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REMARKS ON QUASIGROUPS OBEYING A GENERALIZED ASSOCIATIVE LAW

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T. Evans in [4] introduced a concept generalizing the associative law for groups. Certain very restricted cases of this generalization had occurred, somewhat as side results, in the considerations of A. K. Suškevič [7], D. C. Murdoch [5], [6], and R. H. Bruck [2] associated with quasigroups which obey theorems generalizing certain results in group theory.

In this paper we consider quasigroups obeying an instance of Evans' generalization. We show two means of constructing such quasigroups, and we demonstrate these constructions with several examples. The principal result of this paper is a theorem which reveals a peculiar trait of quasigroups obeying an instance of the generalized associative law we are considering. Such a quasigroup obeys a law expressed with permutations in which the nontrivial permutations involved either are both automorphisms of the quasigroup or else both fail to distribute over any product from the quasigroup. The examples demonstrate that, indeed, both cases do occur.

In this paper the language is, excepting a minor introduction of terminology for the simplification of the statements, in accordance with that of Bruck [3].

I. A quasigroup is an ordered pair (G, θ) for which G is a non-empty set and θ is a binary operation on G which conforms to the requirement that for each ordered pair (x, y) with x, y in G there is a unique ordered pair (u, v) with u, v in G such that $x\theta u = v\theta x = y$. For discussion on quasigroups one may refer to Bruck [3].

Let $X = (G, \theta)$ and $Y = (H, \phi)$ be two quasigroups. We say that Y is an isotope of X provided there exist three one-to-one correspondences A, B, C from the set G to the set H such that $(x\theta y)A = (xB)\phi(yC)$ whenever x, y are in G. Y is a principal isotope of X provided that G and H are the same set, and A, B, C exist having the properties stipulated above and A is the identity map on G. It has been shown (A. A. Albert [1]) that each isotope of a given quasigroup X is a quasigroup isomorphic to some principal isotope of X. Thus, we may restrict our attention to principal isotopes of X. For discussion on isotopy one may refer to Albert [1] and Bruck [3].

Let $X = (G, \theta)$ be a quasigroup. We say that X obeys the generalized associative law (restricted to permutations) of T. Evans [4] if and only if there exist permutations A, B, C, D, E, F on G such that

$$x\theta((y\theta z)D) = ((((xA)\theta(yB))E)\theta(zC))F \tag{1}$$

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for all x, y, z in G. We are interested in the case in which D, E, F are the identity map, and we shall say that X obeys the *associative law of the form* (A, B, C) iff (1) holds with D, E, F all equal to the identity map on G.

A group is simply a quasigroup which obeys the *associative law of the form* (I, I, I) where I is the identity map. Evans [4] shows that a necessary and sufficient condition that a quasigroup X be the principal isotope of some group is simply that X obey (1) for some permutations A, B, C, D, E, F. From Albert ([1] p. 511) we know that whenever R is a group and S is an isotope of R such that S has a two-sided identity element, then S is a group isomorphic to R. Thus, any quasigroup which obeys the *associative law of the form* (A, B, C) for some A, B, C is, up to isomorphism, an isotope of exactly one group. Furthermore, any such quasigroup which possesses a two-sided identity element is a group

Let $X=(G, \theta)$ be a quasigroup, with permutations A, B on G, and let $Y=(G, \phi)$ be that principal isotope of X obtained by having $x\phi y=(xA)\theta(yB)$ for all x, y in G. We shall refer to Y as *the isotope of* X *by* (A, B). If A, B are automorphisms of X, we shall call Y the *natural* isotope of X by (A, B). If X possesses a unique idempotent element and if Z is both the *natural* isotope of X by (C, D) and the *natural* isotope of X by (E, F), then, clearly, we must have $C=E$ and $D=F$. Thus, a given quasigroup can be a *natural* isotope of a given group in at most one way. If we suppose that Y is the *natural* isotope of X by (A, B), then any automorphism C of X is also an automorphism of Y if $AC=CA$ and $BC=CB$, where we represent mapping composition by juxtaposition—i.e., $xAB=(xA)B$.

II. The following statement enables us to construct from groups non-trivial quasigroups which obey the associative law of the form (A, B, C). Once such quasigroups are constructed, one may then proceed to use them to continue forming more complicated quasigroups.

Statement 1. Let X be a quasigroup which obeys the associative law of the form (A, B, C). Let L, M be two automorphisms of X, and let Y be the natural isotope of X by (L^{-1}, M). Then

(i) Y obeys the associative law of the form $(L^{-1}AL^2, L^{-1}MBLM^{-1}, M^2CM^{-1})$.
(ii) If X is a group and not both L and M are the identity map, then Y is not a group.

The case of Statement 1 for $A=B=C=I$, the identity map, is in Suškevič [7], and both that case and the case for $A=B=C=L^{-1}M^{-1}LM=I$ are in Murdoch [5, 6] In [5], [6] and [7], the quasigroups considered possess many resemblances to groups—e.g., they obey an extension to quasigroups of the Lagrange Theorem for Finite Groups.

Of interest in Murdoch [5, 6] is the *abelian* quasigroup—i.e., a quasigroup $X=(G, \theta)$ such that $(x\theta y)\theta(z\theta u)=(x\theta z)\theta(y\theta u)$ for all x, y, z, u in G. In [6] Murdoch shows that such an "abelian" quasigroup must be an isotope of some abelian group (abelian groups are also *abelian* quasigroups), and that if X possesses an idempotent element, then for some abelian group Y, X is the natural isotope of Y by (A^{-1}, B) for some A, B such that $AB=BA$, whence X obeys the associative law of the form (A, I, B), where I is the identity map. Additionally, we find that if Z is the natural isotope of an abelian group U by (A, B), then Z is an abelian quasigroup, provided that $AB=BA$.

Bruck [2] further considers the abelian quasigroup. If $X=(G, *)$ is an abelian group with f in G and if L, M are two automorphisms of X such that $LM=ML$ and N is the mapping from G to G defined by $xN=f*x$, then the isotope U of X by (L^{-1}, MN) is an abelian quasigroup. Every abelian quasigroup V can so be obtained from some abelian group X (Bruck [2] p. 46). Furthermore, Statement 1 shows that U obeys the associative law of the form (L, I, M), where I is the identity map, if LM leaves f fixed. The following statement is a partial generalization of this construction and enables us to construct an isotope Y in example 1 of a non-abelian group such that Y obeys the associative law of the form (L, I, M), where I is the identity map; L, M fail to distribute over each product in Y, and, of course, Y will not be an abelian quasigroup.

Statement 2. Let $X=(G, *)$ be a group. Let f in G satisfy $f*x=x*f$ for all x in G. Let L, M be two automorphisms of X such that $fLM=f$ and $LM=ML$, and let N be the mapping from G to G defined by $xN=f*x$ for all x in G. Then the isotope $Y=(G, \theta)$ of X by (L^{-1}, MN) obeys the associative law of the form (L, I, M), where I is the identity map on G. Furthermore, L and M are automorphisms of Y if and only if $fL=f$. If $fL\neq f$, then $(x\theta y)L\neq(xL)\theta(yL)$ and $(x\theta y)M\neq(xM)\theta(yM)$ for all x, y in G.

THEOREM. *Let $X=(G, \theta)$ be a quasigroup which obeys the associative law of the form (A, I, B) where I is the identity map on G.*

Then either A and B are both automorphisms of X and $AB=BA$ or else

$$(x\theta y)A \neq (xA)\theta(yA) \quad \textit{and} \quad (x\theta y)B \neq (xB)\theta(yB)$$

for all x, y in G.

Proof. One may verify by several applications of the associative law under consideration that

$$((xA^{-1})\theta(yA^{-1}))\theta((z\theta u)B) = ((x\theta y)A^{-1})\theta((zB)\theta(uB)) \tag{2}$$

for all x, y, z, u in G. Since X is a quasigroup, from (2) it readily follows that either A and B are both automorphisms of X or else $(x\theta y)A\neq(xA)\theta(yA)$ and $(x\theta y)B\neq(xB)\theta(yB)$ for all x, y in G.

Suppose that A and B are both automorphisms of X. Then for each x in G we have

$$\begin{aligned}((x\theta x)A)\theta(xBA) &= ((x\theta x)\theta(xB))A = ((xA^{-1})\theta(x\theta x))A \\ &= x\theta((x\theta x)A) = x\theta((xA)\theta(xA)) = ((xA)\theta(xA))\theta(xAB) \\ &= ((x\theta x)A)\theta(xAB),\end{aligned}$$

and therefore $xBA=xAB$. Thus, $AB=BA$. Q.E.D.

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If A and B are both automorphisms in the theorem, then it follows (see Murdoch [6], p. 408) that X is a natural isotope of some group $(G, *)$ by (A^{-1}, B). Such quasigroups are briefly studied by Murdoch in [6].

If we suspect that a quasigroup X obeys the associative law of the form (A, I, B), where I is the identity map, we would reject this suspicion if we would find that A distributes over one product in X and fails to distribute over another product in X. If we know that X obeys the associative law of the form (A, I, B), we would determine, by the use of our theorem, whether or not A and B are automorphisms of X by merely picking a product in X and determining whether or not A distributes over that product.

Examples. The following examples will make clear that our theorem is not vacuous. From the second example we shall see that it is possible for an abelian quasigroup to obey the associative law of the form (L, I, M), where I is the identity map, for non-trivial L, M, where L and M fail to be automorphisms. The first example demonstrates the possibility for non-abelian quasigroups. Again, in the second example, we have a quasigroup which obeys the associative law of the form (R, M, N) where R, M, N are all different from the identity map and such that N is an automorphism while R and M fail to be automorphisms.

1. Let $Q=(H, *)$ be the quaternion group of order 8 with $H=\{1, -1, i, -i, j, -j, k, -k\}$ and with the usual relations holding—i.e., $(-1)*(-1)=1, i*i=j*j=k*k=-1, i*j=k, j*i=-k$, and with 1 being the identity element. Let $P=Q \otimes Q \otimes Q \otimes Q=(K, \phi)$ be the direct product of four factors Q. Take $f=(-1, 1, 1, 1)$ and let the mapping N from K to K be defined by $xN=f\phi x$ for all x in K. Furthermore, let L be that automorphism of P such that $(x, y, z, u)L=(y, z, x, u)$ for all x, y, z, u in H, and let $M=L^2$. We apply Statement 2 and conclude that the isotope $Y=(K, \theta)$ of P by (M, MN) obeys the associative law of the form (L, I, M), where I is the identity map. Furthermore, we conclude that $(x\theta y)L \neq (xL)\theta(yL)$ and $(x\theta y)M \neq (xM)\theta(yM)$ for all x, y in K. Since Y is an isotope of the non-abelian group P, we conclude that Y is not an abelian quasigroup.

2. Let $X=(T, *)$ be the group on five elements with $\{1, 2, 3, 4, 5\}$ and $x*y=x+y-1$ (modulo 5). Let A be the permutation (2453), B be (25)(34), C be (13524) and D be (1452). We note that A and B are automorphisms of X. Thus, we apply Statement 1 to the isotope Y of X by (A^{-1}, B) and conclude that Y obeys the associative law of the form (A, I, B) where I is the identity map. Furthermore, since $AB=BA$, Y is abelian. We note that A, B, C, D are all automorphisms of Y. We define U to be the isotope of Y by (C^{-1}, D) and $Z=(T, \theta)$ to be the isotope of Y by (C^{-1}, C). By applying Statement 1, we conclude that U obeys the associative law of the form (R, M, N) where $R=(1435)$, $M=(15432)$ and $N=(1523)$. We note that N is an automorphism of U while M, R fail to be automorphisms of U. Furthermore, Z is abelian since $C^{-1}C=CC^{-1}$ and Y is abelian, but Z possesses no idempotent elements and hence is not a natural isotope of X or any other group.

Finally, we find by Statement 1 that Z obeys the associative law of the form (R, I, S) where $S=(15)(24)$, but

$$(1\theta 3)R = 5 \neq 2 = (1R)\theta(3R),$$

so that R fails to be an automorphism of Z. By our theorem, we see that this means S must fail to be an automorphism of Z; in fact, both R and S *must* fail to distribute over every product in Z!

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