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LATTICE COVERINGS AND THE DIAGONAL GROUP

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Let M be any bounded set in n-dimensional Euclidean space. Then almost all n-dimensional lattices L with determinant 1 have the following property: There exists a diagonal transformation D with determinant 1 (depending on L) such that L does not cover space with DM. Moreover, if M has non-empty interior, the exceptional (null-) set contains at least enumerably many diagonally non-equivalent lattices.

1. Let $\mathcal{L}$ denote the space of lattices in n-dimensional Euclidean space $\mathbb{R}^n$ with determinant 1, equipped with the usual measure and topology (see [11] Section 17, Section 19), and let $\mathcal{D}$ be the group of nonsingular diagonal $n \times n$-matrices. The main purpose of this note is to prove the following result.

PROPOSITION. *Let M be a bounded set however large in $\mathbb{R}^n$. Then all lattices $L \in \mathcal{L}$ except those from a null-set in $\mathcal{L}$ (in the sense of the measure introduced) have the following property: L can be made a non-covering lattice for DM by applying a suitable diagonal transformation $D \in \mathcal{D}$ (depending on L) with $|\det D| = 1$.*

We will obtain this as a corollary of the Theorem to be stated below.

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Also it will become clear from (1) that our assertion is certainly not trivial at least for sets M with non-empty interior. The present result can be regarded as a metrical contribution to the following general inhomogeneous problem of the "first type" (in Bambah's terminology [1] p.120; see also [7] p.208, and [11] p.407): Given some class $\mathcal{M}$ of (measurable) sets in $\mathbb{R}^n$ and a subgroup G of the general linear group on $\mathbb{R}^n$, determine the infimum $\delta(\mathcal{M})$ of all positive numbers δ with the property that, for any $M \in \mathcal{M}$ with $\mathrm{vol}(M) \leq \delta$ and any lattice $L \in \mathcal{L}$, there exists a transformation $G \in G$ such that the Minkowski sum $GM + L$ does not cover $\mathbb{R}^n$. Taking in particular (in view of our present objective) the diagonal group $\mathcal{D}$, we reformulate this by introducing the function

$$\delta_M(L) = \sup\{vol(DM) \mid D \in \mathcal{D}, \ L \text{ not a covering lattice for } DM\} .$$

For bounded M this quantity is positive, possibly infinite (Note that $\mathbb{R}^n$ is not covered by $DM + L$ if and only if L is strictly admissible for some translate of DM). Then obviously

$$\delta(\mathcal{M}) = \inf\{\delta_M(L) \mid M \in \mathcal{M} , L \in \mathcal{L}\} .$$

This should be compared with the corresponding covering problem of the second type, involving the quantity

$$\Theta_M(L) = \inf \{vol(DM) \mid D \in \mathcal{D}, L \text{ a covering lattice for } DM\} .$$

D.B. Sawyer [15] proved that for arbitrary lattice $L \in \mathcal{L}$ one has

$$\sup_K \Theta_K(L) = n^n/n!$$

where the sup is taken over all n-dimensional convex bodies K, not necessarily 0-symmetric (Indeed, by the affine invariance of this class of sets, this need only be proved for the integer lattice $\mathbb{Z}^n$) .

We remark that there is an analogous packing problem in connection with the diagonal group which was suggested by Mordell in 1936 and has attracted much interest since (see for example [11] Section 24, [7] p.191 f.). It is concerned with the function

$$\rho_K(L) = \sup\{vol(DK) \mid D \in \mathcal{D}\ ,\ \ L \text{ a packing lattice for } DK\}\ .$$

If K is restricted to the class $\mathcal{K}$ of O-symmetrical bounded convex bodies in $\mathbb{R}^n$, then trivially $\rho_K(L) \le 1$, by Minkowski's theorem. The question is to decide whether the values of this function are bounded away from 0 when K and L are varying over $\mathcal{K}$ and $\mathcal{L}$. Rogers [14] confirmed this by finding explicit positive lower bounds in all dimensions. These estimates can be improved when further conditions are imposed on the bodies considered. This was done by Hlawka [10] and Davenport for the classes of parallelepipeds DB ($D \in \mathcal{D}$, B the unit cube) and ellipsoids DS (S the unit sphere), respectively.

A conjecture by Gruber claiming that $\rho_B(L)$ takes the value 1 (which is the greatest possible value) at almost all $L \in \mathcal{L}$ is still open for $n \ge 3$ (for metrical and topological results in this direction see [8,12]).

2. Here we obtain an affirmative answer to the inhomogeneous version of this conjecture. The following Theorem clearly implies the above Proposition.

THEOREM. *Let M be any bounded (measurable) set in $\mathbb{R}^n$ with non-empty interior and let $\mathcal{L}'$ denote the set of lattices $L \in \mathcal{L}$ with $\delta_M(L) < \infty$. Then*

(1) $\mathcal{L}'$ contains an enumerable set of diagonally inequivalent lattices; moreover, $\delta(M) = \inf\{\delta_M(L) \mid L \in \mathcal{L}\}$ is a finite positive number.

(2) $\mathcal{L}'$ is a null-set in $\mathcal{L}$.

Proof. We find it convenient to consider the Minkowski sets $B_p : (|x_1|^p + \ldots + |x_n|^p)^{1/p} < 1$, that is the (open) unit balls of the p-norms $(1 \le p \le \infty)$. In particular, letting p tend to ∞, we obtain the unit cube $B = B_\infty$. By Ω_i $(i=1,\ldots,2^n)$ we denote the (open) coordinate orthants, listed in any order. We introduce the auxiliary function

$$\alpha_p(L) = \max_{i=1,\dots,2^n} \sup\{vol(DB_p \cap \Omega_i) \mid D \in \mathcal{D},\ L \text{ admissible for } DB_p \cap \Omega_i\} .$$

We remark that the functions $\alpha_p(L)$ and $\delta_B(L)$, $B = B_p$, can be described in terms of a semiregular continued fraction expansion for two-dimensional lattices (see [13]). We collect some basic relations between the above functions. First, it follows immediately from the definition that $\delta_M(L)$ is diagonally invariant with respect to both M and L : for any set M and any $L \in \mathcal{L}$ we have

$$\delta_{DM}(L) = \delta_M(L) \qquad (D \in \mathcal{D}) , \tag{3a}$$

$$\delta_M(DL) = \delta_M(L) \qquad (D \in \mathcal{D},\ |\det D| = 1). \tag{3b}$$

By use of (3a), it is easily proved that for any bounded set M with non-empty interior (as specified in the Theorem) the inequalities

$$c\ \delta_B(L) \le \delta_M(L) \le c'\ \delta_B(L) \tag{4a,b}$$

hold with some positive constants c, c' independent of L . Here c (respectively c') may be taken as the ratio $\mathrm{vol}(M)/\mathrm{vol}(DB)$ where $D \in \mathcal{D}$ is any diagonal matrix such that a translate of M can be inscribed in DB (respectively DB is contained in some translate of int$M$) . Next we show that

$$n!n^{-n}\ \alpha_1(L) \le \delta_B(L) \le 2^{n-1}\ \alpha_\infty(L) . \tag{5a,b}$$

The first inequality (5a) is easily obtained on comparing the volumes of a (lattice point free) simplex of the form $DT(T\colon x_1+\dots+x_n < 1,$ $x_1,\dots,x_n > 0)$ and a maximal inscribed translate of a parallelepiped of the form $D'B$ $(D, D' \in \mathcal{D})$. For the proof of (5b) take any lattice point free parallelepiped of the form $DB + z$, $z \in R^n$. Eventually enlarge it by moving appropriate facets outward until at least one facet contains a lattice point in its relative interior (the volume will not be decreased by this process). By passing to a suitable translate P' , if necessary, we may assume that the boundary of P' contains the origin 0 . Now P' is the disjoint union of its non-empty intersections P_i (2^{n-1} in number) with the open orthants, and its intersections with the coordinate planes (which do not contribute to the volume). P'

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being lattice point free, the same is true for the subsets P_i . Now (5b) follows from the fact that these subsets are all of the form $D_iB \cap \Omega_i$, $D_i \in \mathcal{D}$.

We proceed to the proof of (1). First we exhibit an enumerable class of lattices L^* generated by real number fields for which $\delta_M(L^*)$ is finite. By (4b) and (5b) it is enough to show that $\alpha_\infty(L^*)$ is finite for these lattices. Let A be an $n \times n$-matrix whose elements $a_{11},\dots,a_{1n}$ form a basis of a totally real number field of degree n , the k-th column consisting of the conjugates of a_{1k} $(k=1,\dots,n)$. Then $L^*=A^*\mathbb{Z}^n$, $A^*=A/|\det A|^{1/n}$, is in $\mathcal{L}$. For the calculation of $\alpha_\infty(L^*)$ it suffices to consider, in each orthant Ω_i , the (enumerable) system of lattice point free (open) parellelepipeds P_{ij} of the form $D_jB \cap \Omega_i$ each of whose facets not contained in a coordinate plane has a lattice point in its relative interior. It is known [3-6] that under the above assumptions this system is periodic in the following sense: For each $i=1,\dots,2^n$, there exists a finite subsystem of parallelepipeds $P_{i1},\dots,P_{im(i)}$, say, such that any P_{ij} is representable as DP_{ik} with some $k \in \{1,\dots,m(i)\}$, $D \in \mathcal{D}$, $|\det D| = 1$. Therefore $\alpha_\infty(L^*) = \max_{i,k} \{\mathrm{vol}(P_{ik})\}$, but this is clearly finite. Thus we have proved that $\delta_M(L^*)$ is finite for any lattice of the type described.

Finally, Hlawka's result [10] ensures the existence of an (explicit) positive lower bound, depending only on the dimension, for the values of $\rho_B(L)$. Together with the inequality (4a) and the trivial estimate $\delta_B(L) \geq \rho_B(L)$ this implies that $\inf\limits_{L} \delta_B(L)$ is positive, which completes the proof of (1).

We turn to the proof of (2). Consider the following conditions for lattices $L \in \mathcal{L}$:

(a) $L\setminus\{\underline{0}\}$ has no points in common with the coordinate planes;

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(b) there exists a sequence of diagonal transformations $D_i \in \mathcal{D}$ with $|\det D_i| = 1$ $(i = 1,2,\ldots)$, depending on L, such that the lattices $L_i = D_i L$ have bases A_i converging (elementwise) to a nonsingular matrix AQ where Q is an $n \times n$-permutation matrix and A contains an $(n-m) \times m$ rectangular block $(1 \le m \le n-1)$ of zeros below the diagonal;

(c) the homogeneous minimum $\inf\{|x_1 x_2 \ldots x_n| \,\big|\, \underline{x} \in L\backslash\{\underline{0}\}\}$ is equal to 0.

It follows from a result of Birch and Swinnerton-Dyer [2] that (c) implies (b). On the other hand, it is well-known that almost all lattices in $\mathcal{L}$ satisfy conditions (a) and (c). Accordingly, for the verification of (2), it will suffice to prove that $\delta_M(L) = \infty$ if L has properties (a) and (b). We do this by showing that $\alpha_1(L_i)$ tends to ∞, as $i \to \infty$. Let $\underline{a}_1^{(i)},\ldots,\underline{a}_n^{(i)}$ be the columns of $A = (a_{kj}^{(i)})$. After renumbering the coordinate axes, if necessary, we may suppose that Q is the identity matrix. Then, for each pair of indices $k,j, k = m+1,\ldots,n;\ j = 1,\ldots,m,$ the elements $a_{kj}^{(i)}$ tend to 0, as $i \to \infty$. Let E_i denote the $(n-1)$-dimensional lattice plane generated by the points $\underline{0},\underline{a}_1^{(i)},\ldots,\underline{a}_{n-1}^{(i)}$. There is a unique vector $\underline{e}^{(i)}$ orthogonal to E_i and normalized by the conditions $|\underline{e}^{(i)}| = 1$, $d_i = (\underline{e}^{(i)}\underline{a}_n^{(i)}) > 0$. Let $h(i)$ be the index of the orthant which contains the point $\underline{e}^{(i)}$. Since $\underline{e}^{(i)}$ belongs to the orthogonal complement of the subspace generated by $\underline{a}_1^{(i)},\ldots,\underline{a}_m^{(i)}$, the components $\underline{e}_j^{(i)}$ $(j = 1,\ldots,m)$ tend to 0, as $i \to \infty$. The points $\underline{a}_1^{(i)},\ldots,\underline{a}_n^{(i)}$ define a cell of L_i, hence the open strip H_i bounded by the lattice planes E_i and $E_i + \underline{a}_n^{(i)}$ is lattice point free. The same is true a fortiori for the simplex $T_i = H_i \cap \Omega_{h(i)}$. Now

$$\mathrm{vol}(T_i) = d_i^n (n!\,|e_1^{(i)} e_2^{(i)} \ldots e_n^{(i)}|)^{-1} \to \infty \quad (i \to \infty),$$

since the sequence d_i has a positive limit and at least one of the components of $\underline{e}^{(i)}$ tends to zero (note that, for $j = 1,\ldots,n,$ the vectors $\underline{a}_j^{(i)}$ tend to a limit $\neq \underline{0}$, and that all components of the unit vectors $\underline{e}^{(i)}$ are non-zero by (a), and remain bounded). It follows that

$$\alpha_1(L_i) \to \infty \quad (i \to \infty) .$$

Using successively the relations (5a),(3b) and (4a), we obtain

$$n!n^{-n}\alpha_1(L_i) \leq \delta_B(L_i) = \delta_B(D_iL) = \delta_B(L) \leq c^{-1}\delta_M(L) .$$

Since this holds for all $i = 1,2,\ldots,$ we end up with $\delta_M(L) = \infty$, as required. This completes the proof of the Theorem.

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