Can. J. Math. Vol. **50** (3), 1998 pp. 605–619.

# HARDY SPACES OF CONJUGATE SYSTEMS OF TEMPERATURES

MARTHA GUZMÁN-PARTIDA AND SALVADOR PÉREZ-ESTEVA

ABSTRACT. We define Hardy spaces of conjugate systems of temperature functions on $\mathbb{R}^{n+1}_+$. We show that their boundary distributions are the same as the boundary distributions of the usual Hardy spaces of conjugate systems of harmonic functions.

**Introduction.** In [4], C. Fefferman and E. M. Stein defined for $(n-1)/n < p < \infty$, the $H^p$ spaces in $\mathbb{R}^{n+1}_+$ as $(n+1)$-tuples of harmonic functions $F = (u_1, \ldots, u_n, u_{n+1})$ on $\mathbb{R}^{n+1}_+$ satisfying the equations of conjugacy

$$\text{(CR)} \qquad \begin{cases} D_{x_i} u_j = D_{x_j} u_i & i, j = 1, \ldots, n+1 \\ \sum_{j=1}^{n+1} D_{x_j} u_j = 0 \end{cases}$$

and the condition

$$\text{(G)} \qquad \sup_{t>0} \int_{\mathbb{R}^n} |F(x,t)|^p \, dx < \infty.$$

Every such $F$ is determined by $u = u_{n+1}$ and finally by the boundary value $f = \lim_{t \to 0^+} u(\cdot, t)$ which exists as a temperate distribution. Then they gave a real-variable characterization of the corresponding space of distributions in terms of maximal functions constructed using any "nice" test function. In particular the Gaussian function $G(x) \equiv \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$ can be used to characterize the $H^p$ spaces as boundary values of temperature functions (solutions of the heat equation). The purpose of this article is to express these spaces in terms of $(n+1)$-tuples of temperatures satisfying certain conjugacy equations and a condition like (CR). The equations introduced by Y. Sagher and E. Kochneff in the case $n = 1$, involve Weyl fractional derivatives to decompose the partial derivative $D_t$ appearing in the heat operator. This paper is the extension to several variables of previous work by one of the authors [7].

The paper will be organized as follows: In the first section we study temperature functions on $\mathbb{R}^{n+1}_+$ satisfying (G). We give growth estimates for these functions at the boundary, based upon Lemma 1.1 which has interest of its own. These estimates will lead to the existence of the fractional derivatives and boundary limits used in the second section. In the second section we define Hardy spaces of $(n+1)$-tuples of temperature functions and the system of equations ruling them; as in the harmonic case their coordinates are linked by the Riesz transforms. In the main theorem we prove that the boundary values are exactly as the classical real $H^p$ spaces.

Received by the editors November 25, 1996.
AMS subject classification: Primary: 42B30, 42A50; secondary: 35K05.




https://doi.org/10.4153/CJM-1998-034-0 Published online by Cambridge University Press

The following notation will be used throughout the paper. If $f \in L^2(\mathbb{R}^n)$, $\hat{f}$ denotes the Fourier transform of $f$ based on the kernel $e^{-2\pi i x\cdot\xi}$.

If $g$ is a real-valued function which is sufficiently regular for $t > 0$, the Weyl's fractional derivative of order $\alpha > 0$ is defined by

$$D^\alpha g(t) = \frac{e^{i\pi\tilde{\alpha}}}{\Gamma(\tilde{\alpha})} \int_t^\infty g^{(n)}(s)(s-t)^{\tilde{\alpha}-1}\,ds,$$

where $\alpha = n - \tilde{\alpha}$, $n \in \mathbb{N}$, $0 < \tilde{\alpha} \le 1$, and the Weyl's fractional integral of order $\alpha > 0$ is

$$D^{-\alpha} g(t) = \frac{e^{i\pi\alpha}}{\Gamma(\alpha)} \int_t^\infty g(s)(s-t)^{\alpha-1}\,ds.$$

The 1-dimensional Gauss-Weierstrass kernel will be denoted by $k(x,t)$, that is,

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}, \quad t > 0$$

and $k(x,t) = 0$ for $t \le 0$. The $n$-dimensional Gauss-Weierstrass kernel is defined as

$$K(x,t) = \prod_{i=1}^n k(x_i,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t},$$

and the theta function for $t > 0$ as

$$\theta(x,t) = \sum_{n=-\infty}^{\infty} k(x+2n,t),$$

and $\theta = 0$ for $t \le 0$.

1. **Hardy spaces of temperature functions.** We will denote by

$$H(\mathbb{R}^{n+1}_+) = \Big\{ u \in C^2(\mathbb{R}^{n+1}_+) : \sum_{j=1}^n D^2_{x_j} u = D_t u \Big\},$$

$$H^p(\mathbb{R}^{n+1}_+) = \{ u \in H(\mathbb{R}^{n+1}_+) : \|u\|_{H^p} < \infty \},$$

where

$$\|u\|_{H^p} = \sup_{t>0} \|u(\cdot,t)\|_p, \quad 0 < p < \infty.$$

We shall write $H$ and $H^p$, respectively, if the context does not cause confusion. The elements of $H$ will be called temperature functions.

It is well known that the following representation holds for $u \in H^p$ when $1 < p < \infty$ (see [5, Theorem 5]): $u \in H^p$ if and only if $u(x,t) = K(\cdot,t) * f(x)$, where $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$.

Moreover, we have the following estimates for $x \in \mathbb{R}^n$ and $t > 0$ [5, Theorem 2(ii), (iv) and (v)]

$$|u(x,t)| \le C\|f\|_p t^{-n/2p}, \tag{1}$$

$$|D_t u(x,t)| \le C\|f\|_p t^{-1-n/2p}. \tag{2}$$

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As a consequence of the above estimates, $u$ and $D_t u$ are bounded in each proper half-space $\{(x,t) \in \mathbb{R}^{n+1}_+ : t \geq t_0 > 0\}$.

The case $0 < p \leq 1$ will be analyzed in Theorem 1.2 below, following the main ideas of the harmonic case [6, p. 172] adapted to our situation. First, we need to state some known facts and some notation.

Let $Q = (0,1)^{n+1}$ and $\Gamma$ its parabolic boundary, that is, $\Gamma$ is the union of the sets $\Gamma_0$ and $\Gamma_{ij}$, for $i = 1,\dots,n$ and $j = 0,1$ where $\Gamma_0 = \{(\xi,0) : \xi \in (0,1)^n\}$ and $\Gamma_{ij} = \{(\xi,\tau) \in \overline{Q} : 0 < \tau < 1, \xi_i = j\}$. $\lambda$ will be the $n$-dimensional Lebesgue measure on $\Gamma$.

We define for $x, \xi \in \mathbb{R}^n$ and $t, \tau > 0$

$$R(x,t;\xi,\tau) = \prod_{i=1}^{n} H(x_i,\xi_i,t-\tau),$$

where

$$H(x_i,\xi_i,t-\tau) = \theta(x_i-\xi_i,t-\tau) - \theta(x_i+\xi_i,t-\tau).$$

We will consider the kernel $K\colon Q \times \Gamma \longrightarrow \mathbb{R}$ defined as follows:

$$\begin{aligned} K(x,t;\xi,0) &= R(x,t;\xi,0),\\ K(x,t;\xi,\tau) &= (-1)^{j+1} D_{x_i} R(x,t;\xi,\tau), \qquad \xi \in \Gamma_{ij}. \end{aligned}$$

Every temperature function on $Q$ and continuous on $\overline{Q}$ can be written on $Q$ as follows:

$$u(x,t) = \int_\Gamma K(x,t;\xi,\tau) u(\xi,\tau)\, d\lambda(\xi,\tau).$$

(Compare with the results in [1] and [3]).

We shall also consider for $0 < r < 1$ the mapping $T_r(x,t) = (x_r,t_r)$ where

$$\begin{aligned} x_r &= rx + \frac{1-r}{2}(1,\dots,1)\\ t_r &= r^2 t + (1-r^2). \end{aligned}$$

Notice that $Q = \{T_r(\xi,\tau) : (\xi,\tau) \in \Gamma, 0 < r < 1\}$.

LEMMA 1.1. *Let $u$ be a temperature function on a $(n+1)$-dimensional parallelepiped $P \subset \mathbb{R}^{n+1}_+$ such that $P = P' \times (c,d)$, where $P'$ is a $n$-dimensional cuboid which satisfies $|P'|^{2/n} = (d-c)$. Suppose also that $u$ is continuous on $\overline{P}$. If $(x_o,d)$ is the center of the upper face of $P$, then for every $p \in (0,1]$*

$$|u(x_0,d)|^p \leq C_p \frac{1}{|P|} \int_P |u(\xi,\tau)|^p \, d\xi\, d\tau,$$

*where $C_p$ is a constant only depending on $p$, and $|P'|$, $|P|$ denote the Lebesgue measure of $P'$ and $P$, on $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively.*

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PROOF. Without loss of generality we can suppose that $P$ is the unit cuboid $Q$, since the mapping $\Phi\colon Q \longrightarrow P$ such that

$$\Phi(\xi,\tau) = \big(|P'|^{1/n}\xi + a, (d-c)\tau + c\big),$$

where $P' = (a_1,b_1)\times\cdots\times(a_n,b_n)$, makes $u\circ\Phi$ a temperature function. Notice also that for $0<r<1$, $u\circ T_r$ is a temperature function on $Q$. For every $r>0$ define

$$m_p(r) = \Big[\int_\Gamma \big|u\big(T_r(\xi,\tau)\big)\big|^p \,d\lambda(\xi,\tau)\Big]^{1/p},$$

and

$$m_\infty(r) = \sup\Big\{\big|u\big(T_r(\xi,\tau)\big)\big| : (\xi,\tau)\in\Gamma\Big\}.$$

We may assume that $\int_Q |u(\xi,\tau)|^p \,d\xi\,d\tau = |Q|$ and $m_\infty(r)\geq 1$ for every $r\in(0,1)$.

Let $q>1$. We notice that

$$m_q(r) \leq m_\infty(r)^{(q-p)/q} m_p(r)^{p/q}. \tag{3}$$

We have the representation

$$u\big(T_r(x,t)\big) = \int_\Gamma K(x,t;\xi,\tau)u\big(T_r(\xi,\tau)\big)\,d\lambda(\xi,\tau).$$

If $(x,t)\in Q_r = T_r(Q)$ then

$$u(x,t) = \int_\Gamma K\big(T_r^{-1}(x,t);\xi,\tau\big)u\big(T_r(\xi,\tau)\big)\,d\lambda(\xi,\tau),$$

and since $T_r\circ T_s = T_{rs}$, then for $(x,t)\in Q_s$, $0<s<r<1$,

$$u(x,t) = \int_\Gamma K\big(T_{s/r}(T_s^{-1}(x,t));\xi,\tau\big)u\big(T_r(\xi,\tau)\big)\,d\lambda(\xi,\tau). \tag{4}$$

We need to estimate $\int_\Gamma K\big(T_\rho(y,\eta);\xi,\tau\big)u\big(T_r(\xi,\tau)\big)\,d\lambda(\xi,\tau)$ for $(y,\eta)\in Q$ and $\rho\in(0,1)$. First, we notice that (compare with [7, Lemma 1])

$$\Big|\int_{(0,1)^n} R\big(T_\rho(y,\eta);\xi,0\big)u\big(T_r(\xi,0)\big)\,d\xi\Big| \leq C(1-\rho)^{-2n}m_q(r). \tag{5}$$

Next, we will analyze

$$\int_{\Gamma_{10}} K(y_\rho,\eta_\rho;\xi,\tau)u\big(T_r(\xi,\tau)\big)\,d\lambda(\xi,\tau). \tag{6}$$

This integral equals

$$\int_0^{\eta_\rho}\int_{[0,1]^{n-1}} 2D_{x_1}\theta(x_1,\eta_\rho-\tau)\prod_{i=2}^n H(x_i,\xi_i,\eta_\rho-\tau)u\big(T_r(0,\xi_2,\ldots,\xi_n,\tau)\big)\,d\xi_2\cdots d\xi_n\,d\tau,$$

and its module is majorized by (see [7, Lemma 1])

$$C(1-\rho)^{-2}m_q(r)\Big[\int_0^{\eta_\rho}\Big(\prod_{i=2}^n\int_0^1 H(x_i,\xi_i,\eta_\rho-\tau)^{q'}\,d\xi_i\Big)\,d\tau\Big]^{1/q'}$$

where $q'$ is the conjugate exponent of $q$.

Since for every $i = 1, \dots, n$

$$\int_0^1 H(x_i, \xi_i, \eta_\rho - \tau)^{q'}\, d\xi \le C \frac{1}{(\eta_\rho - \tau)^{(q'-1)/2}},$$

if we choose $q$ such that $(n-1)(q'-1)/2 < 1$, we can ensure that the expression (6) is majorized by $C(1-\rho)^{-2} m_q(r)$.

A similar argument shows that for every $i = 1, \dots, n$ and $j = 0, 1$

$$\left| \int_{\Gamma_{ij}} K(y_\rho, \eta_\rho; \xi, \tau) u\big(T_r(\xi, \tau)\big)\, d\lambda(\xi, \tau) \right| \le C(1-\rho)^{-2} m_q(r) \tag{7}$$

and from (4), (5) and (7) we get for $\rho = s/r$ and $(x, t) \in Q_s$

$$m_\infty(s) \le C(1 - s/r)^{-2n} m_q(r). \tag{8}$$

If we choose $s = r^a$ with $a > 1$ we obtain from (3) and (8)

$$m_\infty(r^a) \le C(1 - r^{a-1})^{-2n} m_\infty(r)^{(q-p)/q} m_p(r)^{p/q}.$$

Taking logarithms, multiplying by $1/r$ and then integrating respect to $r$, we get

$$\int_{1/2}^1 \log m_\infty(r^a)\, dr/r \le C + \frac{(q-p)}{q} \int_{1/2}^1 \log m_\infty(r)\, dr/r + \frac{1}{q} \int_{1/2}^1 \log m_p(r)^p\, dr/r. \tag{9}$$

Now, we have

$$1 = \int_Q |u(\xi, \tau)|^p\, d\xi\, d\tau = \int_{Q_0} |u(\xi, \tau)|^p\, d\xi\, d\tau + \sum_{i,j} \int_{Q_{ij}} |u(\xi, \tau)|^p\, d\xi\, d\tau, \tag{10}$$

where $Q_0$ and $Q_{ij}$ are defined as follows:

$$Q_0 = \Psi\big(\Gamma_0 \times (0, 1)\big)$$
$$Q_{ij} = \Psi\big(\Gamma_{ij} \times (0, 1)\big),$$

and $\Psi \colon \Gamma \times (0, 1) \longrightarrow Q$ sends $\big((\xi, \tau), r\big) \longrightarrow T_r(\xi, \tau)$.

First, we see that

$$\begin{aligned} \int_{Q_0} |u(\xi, \tau)|^p\, d\xi\, d\tau &= \int_0^1 \int_{(0,1)^n} 2r^{n+1} \Big|u\big(T_r(\xi, 0)\big)\Big|^p\, d\xi\, dr \\ &= \int_0^1 \int_{\Gamma_0} 2r^{n+1} \Big|u\big(T_r(\xi, \tau)\big)\Big|^p\, d\lambda(\xi, \tau)\, dr. \end{aligned}$$

Also, it is easy to see that

$$\int_{Q_{ij}} |u(\xi, \tau)|^p\, d\xi\, d\tau = \int_0^1 \int_{\Gamma_{ij}} \frac{r^{n+1}}{2} \Big|u\big(T_r(\xi, \tau)\big)\Big|^p\, d\lambda(\xi, \tau)\, dr.$$

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Hence, collecting all the above integrals, we obtain from (10)

$$\int_0^1 r^{n+1} m_p(r)^p\, dr \le C \int_0^1 \int_\Gamma \left|u\big(T_r(\xi,\tau)\big)\right|^p d\lambda(\xi,\tau)\, dr \le C.$$

Thus

$$\frac{1}{q}\int_{1/2}^1 \log m_p(r)^p\, dr/r \le C\int_0^1 r^{n+1} m_p(r)^p\, dr \le C. \tag{11}$$

Using inequality (11), making a change of variable on the left hand side of (9) and choosing $a$ in a such way that $\frac{1}{a} - \frac{q-p}{q} > 0$, we can conclude that

$$\int_{1/2^a}^1 \log m_\infty(r)\, dr/r \le C_p,$$

where $C_p$ is a constant depending on $p$ only. This inequality implies that there exists $r_0 \in [1/2^a, 1]$ such that $m_\infty(r_0) \le C_p$ and from the maximum principle the proof follows. ∎

REMARK. Notice that Lemma 1.1 is also true for $1 < p < \infty$, since it is valid for $p = 1$.

THEOREM 1.2. *Let $u \in H^p$, $0 < p \le 1$. Then, there exists a constant $C$ depending only on $p$ such that for every $(x,t) \in \mathbb{R}^{n+1}_+$*

$$|u(x,t)| \le C\|u\|_{H^p} t^{-n/2p}. \tag{12}$$

*In particular, $u(x,t)$ is bounded in each proper half-space $\Omega_0 = \{(x,t) : t \ge t_0 > 0\}$. In fact, $u(x,t) \longrightarrow 0$ if $(x,t) \longrightarrow \infty$ in $\Omega_0$.*

PROOF. Let $(x,t) \in \mathbb{R}^{n+1}_+$ and $R = \left(\prod_{i=1}^n (x_i - \sqrt{t}/2\sqrt{2}, x_i + \sqrt{t}/2\sqrt{2})\right) \times (t/2, t)$. Using Lemma 1.1, the rest of the proof follows exactly as in the harmonic case (see [6, p. 174]). ∎

Now, we analyze the growth of $D_t u$ for $u \in H^p$, $0 < p \le 1$:

Let $t > 0$ and $x \in \mathbb{R}$. Fix $t_0 > 0$ such that $t/2 < t_0 < t$. Estimate (12) implies that

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y, t-t_0) u(y,t_0)\, dy$$

and using (a) in Lemma 2.3 below we obtain

$$\begin{aligned}
|D_t u(x,t)| &\le C\|u\|_{H^p} t_0^{-n/2p} \int_{\mathbb{R}^n} \min\left\{\frac{1}{|x-y|^{n+2}}, \frac{1}{(t-t_0)^{(n+2)/2}}\right\} dy \\
&= C\|u\|_{H^p} t_0^{-n/2p} \int_{|y-x|^{n+2} > (t-t_0)^{(n+2)/2}} \frac{dy}{|y-x|^{n+2}} \\
&\quad + C\|u\|_{H^p} t_0^{-n/2p} \int_{|y-x|^{n+2} < (t-t_0)^{(n+2)/2}} \frac{dy}{(t-t_0)^{(n+2)/2}} \\
&\le C\|u\|_{H^p} (t-t_0)^{-1-n/2p}.
\end{aligned}$$

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Letting $t_0 \longrightarrow t/2$ we obtain

$$|D_t u(x,t)| \leq C\|u\|_{H^p} t^{-1-n/2p}. \tag{13}$$

Next, we examine the boundary behavior of $u \in H^p$, $0 < p < \infty$. For $1 < p < \infty$, $u(\cdot,t) \longrightarrow f$ in the $L^p$ norm as $t \longrightarrow 0$, where $u(\cdot,t) = K(\cdot,t) * f(x)$, $f \in L^p$, therefore the family $\big(u(\cdot,t)\big)_{t>0}$ converges in $S'$ as $t \longrightarrow 0$. If $u \in H^p$, $0 < p \leq 1$, Theorem 1.2 implies that every $u(\cdot,t)$ is a bounded function, hence a tempered distribution. Moreover, as we will state in the following theorem, $\big(u(\cdot,t)\big)_{t>0}$ converges in $S'$.

THEOREM 1.3. *Let $u \in H^p$, $0 < p < \infty$. Then $\lim_{t\to 0} u(\cdot,t) \equiv f$ exists in $S'$ and $f$ uniquely determines $u$.*

PROOF. It remains to prove the case $0 < p \leq 1$. Since estimate (12) holds, taking $a = b = 0$ and $c = n/2p$ in [5, Theorem 17], we get $u(x,t) = K(\cdot,t) * f(x)$ where $f \in S'$. Denoting $F\Psi \equiv \hat{\Psi}$, we see that for every $\varphi \in S$ $\langle u(\cdot,t),\varphi\rangle = \langle e^{-4\pi^2 t|\cdot|^2} Ff, F^{-1}\varphi\rangle \longrightarrow \langle f,\varphi\rangle$ as $t \longrightarrow 0$. It is clear that $f$ uniquely determines $u$. ∎

2**. Hardy spaces of conjugate systems of temperature functions.** Following the idea of conjugacy introduced by Kochneff and Sagher in [9], we will define conjugate systems of temperature functions on $\mathbb{R}^{n+1}_+$.

DEFINITION 2.1. A $C^2$ vector field in $\mathbb{R}^{n+1}_+$, $F = (u_1,\ldots,u_n,u_{n+1})$ will be called a *conjugate system of temperature functions*, if there exist $D_t^{1/2}u_j$, on $\mathbb{R}^{n+1}_+$ for every $j = 1,\ldots,n+1$ and the following equations hold:

(I) $\sum_{j=1}^n D_{x_j}u_j = iD_t^{1/2}u$.

(II) $D_{x_k}u_j = D_{x_j}u_k$, $j,k = 1,\ldots,n$.

(III) $D_{x_j}u = -iD_t^{1/2}u_j$, $j = 1,\ldots,n$.

where $u \equiv u_{n+1}$ and $D_t^{1/2}$ is the Weyl's fractional derivative operator or order $1/2$ respect to $t$. We will write $F \in AH\,(\mathbb{R}^{n+1}_+)$ or simply, $F \in AH$.

It is important to remark that if $F = (u_1,\ldots,u_n,u_{n+1}) \in AH$, then every function $u_k$, $k = 1,\ldots,n+1$ is a temperature function on $\mathbb{R}^{n+1}_+$.

If $R_j$ denotes the $j$-th Riesz transform for $j = 1,\ldots,n$, we will write $S_j(x,t) \equiv R_jK(\cdot,t)(x)$, where $R_j$ is taken with respect to the spacial variable. Since $\hat{K}(\cdot,t)(x) = e^{-4\pi^2 t|x|^2}$, and $\widehat{R_jK}(\cdot,t)(x) = -i\frac{\xi_j}{|\xi|}\hat{K}(\cdot,t)(x)$, from the Fourier inversion formula, we have for $j = 1,\ldots,n$

$$K(x,t) = \int_{\mathbb{R}^n} e^{-4\pi^2 t|\xi|^2}\cos(2\pi\xi\cdot x)\,d\xi \quad \text{and} \tag{14}$$

$$S_j(x,t) = \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|} e^{-4\pi^2 t|\xi|^2}\sin(2\pi\xi\cdot x)\,d\xi.$$

PROPOSITION 2.2.

$$\big(S_1(x,t),\ldots,S_n(x,t),K(x,t)\big) \in AH \quad \textit{on } \mathbb{R}^{n+1}_+.$$

PROOF.

$$\begin{aligned}\widehat{D_{x_k}S_j}(\cdot,t)(\xi) &= (2\pi i\xi_k)\widehat{S_j}(\cdot,t)(\xi)\\ &= 2\pi\frac{\xi_k\xi_j}{|\xi|}e^{-4\pi^2t|\xi|^2}.\end{aligned}$$

Exchanging the roles of $k$ and $j$, we see that

$$\widehat{D_{x_k}S_j}(\cdot,t)(\xi)=\widehat{D_{x_j}S_k}(\cdot,t)(\xi),$$

which proves (II) in Definition 2.1. Now, using the expressions (14) we have

$$\begin{aligned}-iD_t^{1/2}S_j(x,t) &= -\frac{4\pi^2}{\sqrt{\pi}}\int_0^\infty\Big[\int_{\mathbb{R}^n}\xi_j|\xi|e^{-4\pi^2(s+t)|\xi|^2}\sin(2\pi\xi\cdot x)\,d\xi\Big]s^{-1/2}\,ds\\ &= -\frac{4\pi^2}{\sqrt{\pi}}\int_{\mathbb{R}^n}\Big[\int_0^\infty e^{-4\pi^2s|\xi|^2}s^{-1/2}\,ds\Big]\xi_j|\xi|e^{-4\pi^2t|\xi|^2}\sin(2\pi\xi\cdot x)\,d\xi\\ &= -\int_{\mathbb{R}^n}(2\pi\xi_j)e^{-4\pi^2t|\xi|^2}\sin(2\pi\xi\cdot x)\,d\xi\\ &= D_{x_j}K(x,t),\end{aligned}$$

showing (III). With a similar argument, we can complete the proof of the proposition. ∎

In [9, Lemma 2] it is proved that if $0<\alpha<1$, $\beta>2\alpha$, $x>0$ and if for all $t>0$,

$$|D_tw(t)|\le\min\Big\{\frac{1}{x^\beta},\frac{1}{t^{\beta/2}}\Big\},$$

then

$$|D_t^{1-\alpha}w(t)|\le\min\Big\{\frac{1}{x^{\beta-2\alpha}},\frac{1}{t^{(\beta-2\alpha)/2}}\Big\}. \tag{15}$$

With this result we can prove the following Lemma.

LEMMA 2.3. *Let $t>0$ and $x\in\mathbb{R}^n$. Then for $j,k=1,\dots,n$*

*(a)* $D_tK(x,t)|\le C\min\{\frac{1}{|x|^{n+2}},\frac{1}{t^{(n+2)/2}}\}$.

*(b)* $|D_tS_j(x,t)|\le C\min\{\frac{1}{|x|^{n+2}},\frac{1}{t^{(n+2)/2}}\}$.

*(c)* $|D_{x_k}S_j(x,t)|\le C\min\{\frac{1}{|x|^{n+1}},\frac{1}{t^{(n+1)/2}}\}$.

PROOF. To see part (a), we write

$$D_tK(x,t)=\frac{1}{2(4\pi)^{n/2}}\Big[\frac{|x|^2}{2t}-n\Big]\frac{1}{t^{(n+2)/2}}e^{-|x|^2/4t}.$$

Since

$$\frac{1}{t^\beta}e^{-|x|^2/4t}\le\Big(\frac{4\beta}{e}\Big)^\beta\frac{1}{|x|^{2\beta}}\quad\text{for } t>0 \text{ and } \beta>0,$$

we get for $|x|^2>2nt$

$$\begin{aligned}|D_tK(x,t)| &\le C\frac{|x|^2}{t^{(n+4)/2}}e^{-|x|^2/4t}\\ &\le C\frac{1}{|x|^{n+2}},\end{aligned}$$

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and for $|x|^2 \leq 2nt$

$$\begin{aligned} |D_t K(x,t)| &\leq C \frac{1}{t^{(n+2)/2}} e^{-|x|^2/4t} \\ &\leq C \frac{1}{t^{(n+2)/2}}, \end{aligned}$$

showing part (a).

To show (b), it is not difficult to see that for $x \in \mathbb{R}$ fixed

$$D_t S_j(x,t) = O\Big(\frac{1}{t^2}\Big) \quad \text{and}$$
$$D_t^2 S_j(x,t) = O\Big(\frac{1}{t^3}\Big),$$

then $D_t D_t^{1/2} S_j(x,t) \in L^1$ and $D_t^{1/2} D_t^{1/2} S_j(x,t) = D_t S_j(x,t)$. Thus Proposition 2.2 implies

$$D_t S_j(x,t) = i D_t^{1/2} D_{x_j} K(x,t).$$

Now, since

$$D_t D_{x_j} K(x,t) = \frac{x_j}{2t^2} K(x,t) - \frac{x_j}{2t} D_t K(x,t),$$

making a similar analysis as above, we get

$$|D_t D_{x_j} K(x,t)| \leq \min\Big\{ \frac{1}{|x|^{n+3}}, \frac{1}{t^{(n+3)/2}} \Big\}$$

and applying (15) we obtain the desired inequality.

In order to prove (c), first we observe that for $k, j = 1, \ldots, n$

$$|D_t^{1/2} S_j(x,t)| \leq C \min\Big\{ \frac{1}{|x|^{n+1}}, \frac{1}{t^{(n+1)/2}} \Big\} \tag{16}$$

and

$$|D^2_{x_k x_j} K(x,t)| \leq C \min\Big\{ \frac{1}{|x|^{n+2}}, \frac{1}{t^{(n+2)/2}} \Big\}. \tag{17}$$

Thus, using Proposition 2.2, and estimates (16) and (17), we see that

$$\begin{aligned} D_{x_k} S_j(x,t) &= i D_{x_k} D_t^{-1/2} D_{x_j} K(x,t) \\ &= i D_t^{-1/2} D^2_{x_k x_j} K(x,t). \end{aligned}$$

It follows that

$$|D_{x_k} S_j(x,t)| \leq C \min\Big\{ \frac{1}{|x|^{n+1}}, \frac{1}{t^{(n+1)/2}} \Big\},$$

which concludes the proof of the lemma. ∎

PROPOSITION 2.4. *If $g \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, then*

$$\big(g * S_1(x,t), \ldots, g * S_n(x,t), g * K(x,t)\big) \in AH.$$

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PROOF. From Proposition 2.2 we have for $j = 1, \dots, n$

$$\begin{aligned} -iD_t^{1/2}(g * S_j)(x,t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \Big[ \int_{\mathbb{R}^n} D_t S_j(x-y, s+t) g(y)\, dy \Big] s^{-1/2}\, ds \\ &= -i \int_{\mathbb{R}^n} g(y) D_t^{1/2} S_j(x-y,t)\, dy \\ &= D_{x_j}(g * K)(x,t). \end{aligned}$$

The application of Fubini's Theorem is justified by Lemma 2.3 and

$$\begin{aligned} \int_0^\infty D_t S_j(y, s+t) s^{-1/2}\, ds &\le C \int \min\Big\{ \frac{1}{|y|^{n+2}}, \frac{1}{(s+t)^{(n+2)/2}} \Big\} s^{-1/2}\, ds \\ &\le C \min\Big\{ \frac{1}{|y|^{n+1}}, \frac{1}{t^{(n+1)/2}} \Big\}. \end{aligned}$$

The rest of the proof is similar. ∎

DEFINITION 2.5. For $(n-1)/n < p < \infty$ we define the following set

$$\mathbb{H}^p = \Big\{ F = (u_1, \dots, u_n, u_{n+1}) \in AH \ : \sup_{t>0} \int_{\mathbb{R}^n} |F(x,t)|^p\, dx < \infty \Big\}.$$

We remark that estimates (2) and (13) in previous section imply for $u \in H^p$, $0 < p < \infty$

$$|D_t^{1/2} u(x,t)| \le CB(1/2, 1/2 + n/2p) t^{-1/2 - n/2p},$$

where $B$ is the beta function. Hence any $u \in H^p$ has fractional derivative $D_t^{1/2} u$.

THEOREM 2.6. *For* $1 < p < \infty$

$$\mathbb{H}^p = \{ (R_1 u, \dots, R_n u, u) : u \in H^p \},$$

*where* $R_j u$ *is taken with respect to the spacial variable,* $j = 1, \dots, n$.

PROOF. Let $(R_1 u, \dots, R_n u, u)$ in the set on the right hand side. Then, Proposition 2.4 implies that $(R_1 u, \dots, R_n u, u) \in AH$ and since $R_j$ is bounded from $L^p$ to $L^p$ for $p > 1$ and $j = 1, \dots, n$, we have that $(R_1 u, \dots, R_n u, u) \in \mathbb{H}^p$.

Conversely, if $F = (u_1, \dots, u_n, u) \in \mathbb{H}^p$, there exist $f, f_1, \dots, f_n \in L^p$ such that $u(x,t) = K(\cdot,t) * f(x)$, $u_j(x,t) = K(\cdot,t) * f_j(x)$, $j = 1, \dots, n$. Using that $D_{x_j} u = -iD_t^{1/2} u_j$, $j = 1, \dots, n$ we have that

$$\begin{aligned} (2\pi i x_j) e^{-4\pi^2 t |x|^2} \hat{f}(x) &= \big( \widehat{D_{x_j} u(\cdot,t)} \big)(x) \\ &= \big( \widehat{-iD_t^{1/2} u_j(\cdot,t)} \big)(x) \\ &= (-i) D_t^{1/2} \widehat{u_j}(\cdot,t)(x) \\ &= -2\pi |x| e^{-4\pi^2 t |x|^2} \hat{f_j}(x), \end{aligned}$$

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where the second equality is justified by Lemma 2.3 (a). Thus

$$\hat{f}_j(x) = -i\frac{x_j}{|x|}\hat{f}(x) = (\widehat{R_j f})(x)$$

and therefore $f_j(x) = R_j f(x)$, $j = 1, \ldots, n$. ∎

COROLLARY 2.7. *For* $(n-1)/n < p \leq 1$

$$\mathbb{H}^p \subset \{(R_1 u, \ldots, R_n u, u) : u \in H^p\}.$$

PROOF. Let $F = (u_1, \ldots, u_n, u) \in \mathbb{H}^p$ and fix $t_o > 0$. Then

$$u_{j,t_0}(x,t) \equiv u_j(x, t+t_0) = K(\cdot, t) * u_j(\cdot, t_0)(x),$$
$$u_{t_0}(x,t) \equiv u(x, t+t_0) = K(\cdot, t) * u(\cdot, t_0)(x).$$

Theorem 1.2 implies that $u_j(\cdot, t_0), u(\cdot, t_0) \in L^p \cap L^\infty \subset L^q$ for every $q > 1$ and $j = 1, \ldots, n$. Since $(u_{1,t_0}, \ldots, u_{n,t_0}, u_{t_0}) \in AH$, it follows that $(u_{1,t_0}, \ldots, u_{n,t_0}, u_{t_0}) \in \mathbb{H}^q$, therefore Theorem 2.6 implies that $u_j(\cdot, t+t_0) = R_j u(\cdot, t+t_0)$ for every $t > 0$ . Now, $t$ and $t_0$ are arbitrary and we can conclude that for any $s > 0$ $u_j(\cdot, s) = R_j u(\cdot, s)$, $j = 1, \ldots, n$. ∎

For $F \in \mathbb{H}^p$, $(n-1)/n < p < \infty$, we define

$$\|F\|_{\mathbb{H}^p} = \sup_{t>0}\Big[\int_{\mathbb{R}^n} |F(x,t)|^p \, dx\Big]^{1/p},$$

$F \longmapsto \|F\|_{\mathbb{H}^p}$ is a norm for $1 \leq p < \infty$ and $F \longmapsto \|F\|^p_{\mathbb{H}^p}$ is a $p$-norm for $(n-1)/n < p < 1$.

THEOREM 2.8. $\mathbb{H}^p$ *is complete for every* $(n-1)/n < p < \infty$.

PROOF. The proof is an easy consequence of estimates (2) and (12). ∎

Next, we shall characterize our $\mathbb{H}^p$ spaces. We will denote by $\mathbb{H}^p_{\mathrm{arm}}$ the classical Hardy spaces whose elements are the $(n+1)$-tuples of harmonic functions $F = (u_1, \ldots, u_n, u_{n+1})$ on $\mathbb{R}^{n+1}_+$ satisfying the equations of conjugacy (CR) and the condition (G) with the $p$ -norm $\|F\|^p_{\mathbb{H}^p_{\mathrm{arm}}}$ if $(n-1)/n < p < 1$ and the norm $\|F\|_{\mathbb{H}^p_{\mathrm{arm}}}$ if $p \geq 1$.

It is well known that for $1 < p < \infty$, $\mathbb{H}^p_{\mathrm{arm}} \cong L^p$. Theorem 2.6, [5, Theorem 2(xi)] and continuity of $R_j$ from $L^p$ to $L^p$ imply

$$(\mathbb{H}^p, \|\cdot\|_{\mathbb{H}^p}) \cong (\mathbb{H}^p_{\mathrm{arm}}, \|\cdot\|_{\mathbb{H}^p_{\mathrm{arm}}}).$$

It remains to analyze the case $(n-1)/n < p \leq 1$. Since each element of the Hardy space $\mathbb{H}^p$ is uniquely determined by its last component, we shall refer to $u = u_{n+1}$ as the element of $\mathbb{H}^p$ instead of the $(n+1)$-tuple $(u_1, \ldots, u_n, u_{n+1})$ in $\mathbb{H}^p$. Moreover, $u$ has a limit $f \in S'$ and this limit uniquely determines $u$. So, we can think $\mathbb{H}^p$ as the space of boundary distributions $f \in S'$ corresponding to the $(n+1)$-tuples $F = (u_1, \ldots, u_n, u) \in \mathbb{H}^p$, with the $p$-norm $f \longmapsto \|F\|^p_{\mathbb{H}^p}$. We will also adopt the same point of view for the space $\mathbb{H}^p_{\mathrm{arm}}$. It can be shown as in the case of $\mathbb{H}^p_{\mathrm{arm}}$ the continuity of the inclusion $\mathbb{H}^p \subset S'$.

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The following two results will be crucial for the proof of the main theorem of this section. The first one is a classical result by Fefferman and Stein [4, Theorem 11] which states:

For $0 < p < \infty$ and $f \in \mathcal{S}'$, the following are equivalent

(a) $u^*(x) = \sup_{|y-x|<t} |\varphi_t * f(y)| \in L^p$ for some $\varphi \in \mathcal{S}$ satisfying $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$.

(b) The distribution $f$ arises as $f = \lim_{t\to 0} u(\cdot, t)$ in $\mathcal{S}'$, where $u \in \mathbb{H}^p_{\mathrm{arm}}$.

Moreover $\|u\|^p_{\mathbb{H}^p_{\mathrm{arm}}} \sim \|u^*\|^p_p$, where $\sim$ means the standard equivalence of norms ($p$-norms).

The second result gives another characterization of $\mathbb{H}^p_{\mathrm{arm}}$ [12, Proposition 3, p. 123]:

If $f \in \mathcal{S}'$ is restricted at infinity (that is, $f * \varphi \in L^r$ for every $\varphi \in \mathcal{S}$ and for all $r < \infty$ sufficiently large) and $(n-1)/n < p < \infty$, then $f \in \mathbb{H}^p_{\mathrm{arm}}$ if and only if

$$\sup_{t>0} \Big\{ \|f * \varphi_t\|_{L^p} + \sum_{j=1}^{n} \|R_j(f) * \varphi_t\|_{L^p} \Big\} \le C < \infty,$$

where $\varphi \in \mathcal{S}$, $\int_{\mathbb{R}^n} \varphi = 1$, $\varphi_t(x) = t^{-n}\varphi(x/t)$ and $C$ is a constant. Here, $R_j(f)$ means the $j$-th Riesz transform of a distribution $f$ that is restricted at infinity (see [12, p. 123]).

Before to state the main result, we need to prove the following lemma.

LEMMA 2.9. *Let* $f \in \mathcal{S}'$ *and* $w(x,t) = K(\cdot,t) * f(x)$ *for* $(x,t) \in \mathbb{R}^{n+1}_+$. *If* $w^+(x) \equiv \sup_{t>0} |w(x,t)| \in L^p$ *then* $w^*(x) \equiv \sup_{|y-x|<t^{1/2}} |w(y,t)| \in L^p$ *and* $\|w^*\|^p_p \sim \|w^+\|^p_p$, *for* $0 < p < \infty$.

PROOF. Let $x \in \mathbb{R}^n$ and take any point $(y,t) \in \{(\xi,\tau) : |\xi - x| < \tau^{1/2}\}$. If $R = \big(\prod_{i=1}^n (x_i - \sqrt{t}/2\sqrt{2}, x_i + \sqrt{t}/2\sqrt{2})\big) \times (t/2, t)$, then according to Lemma 1.1 we will have

$$\begin{aligned} |w(y,t)|^{p/2} &\le C \frac{1}{t^{(n+2)/2}} \int_R |w(z,t')|^{p/2}\,dz\,dt' \\ &\le C \frac{1}{t^{(n+2)/2}} \int_R w^+(z)^{p/2}\,dz\,dt' \\ &\le C \frac{1}{t^{n/2}} \int_S w^+(z)^{p/2}\,dz, \end{aligned}$$

with $S = \prod_{i=1}^n (x_i - \sqrt{t}, x_i + \sqrt{t})$. This implies that

$$(w^*)^{p/2}(x) \le CM\big((w^+)^{p/2}\big)(x),$$

where $M$ is the Hardy-Littlewood maximal function, thus

$$\begin{aligned} \int_{\mathbb{R}^n} (w^*)^p(x) &\le C \int_{\mathbb{R}^n} \Big(M\big((w^+)^{p/2}\big)\Big)^2(x)\,dx \\ &\le C \int_{\mathbb{R}^n} (w^+)^p(x)\,dx. \end{aligned}$$

The other inequality is immediate. ∎

THEOREM 2.10. *For* $(n-1)/n < p \le 1$, $\mathbb{H}^p \cong \mathbb{H}^p_{\mathrm{arm}}$.

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PROOF. Let $f$ a distribution in $\mathbb{H}^p$, then there exists $(u_1, \ldots, u_n, u) \in \mathbb{H}^p$ such that $f = \lim_{t\to 0} u(\cdot, t)$ in $S'$. Fix $t_0 > 0$. If we denote by $G(x) \equiv \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2}$, then $K(x,t) = G_{\sqrt{2t}}(x)$, thus Corollary 2.7 implies $u_{t_0}(\cdot, t) = G_{\sqrt{2t}} * u(\cdot, t_0)$, $u_{j,t_0}(\cdot, t) = G_{\sqrt{2t}} * R_j u(\cdot, t_0)$, $j = 1, \ldots, n$, moreover

$$\sup_{t>0} \Big\{ \|G_{\sqrt{2t}} * u(\cdot, t_0)\|_{L^p} + \sum_{j=1}^{n} \|G_{\sqrt{2t}} * R_j u(\cdot, t_0)\|_{L^p} \Big\} < \infty$$

and since $u(\cdot, t_0) \in L^q$ for any $q > 1$ it is a distribution restricted at infinity, then [12, Proposition 3, p. 123] implies that $u(\cdot, t_0)$ is a distribution in $\mathbb{H}^p_{\mathrm{arm}}$. Also, from Lemma 2.9 and [4, Theorem 11] we have

$$\Big\| G^*_{\nabla}\big(u(\cdot, t_0)\big) \Big\|_p^p \sim \Big\| G^+\big(u(\cdot, t_0)\big) \Big\|_p^p \sim \|u(\cdot, t_0)\|^p_{\mathbb{H}^p_{\mathrm{arm}}}, \tag{18}$$

where $G^*_{\nabla}\big(u(\cdot, t_0)\big)(x) \equiv \sup_{|y-x|<\sqrt{2t}} |G_{\sqrt{2t}} * u(\cdot, t_0)(y)|$ and $G^+\big(u(\cdot, t_0)\big)(x) \equiv \sup_{t>0} |G_{\sqrt{2t}} * u(\cdot, t_0)(x)|$. On the other side, if we define

$$F_{t_0}(x,t) = \Big(R_1\big(P_t * u(\cdot, t_0)\big)(x), \ldots, R_n\big(P_t * u(\cdot, t_0)\big)(x), P_t * u(\cdot, t_0)(x)\Big)$$

where $P_t$ is the Poisson kernel, we obtain a conjugate system of harmonic functions in $\mathbb{H}^p_{\mathrm{arm}}$ which satisfies

$$F_{t_0}(x,t) \longrightarrow \big(R_1 u(x, t_0), \ldots, R_n u(x, t_0), u(x, t_0)\big) \equiv \big(u_1(x, t_0), \ldots, u_n(x, t_0), u(x, t_0)\big)$$

as $t \longrightarrow 0$ a.e. on $\mathbb{R}^n$ and as in [6, Corollary 1.2, p. 233] it can be shown that

$$\|F_{t_0}\|^p_{\mathbb{H}^p_{\mathrm{arm}}} \sim \Big\| \big(u_1(\cdot, t_0), \ldots, u_n(\cdot, t_0), u(\cdot, t_0)\big) \Big\|_p^p.$$

Therefore

$$\begin{aligned} \|u(\cdot, t_0)\|^p_{\mathbb{H}^p_{\mathrm{arm}}} &\equiv \|F_{t_0}\|^p_{\mathbb{H}^p_{\mathrm{arm}}} \\ &\leq C \Big\| \big(u_1(\cdot, t_0), \ldots, u_n(\cdot, t_0), u(\cdot, t_0)\big) \Big\|_p^p \\ &\leq C \|f\|^p_{\mathbb{H}^p}. \end{aligned} \tag{19}$$

Combining (18) and (19) we get

$$\Big\| G^*_{\nabla}\big(u(\cdot, t_0)\big) \Big\|_p^p \leq C \|u(\cdot, t_0)\|^p_{\mathbb{H}^p_{\mathrm{arm}}} \leq C \|f\|^p_{\mathbb{H}^p}$$

and since $G^*_{\nabla}(f) = \lim_{t_0 \to 0} G^*_{\nabla}\big(u(\cdot, t_0)\big)$ on $\mathbb{R}^n$, an application of Fatou's Lemma yields

$$\|G^*_{\nabla}(f)\|_p^p \leq C \|f\|^p_{\mathbb{H}^p},$$

consequently

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$$\|f\|^p_{\mathbb{H}^p_{\mathrm{arm}}} \le C\|f\|^p_{\mathbb{H}^p}, \tag{20}$$

which shows that $f$ is a distribution in $\mathbb{H}^p_{\mathrm{arm}}$.

Conversely, if $f$ is a distribution in $\mathbb{H}^p_{\mathrm{arm}}$, there exists a conjugate system of harmonic functions $F = (u_1, \dots, u_n, u) \in \mathbb{H}^p_{\mathrm{arm}}$ such that $f = \lim_{t\to 0} u(\cdot, t)$ in $S'$. From [12, Proposition 3, p. 123] we have

$$\sup_{t>0}\Big\{\|f * G_{\sqrt{2t}}\|_{L^p} + \sum_{j=1}^{n} \|R_j f * G_{\sqrt{2t}}\|_{L^p}\Big\} < \infty,$$

thus, the function $w(x, t) = K(\cdot, t) * f(x)$ belong to $H^p$. Moreover, for arbitrary $t$, $t_0 > 0$ we can write $w(x, t + t_0) = K(\cdot, t) * w(x, t_0)$ and since $w(\cdot, t_0) \in L^p \cap L^\infty \subset L^q$ for every $q > 1$, it follows that for $j = 1, \dots, n$, $R_j w(\cdot, t_0) \in L^q$. Thus, the function $\big(w_1(x, t + t_0), \dots, w_n(x, t + t_0), w(x, t + t_0)\big) \in AH$, where $w_j(x, t + t_0) \equiv K(\cdot, t) * R_j w(\cdot, t_0)(x)$. Furthermore, $(w_1, \dots, w_n, w) \in \mathbb{H}^p$ because $w_j(x, t + t_0) = K(\cdot, t + t_0) * R_j f(x)$. Now, by Theorem 1.3, $w(\cdot, t)$ converges in $\mathbb{H}^p$ and in $S'$. Since $S'$ is a Hausdorff space, the limit must be $f$.

To finish the proof, it is sufficient to notice from (20) that the bijective linear mapping $T\colon \mathbb{H}^p \longrightarrow \mathbb{H}^p_{\mathrm{arm}}$, $F = (u_1, \dots, u_n, u) \longmapsto f = \lim_{t\to 0} u(\cdot, t)$ in $S'$ is continuous, and the result follows by Open Mapping Theorem. ∎

COROLLARY 2.11. *Let $f \in S'$ and $(n-1)/n < p \le 1$. Then $f$ is a distribution in $\mathbb{H}^p$ if and only if the maximal function*

$$u^*(x) = \sup_{|y-x|<\sqrt{2t}} |G_{\sqrt{2t}} * f(y)|$$

*belongs to $L^p$.*

PROOF. It is sufficient to apply Theorem 2.10 and Lemma 2.9. ∎

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*Instituto de Matemáticas*
*Universidad Nacional Autónoma de México*
*Ciudad Universitaria*
*D. F., 04510*
*México*
*e-mail: martha@matem.unam.mx*

*Instituto de Matemáticas*
*Universidad Nacional Autónoma de México*
*Unidad Cuernavaca*
*Apartado Postal 273-3*
*Administración de correos # 3*
*Cuernavaca, Morelos, 62251*
*México*
*e-mail: chava@matem.unam.mx*