EXTENSION OF A RESULT OF L. LORCH AND P. SZEGO ON HIGHER MONOTONICITY

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1. Introduction and statement of results. We suppose that $\{x_1, x_2, \ldots\}$, $\{\bar{x}_1, \bar{x}_2, \ldots\}$ are increasing sequences of consecutive zeros of non-trivial solutions $y(x)$, $\bar{y}(x)$, of the differential equation

(1) $$y'' + f(x)y = 0$$

on an open interval I, and define, for any fixed $\lambda > -1$,

$$M_k = \int_{x_k}^{x_{k+1}} |y(x)|^{\lambda} dx \qquad (k = 1, 2, \ldots) .$$

Then, with the usual notation for forward differences, i.e.
$\Delta^0 \mu_k = \mu_k$, $\Delta\mu_k = \mu_{k+1} - \mu_k$, $\Delta^n \mu_k = \Delta(\Delta^{n-1}\mu_k)$ $(n = 2, 3, \ldots)$, $(k = 1, 2, \ldots)$, we have the following result:

THEOREM. Let $p(x) = y_1^2(x) + y_2^2(x)$, where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1) over the closure I* of I, and suppose that

(2)
$$(-1)^n p^{(n)}(x) > 0 \qquad (n = 0, 1),$$
$$(-1)^n p^{(n)}(x) \geq 0 \qquad (n = 2, \ldots, N),$$

where the N^{th} derivative exists in the open interval I and the lower order derivatives are continuous in I*. Then

(3) $$(-1)^n \Delta^n M_k > 0 \qquad (n = 0, \ldots, N; \quad k = 1, 2, \ldots),$$

so that, in particular (on taking $\lambda = 0$),

(4) $$(-1)^n \Delta^{n+1} x_k > 0 \qquad (n = 0, \ldots, N; \quad k = 1, 2, \ldots).$$

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Moreover, if $x_1 > \bar{x}_1$, then

(5) $$(-1)^n \Delta^n (x_k - \bar{x}_k) > 0 \qquad (n = 0, \ldots, N; \quad k = 1, 2, \ldots).$$

The theorem remains true if the factor $(-1)^n$ is deleted simultaneously from (2), (3), (4) and (5).

L. Lorch and P. Szego [1, Theorem 2.1] prove these results with the slightly stronger assumptions

(6) $$(-1)^n p^{(n)}(x) > 0 \qquad (n = 0, 1, \ldots, N),$$

instead of (2).

In §3, we give some examples to which the present theorem, though not that of [1], is applicable. We apply the theorem with the modification noted in its last sentence, i.e., with the factor $(-1)^n$ deleted. The applications of Theorem 2.1 [1] made in [1] and [2], were all to cases where the $(-1)^n$ factor is retained. We cannot weaken (2) further by replacing $p'(x) < 0$ by $p'(x) \leq 0$, as the example $f(x) \equiv 1$, $y_1(x) = \sin x$, $y_2(x) = \cos x$ shows. Strict positivity of $p(x)$ is also necessary [1, p.70, Remark 1]. As in [1], all quantities considered here are real.

2. Proof of the theorem. Only minor changes are required in the proof of Theorem 2.1, as given in [1]. It is shown there that (3) and (5) depend on the inequalities

(7) $$(-1)^n D_t^{(n)} \{[x'(t)]^\sigma\} > 0 \qquad (n = 0, \ldots, N),$$

where $x'(t) = p(x)$ and $\sigma > 0$. To prove (7), it is shown in [1; Lemmas 2.1 and 2.2] that its left-hand-side is a homogeneous form in $p^{(0)}(x), p^{(1)}(x), \ldots, p^{(n)}(x)$, each of whose terms is positive, the positivity following from (6). The weaker condition (2) will, in general, only imply non-negativity of these terms. However, (7) still follows because, for each n, the homogeneous form mentioned includes the positive term $(-1)^n [p^{(0)}(x)]^\sigma [p^{(1)}(x)]^n$, as an easy induction shows. The final sentence of the theorem follows on making obvious modifications in the above proof.

3. Applications. (i) When $|a| > 1/2$, the Cauchy-Euler equation

(8) $$y'' + (a^2/x^2)y = 0, \quad 0 < x < \infty,$$

has linearly independent solutions

$$y_1(x) = x^{1/2}\cos(s \log x), \quad y_2(x) = x^{1/2}\sin(s \log x), \quad (s = \sqrt{a^2 - 1/4})$$

on the interval $0 < x < \infty$. Thus $p(x) = x$, and so our theorem[1] (modified as in its last sentence) shows that if $\{x_1, x_2, \ldots\}$, $\{\bar{x}_1, \bar{x}_2, \ldots\}$ are increasing sequences of consecutive positive zeros of nontrivial solutions $y(x)$, $\bar{y}(x)$ of (8), and $x_1 > \bar{x}_1$, then we have, for $k = 1, 2, \ldots$

(9) $$\Delta^n M_k > 0 \qquad (n = 0, 1, \ldots),$$

(10) $$\Delta^n (x_k - \bar{x}_k) > 0 \quad (n = 0, 1, \ldots).$$

A direct proof of (9) may be given as follows. A nontrivial solution of (8) must have the form

$$y(x) = Ax^{1/2}\sin(s \log x + b) \qquad (A \neq 0).$$

Thus we get, for $k = 1, 2, \ldots,$

$$M_k = |A|^\lambda \int_{x_k}^{x_{k+1}} x^{\lambda/2} |\sin(s \log x + b)|^\lambda \, dx$$

$$= c \int_{t_k}^{t_{k+1}} |\sin t|^\lambda \exp \frac{(\lambda+2)t}{2s} \, dt,$$

where $c(> 0)$ is independent of k, and $t_{k+1} = t_k + \pi$. Thus we get (cf. [1, p.60][2]), for $n = 0, 1, \ldots,$

$$\Delta^n M_k = c \int_{t_k}^{t_{k+1}} |\sin t|^\lambda \Delta_\pi^n \left[\exp \frac{(\lambda+2)\, t}{2s}\right] dt$$

$$= c\, \pi^n \int_{t_k}^{t_{k+1}} |\sin t|^\lambda D_t^{(n)} \left[\exp \frac{(\lambda+2)(t + \theta n\pi)}{2s}\right] dt,$$

1. We apply the theorem to an interval $\varepsilon < x < \infty$ $(\varepsilon > 0)$, containing the zeros in question.

2. As usual, $\Delta_\pi F(t) = F(t + \pi) - F(t)$ and $\Delta_\pi^n F(t) = \Delta_\pi (\Delta_\pi^{n-1} F(t))$, $n = 2, 3, \ldots$.

where $0 < \theta < 1$, the last equality following from a mean value theorem for higher derivatives and differences [3, p.55, No. 98]. Since the successive derivatives of the quantity in square brackets are positive we get (9). A similar argument may be used to prove (10).

The Cauchy-Euler equation (8) is (except for linear changes in the variable x) the only one for which $p^{(n)}(x) > 0$ $(n = 0, 1)$ and $p^{(n)}(x) \equiv 0$ $(n = 2, 3, \ldots)$. This follows from [4, Theorem 3, Remark 1], where it is shown that if $p(x)$ (there called $z(x)$) is a polynomial of degree ≤ 2, then $f(x) = d/[p(x)]^2$, where d is a constant.[3]

(ii) We consider the equation

$$(11) \qquad y'' + [\nu^2 x^{-2\nu-2} - (\nu^2 - 1)/(4x^2)]y = 0,$$

where $\nu > 0$. It has linearly independent solutions $y_1(x) = x^{(\nu+1)/2} \sin(x^{-\nu})$ $y_2(x) = x^{(\nu+1)/2} \cos(x^{-\nu})$ on $0 < x < \infty$ so that $p(x) = x^{\nu+1}$. Thus if ν is a positive integer, $p(x)$ satisfies (2), with $(-1)^n$ deleted, for $N = \infty$. We note, however, that each sequence of consecutive positive zeros of a solution y of (11) terminates. We apply the theorem to an interval (a, b), $a > 0$, containing a sequence of such zeros. The theorem is, of course, valid for finite sequences, provided we restrict attention to those higher differences which have meaning for the sequence in question. Thus we find that <u>if ν is a positive integer</u> and if $\{x_1, x_2, \ldots, x_K\}$ is a sequence of successive positive zeros of a nontrivial solution y of (11), then

$$(12) \qquad \Delta^n M_k > 0 \qquad (k = 1, \ldots, K-n-1; n = 0, \ldots, K-2).$$

A corresponding suitably modified analogue of (5) also holds.

A direct proof of (12) is possible. In fact, the direct proof shows that the restriction that ν be a positive <u>integer</u> is not necessary. The direct proof is similar to the direct proof of (9). Each nontrivial solution of (11) has the form $y = ax^{(\nu+1)/2} \sin(x^{-\nu} + b)$, where $a \neq 0$. Thus we get for $k = 1, \ldots, K-1$,

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3. In the special case where $p(x) = a_0 + a_1 x$, $a_0^2 + a_1^2 > 0$, we get $d = W^2 + \frac{1}{4} a_1^2$, W being the (constant) Wronskian of y_1 and y_2.

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$$M_k = |a|^\lambda \int_{x_k}^{x_{k+1}} x^{(\nu+1)\lambda/2} |\sin(x^{-\nu}+b)|^\lambda \, dx$$

$$= -c \int_{t_k}^{t_{k+1}} |\sin(t+b)|^\lambda \, t^{-(\nu+1)(\lambda+2)/(2\nu)} dt$$

where $c(>0)$ is independent of k, $t_{k+1} = t_k - \pi$ $(k = 1, \ldots, K-1)$ and $t_K = x_K^{-\nu} > 0$. Hence, for $k = 1, \ldots, K-n-1$, $n = 0, \ldots, N-2$,

$$\Delta^n M_k = c(-1)^n \int_{t_{k+n+1}}^{t_{k+n}} |\sin(t+b)|^\lambda \, \Delta_\pi^n [t^{-(\nu+1)(\lambda+2)/(2\nu)}] \, dt$$

$$= c(-1)^n \pi^n \int_{t_{k+n+1}}^{t_{k+n}} |\sin(t+b)|^\lambda \, D_t^{(n)} [(t+\theta n\pi)^{-(\nu+1)(\lambda+2)/(2\nu)}] \, dt$$

where $0 < \theta < 1$, by a mean value theorem for higher derivatives and differences [3, p.55, No.98]. Now, since $(-1)^n D_t^{(n)} t^{-(\nu+1)(\lambda+2)/(2\nu)} > 0$ $(t > 0)$, we find that (12) holds, for solutions of (11) where ν is any positive number.

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