A CLASSIFICATION OF n-ABELIAN GROUPS

J. L. ALPERIN

1. Introduction. The concept of an abelian group is central to group theory. For that reason many generalizations have been considered and exploited. One, in particular, is the idea of an n-abelian group (**6**). If n is an integer and $n > 1$, then a group G is n-abelian if, and only if,

$$(xy)^n = x^n y^n$$

for all elements x and y of G. Thus, a group is 2-abelian if, and only if, it is abelian, while non-abelian n-abelian groups do exist for every $n > 2$.

Many results pertaining to the way in which groups can be constructed from abelian groups can be generalized to theorems on n-abelian groups (**1**; **2**). Moreover, the case of $n = p$, a prime, is useful in the study of finite p-groups (**3**; **4**; **5**). Moreover, a recent result of Weichsel (**9**) gives a description of all p-abelian finite p-groups. It is this classification that we wish to extend and simplify. We shall prove the following result.

THEOREM 1. *A group is n-abelian if, and only if, it is a homomorphic image of a subgroup of the direct product of an abelian group, a group of exponent dividing n, and a group of exponent dividing $n - 1$.*

Abelian groups and groups of exponent dividing n are clearly n-abelian. Moreover, if G is a group of exponent dividing $n - 1$ and x is an element of G, then $x^n = x$, the nth power map is the identity map and again G is n-abelian. Furthermore, direct products, subgroups, and homomorphic images of n-abelian groups are also n-abelian so that half of the theorem is now obvious. It remains to show that an arbitrary n-abelian group can be so described. This can also be rephrased in terms of varieties of groups: The join of the varieties of abelian groups, groups of exponent dividing n, and groups of exponent dividing $n - 1$ is the variety of n-abelian groups. Here, as always, we implicitly assume that $n > 1$.

We shall also derive two consequences of the above theorem for finite groups.

COROLLARY 1. *A finite group is n-abelian if, and only if, it is a homomorphic image of a subgroup of the direct product of a finite abelian group, a finite group of exponent dividing n and a finite group of exponent dividing $n - 1$.*

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While the statement of the preceding result is just that of the theorem with "group" replaced by "finite group", it is not an immediate consequence and requires a lemma which may be of some independent interest.

The next assertion is a direct consequence of the corollary just given.

COROLLARY 2 (Weichsel (**9**)). *A finite p-group is p-abelian if, and only if, it is a homomorphic image of a subgroup of the direct product of a finite abelian p-group and a finite p-group of exponent p.*

Our notation is all quite standard. For example, if x and y are elements of the group G, then

$$[x, y] = x^{-1}y^{-1}xy, \qquad x^y = y^{-1}xy.$$

Moreover, if k is a positive integer, then G^k is the subgroup of G generated by all kth powers of elements of G.

The remainder of this paper is organized in the following manner. Section 2 contains a proof of Theorem 1, § 3 is devoted to a useful lemma and the derivation of the two corollaries, while § 4 contains a short and direct proof of the last of these corollaries.

2. The proof of Theorem 1. The heart of the argument is contained in two lemmas, the first of which consists of a number of identities while the second is, in fact, a special case of the theorem.

LEMMA 1. *If x and y are elements of an n-abelian group, then*
(a) $(xy)^{n-1} = y^{n-1}x^{n-1}$,
(b) $[x, y]^n = [x^n, y]$,
(c) $[x, y]^{n-1} = [y, x^{-(n-1)}]$,
(d) $[x^n, y^{n-1}] = 1$,
(e) $[x, y]^{n(n-1)} = 1$.

The results in this lemma are more or less contained in (**1**; **2**); we give the proofs for the convenience of the reader.

Proof. Each assertion follows from a direct and simple calculation.

(a) $$(xy)^{n-1} = ((yx)^y)^{n-1} = ((yx)^{n-1})^y = ((yx)^n x^{-1}y^{-1})^y$$
$$= (y^n x^n x^{-1} y^{-1})^y = y^{n-1}x^{n-1}.$$

(b) $$[x, y]^n = (x^{-1}x^y)^n = x^{-n}(x^y)^n = (x^n)^{-1}(x^n)^y = [x^n, y].$$

(c) $$[x, y]^{n-1} = (x^{-1}x^y)^{n-1} = (x^y)^{n-1}x^{-(n-1)} \text{ (by (a))}$$
$$= (x^{n-1})^y x^{-(n-1)} = [y, x^{-(n-1)}].$$

(d) $(x^n)^y = (x^y)^n = (y^{-1}xy)^n = (y^n)^{-1}x^n y^n$ so that $y^{n-1}x^n = x^n y^{n-1}$, as desired.

(e) $$[x, y]^{n(n-1)} = [y, x^{-(n-1)}]^n \text{ (by (c))} = [y^n, x^{-(n-1)}] \text{ (by (b))} = 1 \text{ (by (d))}.$$

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LEMMA 2. *If G is an n-abelian group and G/G' is torsion-free, then*

$$G' \cap G^n \cap G^{n-1} = 1.$$

Proof. Let g be any element of the intersection; we shall prove that g is the identity. First, $g \in G^n$, hence g equals a product of nth powers of elements of G. However, G is n-abelian, thus g is an nth power and $g = h^n$, for some h in G. Moreover, $h \in G^{n-1}$. Indeed, if $h \notin G^{n-1}$, then $g = h^n \notin G^{n-1}$ since n and $n - 1$ are relatively prime. Thus, h equals a product of $(n - 1)$st powers, thus $h = k^{n-1}$, $k \in G$, by Lemma 1 (a). Furthermore, $k \in G'$ as $g \in G'$, $k^{n(n-1)} = g$ and G/G' is torsion-free. Thus, there is a positive integer r and elements x_i, y_i, $1 \leqq i \leqq r$, of G such that

$$k = [x_1, y_1] \cdots [x_r, y_r].$$

Hence, as G is n-abelian,

$$k^n = [x_1, y_1]^n \cdots [x_r, y_r]^n$$

and, by Lemma 1 (a),

$$k^{n(n-1)} = [x_r, y_r]^{n(n-1)} \cdots [x_1, y_1]^{n(n-1)}.$$

However, each factor of this product is the identity, by Lemma 1 (e), thus $g = k^{n(n-1)} = 1$, as desired. This proves the lemma.

Proof of Theorem 1. Let H be an arbitrary n-abelian group and let F be a free group which has H as a homomorphic image. Furthermore, let R be the least normal subgroup of F with n-abelian quotient group and set $G = F/R$. Since H is n-abelian, it follows that H is a homomorphic image of G. Hence, we need only prove that G is isomorphic with a subgroup of the direct product of an abelian group, a group of exponent dividing n, and a group of exponent dividing $n - 1$.

However, $F' \supseteq R$ as F/F' is certainly n-abelian, thus $G/G' \simeq F/F'$ and is torsion-free. Thus, Lemma 2 yields

$$G' \cap G^n \cap G^{n-1} = 1.$$

Thus, the kernel of the homomorphism of H into the direct product of G/G', G/G^n, and G/G^{n-1} is trivial, and the proof is complete.

3. Derivation of the corollaries. At this point we need to recall a few elementary definitions and facts. A quotient of a subgroup of a group G is called a *section* of G. Moreover, if H is a group isomorphic with a section of G, then we shall say that H *is* a section of G.

We also have to review the description of subgroups of a direct product. Let A_1 and A_2 be groups and assume that H is a subgroup of the direct product $A_1 \times A_2$. The projection of H in A_1 is the subgroup P_1 of A_1 consisting of all $a_1 \in A$ such that there is $a_2 \in A_2$ with $(a_1, a_2) \in H$. The intersection I_1 of H

and A_1 is the subgroup of A_1 consisting of all $a_1 \in A_1$ with $(a_1, 1) \in H$. The projection P_2 and intersection I_2 of H with A_2 are defined similarly. It follows that I_i is normal in P_i, $i = 1, 2$, the corresponding quotients are isomorphic, and there is an isomorphism θ of P_1/I_1 onto P_2/I_2 such that (a_1, a_2) is in H if, and only if, $a_i \in P_i$, $i = 1, 2$, and $(I_1a_1)\theta = I_2a_2$.

The key step in the derivation of Corollary 1 from Theorem 1 is an application of the following result.

Lemma 3. *If G is a finite group which is a section of the direct product $A_1 \times A_2$ of groups A_1 and A_2, then G is a section of the direct product of finite sections S_1 and S_2 of A_1 and A_2, respectively, provided any finitely generated group which is a section of A_1 and A_2 is finite.*

We have been unable to remove the condition on finitely generated groups and we strongly suspect the lemma is false without some such hypothesis.

Proof. Let G be isomorphic with X/K, where X is a subgroup of $A_1 \times A_2$ and K is a normal subgroup of X. We assert that we may assume that X is finitely generated. Indeed, given an isomorphism of X/K onto G, choose for each $g \in G$ an element of X mapped to g and let X_0 be the subgroup of X generated by this finite set of elements. It follows that the given map of X/K onto G induces an isomorphism of $X_0/X_0 \cap K$ onto G so that we may replace X by X_0.

Let X_i and Y_i be the projection and intersection, of X with A_i, $i = 1, 2$, respectively. Let $Y = Y_1 \times Y_2$ so that Y is a normal subgroup of X as Y_i is normal in X_i, $i = 1, 2$, and $X \subseteq X_1 \times X_2$. Let $L = Y \cap K$ so that L is also a normal subgroup of X. Moreover, let L_i be the intersection of L and A_i, $i = 1, 2$, thus $L_i \subseteq Y_i$ as $L \subseteq Y$. We now claim that the following assertions hold:

(1) L_i is a normal subgroup of X_i, $i = 1, 2$;

(2) The index of L_i in X_i is finite, $i = 1, 2$;

(3) G is a section of $X_1/L_1 \times X_2/L_2$.

Once these three statements are established, our proof will be complete by taking $S_i = X_i/L_i$, $i = 1, 2$.

The last assertion is easy to prove. Indeed $L_1 \times L_2 \subseteq L \subseteq K$, thus we have

$$L_1 \times L_2 \subseteq K \subseteq X \subseteq X_1 \times X_2$$

and X/K is a section of $X_1 \times X_2/L_1 \times L_2$ which is isomorphic with $X_1/L_1 \times X_2/L_2$.

As for the first statement, $L_i \subseteq X_i$ as we have just seen. Moreover,

$$L_1 \times 1 = (A_1 \times 1) \cap L,$$

hence $L_1 \times 1$ is normal in X. Thus, the image X_1 of X in the projection on A_1 normalizes the image L_1 of $L_1 \times 1$. Similarly, L_2 is normal in X_2.

Finally, we establish (2). First, we know that X_1/Y_1 and X_2/Y_2 are isomorphic, by the remarks preceding the statement of the lemma. Moreover, X

is finitely generated, hence X_i is also, $i = 1, 2$, being a homomorphic image of X. Thus, by our hypothesis, X_i/Y_i is finite. However, $L \subseteq Y$ hence $L_i \subseteq Y_i$ and we, therefore, only need to demonstrate that Y_i/L_i is finite.

For this purpose, let M_i be the projection of L on A_i, $i = 1, 2$, thus $L_i \subseteq M_i \subseteq Y_i$, L_i is normal in M_i, and there is an isomorphism θ of M_1/L_1 onto M_2/L_2 so that $(m_1, m_2) \in M_1 \times M_2$ lies in L if, and only if, $(L_1m_1)\theta = L_2m_2$. We set $M = M_1 \times M_2$.

Now Y/L is finite since $Y/L = Y/Y \cap K$ is isomorphic with a subgroup of X/K. Hence, the projection M_i of L on A_i has finite index in the projection Y_i of Y on A_i. Moreover, $L_i \subseteq M_i \subseteq Y_i$, thus it remains only to show that M_i/L_i is finite, $i = 1, 2$. However, M/L is finite, being a subgroup of Y/L. Hence, it suffices to show that there is a one-to-one map of M_1/L_1 into M/L.

If $t \in M_1$, we map $L_1t \in M_1/L_1$ to $L(t, 1)$ and $L(t, 1) \in M/L$ since $M = M_1 \times M_2$. This is well-defined since $x \in L_1$ yields $L(xt, 1) = L(t, 1)$ as $(x, 1) \in L$. Moreover, suppose that $s \in M_1$ and $L(t, 1) = L(s, 1)$. This implies that $(ts^{-1}, 1) \in L$. Hence $(L_1ts^{-1})\theta = L_2$, and thus $ts^{-1} \in L_1$ and $L_1s = L_1t$ and the map is one-to-one. This proves the second assertion and establishes the lemma.

We are now ready to prove the corollaries.

Proof of Corollary 1. As for the theorem, we need only show that a finite n-abelian group has the desired structure. However, if G is such a group, then G is a section of the direct product of an abelian group A, a group A_n of exponent dividing n, and a group A_{n-1} of exponent dividing $n - 1$. A finitely generated group which is a section of A and a section of $A_n \times A_{n-1}$ is finite as it is finitely generated, abelian, and of finite exponent. Thus, by Lemma 3, G is a section of $B \times C$, where B is a finite section of A and C is a finite section of $A_n \times A_{n-1}$. Moreover, any group which is a section of A_n and a section of A_{n-1} is the identity, as n and $n - 1$ are relatively prime, thus again by Lemma 3, C is a section of $B_n \times B_{n-1}$, where B_n is a finite section of A_n and B_{n-1} is a finite section of A_{n-1}. Thus, G is a section of $B \times B_n \times B_{n-1}$, and the result is established.

Proof of Corollary 2. Let P be a p-abelian finite p-group. By the previous result, P is a section of the direct product of a finite abelian group B, a finite group B_p of exponent dividing p, and a finite group B_{p-1} of exponent dividing $p - 1$. Now B is the direct product of a p-subgroup B_0 and a subgroup B' of order prime to p, thus P is a section of $A_p \times A'$, where $A_p = B_0 \times B_p$ and $A' = B' \times B_{p-1}$. We need only see that P is a section of A_p.

However, suppose that P is isomorphic to X/Y, where X and Y are subgroups of $A_p \times A'$ and Y is normal in X. However, A_p and A' are of coprime orders, thus $X = X_p \times X'$ and $Y = Y_p \times Y'$, where X_p and X' are the projections of X on A_p and A', and Y_p and Y' are the projections of Y on A_p and A'. Thus,

$$P \simeq X_p/Y_p \times X'/Y'.$$

However, X'/Y' has order prime to p, thus $Y' = X'$, $P \simeq X_p/Y_p$, and P is a section of A_p.

As in all the above cases, the other half of the corollary is obvious and the proof is, therefore, complete.

4. Another proof of Corollary 2. The arguments we have developed give a short direct proof of Weichsel's theorem and avoid the appeal to stronger results. For this reason, we sketch the quick argument.

Let P be a p-abelian finite p-group. Suppose that P is a d-generator group of class of nilpotence c and exponent p^e, where c, d, and e are positive integers. Let F be a free group on a set of d free generators. Let R be the least normal subgroup of F whose quotient is nilpotent of class at most c, has exponent dividing p^e, and is p-abelian. Let $G = F/R$, thus G is a finite p-group as it is finitely generated, nilpotent, and of exponent dividing p^e. Moreover, P is a homomorphic image of G since P is an image of F and the corresponding kernel must contain R. Hence, we need only show that G is isomorphic with a subgroup of the direct product of a finite abelian p-group and a finite p-group of exponent dividing p.

However, there is a homomorphism of G into $G/G' \times G/G^p$ constructed from the natural maps of G onto G/G' and G onto G/G^p. The kernel of this homomorphism is $G' \cap G^p$, thus we need only prove that $G' \cap G^p = 1$ inasmuch as G/G' is a finite abelian p-group and G/G^p is finite and of exponent p.

To do this we first remark that G/G' is the direct product of d cyclic groups of order p^e. Indeed, such a direct product is a d-generator p-abelian group of class one and exponent p^e and hence is a homomorphic image of G and consequently of G/G'. On the other hand, G/G' is an abelian p-group on d generators and of exponent dividing p^e, thus it is as described. In particular, any element of G/G' of order dividing p is equal to the p^{e-1}st power of an element of G/G'.

Finally, suppose that $g \in G' \cap G^p$; we shall show that $g = 1$ and conclude the proof. Since $g \in G^p$ and G is p-abelian, we have: $g = h^p$ for some h in G. However, $g \in G'$, thus $(G'h)^p = 1$ and $h = x^{p^{e-1}}y$, where $x \in G$, $y \in G'$, by our remarks above. Hence $h^p = x^{p^e}y^p = y^p$ as G is p-abelian of exponent p^e. However,

$$y = [x_1, y_1] \cdots [x_r, y_r]$$

for suitable x_i, y_i in G, $1 \leqq i \leqq r$. Thus

$$g = h^p = y^p = [x_1, y_1]^p \cdots [x_r, y_r]^p.$$

However, $[x_i, y_i]^p = 1$, by Lemma 1 (e), since G is a p-group. Thus, $g = 1$, as desired.

5. Concluding remarks. Our main result also has some consequences in a slightly different direction. Indeed, it is an easy consequence of Theorem 1 that a group, in which the taking of nth powers is an automorphism, is a homomorphic image of a subgroup of the direct product of an abelian group and a group of exponent dividing $n - 1$. This result, in turn, leads to easy derivations of the results of Trotter (**8**).*

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Northeastern University,
Boston, Massachusetts;
University of Chicago,
Chicago, Illinois

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