\}$ for all $n \geq 0$". This can be shown in a completely analogous manner and is left to the reader.

Let us return for a moment to the example of eleven points in $\mathbb{P}^2$ considered in Section 3.

EXAMPLE. Let $X = \{P_1, \ldots, P_{11}\} \subseteq \mathbb{P}^2$ be the scheme defined in the example after Theorem 3.2. We have seen that the multiplication map $R_2 \otimes (\omega_R)_{-2} \longrightarrow (\omega_R)_0$ is not biinjective. Since no three points of X are on a line, X is in linearly general position. Therefore the multiplication map $R_1 \otimes (\omega_R)_{-1} \longrightarrow (\omega_R)_0$ is biinjective. Using the method described after Theorem 3.1, it is posiible to check this directly.

Altogether we conclude that X is in linearly general position, but not in 2-nd-order general position. The latter statement corresponds geometrically to the fact that the six points $\{P_1, \ldots, P_6\}$ of X are contained in a conic.

Theorem 4.4 has the following consequences for Hilbert functions of schemes in i-th-order general position.

COROLLARY 4.5. *Let $i \geq 0$, let X be in i-th-order general position, let $\alpha_X \geq i+1$, and let $m \in \{1,\ldots,i\}$. Then the sum of m consecutive terms of the sequence $\{\Delta H_X(1),\ldots,\Delta H_X(\sigma_X)\}$ is at least $\binom{m+d}{d}-1$.*

The proof of this corollary is obtained by applying the Biinjective Map Lemma to Theorem 4.4.c.

EXAMPLE. If $X \subseteq \mathbb{P}^3$ is a 0-dimensional subscheme of degree $\deg X \geq 10$, and if no subscheme of degree 10 of X is contained in a quadric surface, then the Hilbert function of X cannot satisfy $\Delta H_X(\sigma_X - 1) = 4$ and $\Delta H_X(\sigma_X) \leq 4$. This follows from the corollary, because X is in 2-nd-order general position and $\Delta H_X(\sigma_X - 1) + \Delta H_X(\sigma_X) \leq 8 < 9 = \binom{5}{3} - 1$.

Finally, we want to explain the connection of our notion of "i-th-order general position" with the notion "imposes independent conditions on forms of degree i". The following definition is adapted from [EK].

DEFINITION. Let $i \geq 1$. We say that X *imposes independent conditions on forms of degree i*, if $\deg X \leq H_{\mathbb{P}^d}(i)$ implies $H_X(i) = \deg X$, and if $\deg X \geq H_{\mathbb{P}^d}(i)$ implies $H_Y(i) = H_{\mathbb{P}^d}(i)$ for all subschemes $Y \subseteq X$ of degree $\deg Y = H_{\mathbb{P}^d}(i)$.

PROPOSITION 4.6. *Let $i \geq 0$. The following conditions are equivalent.*

a) X is in i-th-order general position.

b) X imposes independent conditions on forms of degree j for every $j \in \{1,\ldots,i\}$.

PROOF. a) $\Rightarrow$ b): Let $j \in \{1,\ldots,i\}$. We consider two cases.

1) If $\deg X \leq H_{\mathbb{P}^d}(j)$, we choose $Y = X$ in the definition of i-th-order general position and get $H_X(n) = \min\{H_{\mathbb{P}^d}(n), \deg X\}$ for $n \geq 0$. In particular, $H_X(j) = \deg X$.

2) If $\deg X \geq H_{\mathbb{P}^d}(j)$, we choose a subscheme $Y \subseteq X$ of degree $H_{\mathbb{P}^d}(j)$ in the definition of i-th-order general position. We get $H_Y(j) = \min\{H_{\mathbb{P}^d}(j), \deg Y\} = H_{\mathbb{P}^d}(j)$.

Consequently, X imposes independent conditions on forms of degree j.

b) $\Rightarrow$ a): Let $Y \subseteq X$ be a subscheme of degree $\deg Y \leq H_{\mathbb{P}^d}(i)$. Since we can exclude the trivial case $\deg Y = 1$, we can find $j \in \{1,\ldots,i\}$ such that $H_{\mathbb{P}^d}(j-1) < \deg Y \leq H_{\mathbb{P}^d}(j)$. Choose subschemes $Z, Z' \subseteq X$ such that $\deg Z = H_{\mathbb{P}^d}(j-1)$, $\deg Z' = H_{\mathbb{P}^d}(j)$, and $Z \subseteq Y \subseteq Z' \subseteq X$.

Since X imposes independent conditions on forms of degree j, we have $H_{Z'}(j) = H_{\mathbb{P}^d}(j) = \deg Z'$. Hence $\sigma_Y \leq \sigma_{Z'} = j-1$, and henceforth $H_Y(n) = \deg Y = \min\{H_{\mathbb{P}^d}(n), \deg Y\}$ for $n \geq j$. Since X imposes independent conditions on forms of degree $j-1$, we have $H_Z(j-1) = H_{\mathbb{P}^d}(j-1)$. Thus also $H_Y(j-1) = H_{\mathbb{P}^d}(j-1)$, implying $H_Y(n) = H_{\mathbb{P}^d}(n) = \min\{H_{\mathbb{P}^d}(n), \deg Y\}$ for all $n \in \{0,\ldots,j-1\}$.

Altogether we see that X is in i-th-order general position. ∎

5. **The projective resolution.** The last topic of this paper is to exhibit some connections between the canonical module ω_R and the minimal graded free resolution of R. Here we consider R as a module over the polynomial ring $A := k[X_0, \ldots, X_d]$. Since R is a 1-dimensional Cohen-Macaulay ring, its resolution is of the form

$$0 \longrightarrow \bigoplus_{i=1}^{\beta_d} A(-\alpha_{di}) \xrightarrow{\Phi_d} \bigoplus_{i=1}^{\beta_{d-1}} A(-\alpha_{d-1i}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{\beta_1} A(-\alpha_{1i}) \xrightarrow{\Phi_1} A \longrightarrow R \longrightarrow 0$$

where $\alpha_{ij} \in \mathbb{N}$ and $\beta_1, \ldots, \beta_d \in \mathbb{N}$ are the *Betti numbers* of X.

W. l. o. g. we can assume that $\alpha_{i1} \leq \cdots \leq \alpha_{i\beta_i}$ for $i = 1, \ldots, d$. Let $\mathfrak{A}_i$ be the matrix of Φ_i for $i = 1, \ldots, d$. As the above resolution is minimal, no entry of any of the matrices $\mathfrak{A}_i$ is a nonzero element of k. Hence $\alpha_{11} < \cdots < \alpha_{d1}$. Also notice that $\alpha_{11} = \alpha_X$, where $\alpha_X := \min\{n \in \mathbb{N} : (I_X)_n \neq 0\}$ denotes the least degree of a hypersurface containing X.

Now we dualize the above resolution and observe that $\mathrm{Ext}^i_A(R, A) = 0$ for $i = 0, \ldots, d-1$ and $\mathrm{Ext}^d_A(R, A) \cong \omega_R(d+1)$, *cf.* [GW]. We obtain a homogeneous exact sequence

$$0 \longrightarrow A \xrightarrow{\Phi_1^\vee} \bigoplus_{i=1}^{\beta_1} A(\alpha_{1i}) \longrightarrow \cdots \longrightarrow \bigoplus_{i=1}^{\beta_{d-1}} A(\alpha_{d-1i}) \xrightarrow{\Phi_d^\vee} \bigoplus_{i=1}^{\beta_d} A(\alpha_{di}) \longrightarrow \omega_R(d+1) \longrightarrow 0.$$

Since also this resolution is minimal, we conclude that $\alpha_{1\beta_1} < \cdots < \alpha_{d\beta_d}$. Notice that $\sigma_X = -\min\{n \in \mathbb{Z} : (\omega_R)_n \neq 0\} = \alpha_{d\beta_d} - d - 1$.

DEFINITION. Let $n \geq 1$.

a) We say that X has a *resolution of order n*, if $\alpha_{i\beta_i} \leq \alpha_X + i + n - 2$ for $i = 1, \ldots, d$.

b) We say that X has a *resolution almost of order n*, if $\alpha_{i\beta_i} \leq \alpha_X + i + n - 2$ for $i = 1, \ldots, d-1$.

In particular, if $n = 1$ and a) (resp. b)) is satisfied, we say that X has a *linear* (resp. *almost linear*) *resolution.*

Notice that if X has a resolution of order n (resp. almost of order n), then for $i = 2, \ldots, d$ (resp. for $i = 2, \ldots, d-1$) each matrix $\mathfrak{A}_i$ contains only homogeneous polynomials of degree at most n.

The following proposition generalizes the analogous statement for linear resolutions in [S] and follows also from [L].

PROPOSITION 5.1. *Let $n \geq 1$.*

a) X has a resolution of order n if and only if $\sigma_X \leq \alpha_X + n - 3$.

b) X has a resolution almost of order n if and only if $\alpha_{d-1\beta_{d-1}} \leq \alpha_X + n + d - 3$.

PROOF. a): In view of the definition, it suffices to show "$\Leftarrow$". From $\alpha_{d\beta_d} = \sigma_X + d + 1 \leq \alpha_X + d + n - 2$ and $\alpha_{d\beta_d} \geq \alpha_{d-1\beta_{d-1}} + 1 \geq \cdots \geq \alpha_{1\beta_1} + d - 1$ we obtain the desired inequalities.

"b)" follows from $\alpha_{1\beta_1} < \cdots < \alpha_{d-1\beta_{d-1}} \leq \alpha_X + n + d - 3$. ∎

COROLLARY 5.2. *The following conditions are equivalent.*

a) X has a linear resolution.

b) $\sigma_X = \alpha_X - 2$

c) $H_X(n) = \min\{\binom{n+d}{d}, \binom{\alpha_X+d-1}{d}\}$ *for all* $n \in \mathbb{N}$.

In particular, in this case we have $\deg X = \binom{\alpha_X+d-1}{d} = \binom{\sigma_X+d+1}{d}$.

PROOF. a)$\Leftrightarrow$b): Because of the proposition we only have to show $\sigma_X \geq \alpha_X - 2$. This follows from $\sigma_X = \alpha_{d\beta_d} - d - 1 \geq \alpha_{d1} - d - 1 \geq \alpha_{d-11} - d \geq \cdots \geq \alpha_{11} - 2 = \alpha_X - 2$.

a) $\Rightarrow$ c): From the presentation $A(-\alpha_X)^{\beta_1} \longrightarrow A \longrightarrow R \longrightarrow 0$ we obtain that $H_X(n) = \binom{n+d}{d}$ for $0 \leq n \leq \alpha_X - 1$. Since $\deg X = H_X(\sigma_X+1) = H_X(\alpha_X-1) = \binom{\alpha_X+d-1}{d}$, the conclusion follows.

"c) $\Rightarrow$ b)" is clear from the definition of σ_X. ■

Our next result contains Theorem 6 of the introduction. We characterize schemes with almost linear resolutions using the algebraic structure of their canonical module. The reader may consult [EG] and [L] for related results.

THEOREM 5.3. *Let* $\{\varphi_1, \ldots, \varphi_{\beta_d}\}$ *with* $\varphi_i \in (\omega_R)_{-\alpha_{di}+d+1}$ *be a minimal homogeneous system of generators of* ω_R. *The following conditions are equivalent.*

a) X *has an almost linear resolution.*

b) If for $i = 1, \ldots, \beta_d$ *there are elements* $r_i \in R_{\alpha_{di}-\alpha_X-d+1}$ *such that* $r_1\varphi_1 + \cdots + r_{\beta_d}\varphi_{\beta_d} = 0$, *then* $r_1 = \cdots = r_{\beta_d} = 0$.

In particular, if X *has an almost linear resolution, then the multiplication maps* $R_n \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+n}$ *are injective for* $n = 0, \ldots, \sigma_X - \alpha_X + 2$.

PROOF. a) $\Rightarrow$ b): Consider the minimal homogeneous presentation of ω_R induced by $\epsilon\colon \bigoplus_{i=1}^{\beta_d} A(\alpha_{di} - d - 1) \twoheadrightarrow \omega_R$ with $\epsilon(e_i) = \varphi_i$ for $i = 1, \ldots, \beta_d$. Because of what we know about the graded Betti numbers of X it has the shape

$$\bigoplus_{i=1}^{\beta_{d-1}} A(\alpha_X - 3) \longrightarrow \bigoplus_{i=1}^{\beta_d} A(\alpha_{di} - d - 1) \xrightarrow{\epsilon} \omega_R \longrightarrow 0.$$

Let K be the kernel of ϵ. Then $K_{-\alpha_X+2} = 0$ implies that any relation $r_1\varphi_1 + \cdots + r_{\beta_d}\varphi_{\beta_d} = 0$ of degree $-\alpha_X + 2$ (*i.e.* with $r_i \in R_{-\alpha_X+2-\deg\varphi_i} = R_{\alpha_{di}-\alpha_X-d+1}$) is trivial.

b) $\Rightarrow$ a): Consider the minimal homogeneous presentation

$$\bigoplus_{i=1}^{\beta_{d-1}} A(\alpha_{d-1i} - d - 1) \longrightarrow \bigoplus_{i=1}^{\beta_d} A(\alpha_{di} - d - 1) \xrightarrow{\epsilon} \omega_R \longrightarrow 0$$

induced by $\epsilon(e_i) = \varphi_i$ for $i = 1, \ldots, \beta_d$. Let K be the kernel of ϵ. The hypothesis implies that $K_{-\alpha_X+2} = 0$. Hence we have $-\alpha_{d-1i} + d + 1 > -\alpha_X + 2$ for $i = 1, \ldots, \beta_{d-1}$. In particular, we have $\alpha_{d-1\beta_{d-1}} \leq \alpha_X + d - 2$, so that Proposition 5.1 shows that X has an almost linear resolution.

Finally we prove the additional claim. Let $\Delta := \Delta_X = \dim_k(\omega_R)_{-\sigma_X}$. We conclude from b) that $r_1\varphi_{\beta_d-\Delta+1} + \cdots + r_\Delta\varphi_{\beta_d} = 0$ with $r_1, \ldots, r_\Delta \in R_{\sigma_X-\alpha_X+2}$ implies $r_1 = \cdots = r_\Delta = 0$. Since $\{\varphi_{\beta_d-\Delta+1}, \ldots, \varphi_{\beta_d}\}$ is a k-basis of $(\omega_R)_{-\sigma_X}$, this means that $R_{\sigma_X-\alpha_X+2} \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\alpha_X+2}$ is injective. The injectivity of the other multiplication maps follows now easily from the fact that x_0 is not a zero divisor of R. ■

EXAMPLE. Let us return to the example of eleven points $X = \{P_1, \ldots, P_{11}\}$ in $\mathbb{P}^2$ given after Theorem 3.2 for one last time. The projective resolution of X is

$$0 \longrightarrow A^2(-5) \oplus A(-6) \longrightarrow A^4(-4) \longrightarrow A \longrightarrow R \longrightarrow 0.$$

Therefore X has an almost linear resolution, and Theorem 5.3 shows that the multiplication map $(\omega_R)_{-3} \otimes R_1 \longrightarrow (\omega_R)_{-2}$ is injective. Of course, in the present example this follows also from the fact that X is a Cayley-Bacharach scheme with $\Delta_X = 1$.

By combining the various informations which we obtained in the last three sections, we have now a clear picture of the multiplication maps of ω_R in degrees ≤ 0:

1) For $i = 1, 2, 3$ the multiplication maps $(\omega_R)_{-3} \otimes R_i \longrightarrow (\omega_R)_{-3+i}$ are injective.
2) For $i = 1, 2$ the multiplications maps $(\omega_R)_{-i} \otimes R_1 \longrightarrow (\omega_R)_{-i+1}$ are biinjective. They cannot be injective because of dimension reasons.
3) The map $(\omega_R)_{-2} \otimes R_2 \longrightarrow (\omega_R)_0$ is neither injective nor biinjective.

Of course, also the previous theorem admits a generalization for schemes with almost quadratical or higher order resolutions. Since the proof of our final theorem is completely analogous to the one given above, we leave it to the reader.

THEOREM 5.4. *Let $n \in \{1, \ldots, \sigma_X - \alpha_X + 3\}$, and let $\{\varphi_1, \ldots, \varphi_{\beta_d}\}$ be a minimal homogeneous system of generators of ω_R, where $\varphi_i \in (\omega_R)_{-\alpha_{di}+d+1}$ for $i = 1, \ldots, \beta_d$. Choose $\nu \in \{1, \ldots, \beta_d - \Delta_X + 1\}$ such that $\{\varphi_\nu, \ldots, \varphi_{\beta_d}\}$ are precisely those elements in $\{\varphi_1, \ldots, \varphi_{\beta_d}\}$ which have degree at most $-\alpha_X - n + 3$. (This is possible because of $n \leq \sigma_X - \alpha_X + 3$.) The following conditions are equivalent.*

a) X has a resolution almost of order n.

b) If for $i = 1, \ldots, \beta_d - \nu + 1$ we have elements $r_i \in R_{\alpha_{di}-\alpha_X-d-n+2}$ such that $r_1\varphi_\nu + \cdots + r_{\beta_d-\nu+1}\varphi_{\beta_d} = 0$, then $r_1 = \cdots = r_{\beta_d-\nu+1} = 0$.

In particular, if X has a resolution almost of order n, then the multiplication maps $R_m \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+m}$ are injective for $m = 1, \ldots, \sigma_X - \alpha_X - n + 3$.

The injectivity claim in Theorem 5.4 implies strong inequalities for the Hilbert function of X.

COROLLARY 5.5. *Let $n \in \{1, \ldots, \sigma_X - \alpha_X + 3\}$, and suppose that X has a resolution almost of order n. Then we have*

$$\Delta_X \cdot H_X(m) + H_X(\sigma_X - m) \leq \deg X$$

for every $m \in \{1, \ldots, \sigma_X - \alpha_X - n + 3\}$.

Clearly every 0-dimensional scheme $X \subseteq \mathbb{P}^d$ has a resolution almost of order $\sigma_X - \alpha_X + 3$. The next lower case is somewhat more interesting.

COROLLARY 5.6. *The following conditions are equivalent.*

a) X has a resolution almost of order $\sigma_X - \alpha_X + 2$.

b) $\alpha_{d-1\beta_{d-1}} \leq \sigma_X + d - 1$

c) The multiplication map $R_1 \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+1}$ is injective.

PROOF. In view of Proposition 5.1 and Theorem 5.4 we only have to show "c) $\Rightarrow$ a)". Choose $\nu \in \{1,\ldots,\beta_d\}$ as in 5.4, and let $r_i \in R_{\alpha_{di}-\sigma_X-d}$ be such that $r_1\varphi_\nu + \cdots + r_{\beta_d-\nu+1}\varphi_{\beta_d} = 0$. For $i = 1,\ldots,\Delta_X$ we have $r_i \in R_0 = k$. Since $\{\varphi_1,\ldots,\varphi_{\beta_d}\}$ is minimal, this implies $r_1 = \ldots = r_\Delta = 0$. For $i = \Delta_X + 1,\ldots,\beta_d - \nu + 1$ we have $r_i \in R_1$. Since $R_1 \otimes (\omega_R)_{-\sigma_X} \longrightarrow (\omega_R)_{-\sigma_X+1}$ is injective, the relation $r_{\Delta_X+1}\varphi_{\beta_d-\Delta_X+1} + \cdots + r_{\beta_d-\nu+1}\varphi_{\beta_d} = 0$ implies $r_{\Delta+1} = \cdots = r_{\beta_d-\nu+1} = 0$. An application of the theorem now finishes the proof. ■

The following corollary is a special case of Corollary 5.5.

COROLLARY 5.7. *If $X \subseteq \mathbb{P}^d$ is nondegenerate and has a resolution almost of order $\sigma_X - \alpha_X + 2$, then $\Delta H_X(\sigma_X) \geq d \cdot \Delta_X$.*

REMARK. For subschemes $X \subseteq \mathbb{P}^2$ the projective resolution is particularly short. In this case "X has a resolution almost of order $\sigma_X - \alpha_X + 2$" easily translates into "I_X is generated by elements of degree $\leq \sigma_X+1$". Notice that I_X is always generated by elements of degree $\leq \sigma_X+2$ because of what we explained at the beginning of this section. If I_X is generated by elements of degree $\leq \sigma_X + 1$, Corollary 5.7 yields that the Hilbert function of X satisfies $\Delta H_X(\sigma_X) \geq 2\Delta_X$.

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