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Computation of the stationary distribution of an infinite stochastic matrix of special form

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An algorithm is presented for computing the unique stationary distribution of an infinite regular stochastic matrix of a structural form subsuming both upper-Hessenberg and generalized renewal matrices of this kind. Convergence is elementwise, monotone from above, from information within finite truncations, of increasing order.

1. Finite stochastic matrices

Let $P = \{p_{ij}\}_{i,j=1}^{n}$ be a finite stochastic matrix. It is well known that there is at least one essential class, J, of indices, $J \subset \{1, 2, \ldots, n\}$, and that there is at least one stationary distribution, namely, a vector $\mathrm{v} = \{v_i\}$, satisfying

$$\mathrm{v} \geq 0 \ , \quad \mathrm{v}'P = \mathrm{v}' \ , \quad \mathrm{v}'\mathbf{1} = 1 \ . \tag{1.1}$$

In the case when there is *only one* essential class (other indices, if any, being inessential) there is a unique stationary distribution. Such a P is called *regular*. The vector v is then in fact completely determined as the unique solution of the last two expressions of (1.1) alone, that is, of

$$\mathrm{v}'\{\mathbf{1}, I-P\} = \{1, \mathbf{0}'\} \ , \tag{1.2}$$

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as follows, for example, from the Perron-Frobenius theory of non-negative matrices, where the matrix $\{\mathbf{1}, I-P\}$ is $n \times (n+1)$. The uniqueness of solution implies this matrix is of rank n, and hence contains n linearly independent columns. Clearly the last n cannot be taken as such, for $\{I-P\}\mathbf{1} = \mathbf{0}$; and consequently, it is easily seen that the vector $\mathbf{1}$ combined with any $n-1$ columns of $I-P$ gives n linearly independent vectors. *Thus, for a regular* P*, it follows that any* $(n-1)$ *of the stationary equations* $\mathbf{v}'P = \mathbf{v}'$ *are sufficient to determine the stationary distribution vector to a constant multiple* (and the additional condition $\mathbf{v}'\mathbf{1} = 1$ then specifies it completely). These ideas are not new (for example, [3]) although of historical origin unknown to us.

We shall find these facts useful in the construction of an algorithm for computing successive finite approximations, converging elementwise from above, to the unique stationary distribution of certain kinds of *infinite* stochastic matrices. In an earlier memoir [1] we considered the problem for infinite Markov matrices, and produced an algorithm for convergence elementwise from below.

We shall call an infinite stochastic matrix $P = \{p_{ij}\}_{i,j=1}^{\infty}$ *regular* if its index set $\{1, 2, \ldots\}$ contains precisely one positive-recurrent class of indices (any other indices being null-recurrent or transient). This is a necessary and sufficient condition for a unique stationary distribution (that is, a vector $\mathbf{v}$ satisfying (1.1)) to exist, and so is exactly consistent with the definition of regularity in the finite case. We write ${}_{(n)}P$ for the $n \times n$ northwest corner truncation of a stochastic matrix P.

2. Infinite stochastic matrices of special form

We consider in this section *infinite stochastic matrices* P *which are assumed regular*, and are further either of upper-Hessenberg, or of generalized renewal, form. (The assumption of regularity of P may be partially relaxed for these special kinds of matrices, to give not uninteresting consequences, as the interested reader will readily perceive.)

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(a) Upper-Hessenberg matrices

A matrix of this type satisfies

$$p_{ij} = 0 \, , \quad i > j + 1 \, , \tag{2.1}$$

so that its entries below the subdiagonal are all zero. It follows that its stationary equations $\mathbf{v}'P = \mathbf{v}'$ may be written

$$\sum_{i=1}^{j+1} v_i p_{ij} = v_j \, , \quad j = 1, 2, \ldots \, . \tag{2.2}$$

Let us assume that $\mathbf{v}'\mathbf{1} = 1$.

Let ${}_{(n)}\tilde{P}$ denote the truncation ${}_{(n)}P$ after it has been made stochastic by replacing p_{in} by

$$\tilde{p}_{in} = p_{in} + \sum_{j=n+1}^{\infty} p_{ij} \, , \quad i = 1, \ldots, n \, ,$$

so that only the last column of ${}_{(n)}P$ is affected. If we *assume for the moment* that ${}_{(n)}\tilde{P}$ is regular, it is readily seen that the *first* $n - 1$ of the n stationary equations ${}_{(n)}\mathbf{v}'{}_{(n)}\tilde{P} = {}_{(n)}\mathbf{v}'$ which its unique stationary vector satisfies *coincide* with the first $n - 1$ of the equations (2.2). Thus

$${}_{(n)}v_i = c(n)v_i \, , \quad i = 1, \ldots, n \, ,$$

where $c(n)$ is a constant, and since we take ${}_{(n)}\mathbf{v}$ to satisfy ${}_{(n)}\mathbf{v}'\mathbf{1} = 1$,

$${}_{(n)}v_i = v_i \Big/ \sum_{r=1}^{n} v_r \, , \quad i = 1, \ldots, n \, . \tag{2.3}$$

Thus it follows that if we can find an infinite sequence $\left\{{}_{(n_k)}P\right\}$ of truncations with each ${}_{(n_k)}\tilde{P}$ regular, then, elementwise, as $k \to \infty$,

${}_{(n_k)}v_i \downarrow v_i \, , \quad i = 1, 2, \ldots \, .$

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(b) Generalized renewal matrices

A matrix of this type satisfies

$$p_{ij} = 0 \ , \quad i > j > 1 \ . \tag{2.4}$$

The stationary equations $\mathbf{v}'P = \mathbf{v}'$ may be written

$$\begin{cases} \sum_{i=1}^{\infty} v_i p_{i1} = v_1 \ , \\ \\ \sum_{i=1}^{j} v_i p_{ij} = v_j \ , \quad j = 2, 3, \ldots \end{cases} \tag{2.5}$$

where we assume $\mathbf{v}'\mathbf{1} = 1$.

If we consider stochastic ${}_{(n)}\tilde{P}$ formed from ${}_{(n)}P$ by replacing p_{i1} by

$$\tilde{p}_{i1} = p_{i1} + \sum_{j=n+1}^{\infty} p_{ij} \ , \quad i = 1, \ldots, n \ ,$$

so that only the first column of ${}_{(n)}P$ is affected, then, on assumption of the regularity of ${}_{(n)}\tilde{P}$, the *last* $n - 1$ of its stationary equations coincide with equations with indices $j = 2, 3, \ldots, n$ of (2.5), and (2.3) and the subsequent conclusion follow as before.

The following theorem, whose validity was kindly pointed out to us by Tweedie [7] upon reading an earlier version of this note, shows that the assumption of regularity of ${}_{(n)}\tilde{P}$ is not required in those cases where the computational problem is truly one for an infinite matrix (the only cases of present interest).

THEOREM. *Suppose for regular P , the positive-recurrent class consists of an infinite number of indices. Let ${}_{(n)}\tilde{P}$ be stochastic and obtained from ${}_{(n)}P$ by augmenting any specific one column of ${}_{(n)}P$ only to make it stochastic. Then ${}_{(n)}\tilde{P}$ is regular.*

An easy proof results from showing by a contradiction argument that ${}_{(n)}\tilde{P}$ contains only one essential class of indices, using canonical form of

${}_{(n)}\tilde{P}$.

This enables us to state the operative principle in general: *if we can find a sequence* $\left\{{}_{(n_k)}P\right\}$ *of truncations, such that by changing one column only (not necessarily the same for each* k *),* ${}_{(n_k)}P$ *can be made stochastic in such a way that the other* $n-1$ *stationary equations of* ${}_{(n_k)}\tilde{P}$ *coincide with* $n-1$ *of the first* n *stationary equations of infinite regular* P *having infinite positive-recurrent class, then* (2.3) *obtains with* $n = n_k$, $k = 1, 2, \ldots$.

3. Background

As mentioned before, for infinite Markov matrices, we have already an algorithm for convergence elementwise from below. The present note was largely motivated by the wish to obtain an algorithm for convergnece elementwise from above, and this has been achieved for another class of matrices. Clearly there is an overlap in the classes: those generalized renewal matrices with elements of the first column uniformly bounded away from zero. One such is the structurally simple example with

$$p_{ij} = \begin{cases} p_j & , \quad j \geq i \geq 1 \; ; \\ \sum_{r=1}^{i-1} p_r & , \quad j = 1, \; i \geq 2 \; ; \\ 0 & \quad \text{otherwise,} \end{cases}$$

where $\{p_j\}$, $j = 1, 2, \ldots$ is a probability distribution with each element positive. For this example it is readily shown that

$$v_i = \frac{v_1}{\prod_{r=2}^{i} (1-p_r)} p_i \; , \quad i \geq 2 \; ,$$

so that as $n \to \infty$,

$$\sum_{r=n+1}^{\infty} v_r \sim \text{const.} \sum_{r=n+1}^{\infty} p_r \; .$$

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Since the *convergence rate* of ${}_{(n)}v_i$ to v_i with n , for fixed i , is, from (2.3), *in general*

$$\sum_{r=n+1}^{\infty} v_r ,$$

it follows from the example that this rate may be rather slow; although it will be geometrically fast if the distribution $\{v_i\}$ has geometrically decreasing tail. This situation is similar to that of the algorithm in [*1*].

The method of "stochastizing" truncations of an infinite stochastic matrix was suggested by Sarymsakov [*4*], and used by him for other purposes; it is an obvious idea, whose general utility is less clear as regards truncation theory. Upper-Hessenberg structure of infinite stochastic P , with first state *absorbing*, lends itself (on account of the special "column finiteness") to useful analysis [5], [*6*]. Kemeny [*2*] has carried out substantial analysis on irreducible lower-Hessenberg P ("slowly spreading Markov chains"). This structure is not directly amenable to the foregoing type of analysis for stationary distribution.

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