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ON ROBILLARD'S BOUNDS FOR RAMSEY NUMBERS

BY

J. G. KALBFLEISCH

1. **Introduction.** A recent issue of the Bulletin contained a paper by Robillard [13] in which results from the theory of confounded factorial designs were used to obtain some lower bounds for Ramsey numbers. We shall derive, by more elementary methods, bounds which are much better than Robillard's in every example which he considered.

2. **The Ramsey numbers.** Let $r, t, q_1, \ldots, q_t$ be positive integers with $q_i \geq r$ $(i=1, 2, \ldots, t)$. The r-subsets of an n-set are partitioned into t classes $A_1 \cup A_2 \cup \cdots \cup A_t$ (coloured in t colours). The partition is called a $(q_1, \ldots, q_t; r)$ colouring of the n-set if there exists no q_i-subset whose r-subsets all belong to A_i $(i=1, 2, \ldots, t)$. The Ramsey number $N(q_1, \ldots, q_t; r)$ is the least n for which there exists no $(q_1, \ldots, q_t; r)$ colouring of the n-set. We shall write $N_t(q; r)$ for $N(q_1, \ldots, q_t; r)$ with $q_1 = \cdots = q_t = q$. The existence of the Ramsey numbers follows from a theorem of Ramsey [12].

It is difficult to obtain good bounds for the Ramsey numbers, and many papers have been written on this subject. Most of them deal with the case $r=2$, in which a few of the smaller numbers are known [2], [5], [6], [8], [10], [14]. Robillard's examples all have $r=3$. In this case, the only known Ramsey numbers are the trivial values

$$N(q, 3; 3) = N(3, q; 3) = q. \tag{1}$$

It has been shown [4], [7], [9] that

$$13 \leq N(4, 4; 3) \leq 17. \tag{2}$$

Otherwise, the best available upper and lower bounds are far apart.

3. **Colourings of 3-subsets in two colours.** Choose $n \leq N(p, q-1; 3)-1$, and $m \leq N(p-1, q; 3)-1$. Let S be an $(n+m)$-set, and put $S = S_1 \cup S_2$ where $|S_1| = n$ and $|S_2| = m$. We colour the 3-subsets of S as follows:

(i) construct a $(p, q-1; 3)$ colouring of S_1;
(ii) construct a $(p-1, q; 3)$ colouring of S_2;
(iii) assign colour 1 (red) to 3-subsets containing one element of S_1 and two elements of S_2;
(iv) assign colour 2 (blue) to 3-subsets containing one element of S_2 and two elements of S_1.

Now suppose that there exists a p-set P whose 3-subsets are all red. Then P contains at most $p-1$ elements of S_1 by (i), and at most $p-2$ elements of S_2 by (ii). Thus P contains at least one element of S_2 and at least two elements of S_1. By (iv), P has a blue 3-subset. Hence there exists no p-set whose 3-subsets are all red. Similarly, there exists no blue q-set, and the above construction gives a $(p, q; 3)$ colouring of the $(n+m)$-set. It follows that

$$N(p,q;3) > N(p,q-1;3)+N(p-1,q;3)-2. \tag{3}$$

If (3) is applied repeatedly using initial condition (1), one obtains

$$N(p,q;3) > [N(4,4;3)-7]\binom{p+q-8}{p-4}+\binom{p+q-4}{p-2}. \tag{4}$$

4. **Comparison with Robillard's results.** We list the bounds implied by (4) and (2) in the particular numerical examples selected in [13]. Robillard's bounds follow in parentheses.

$N(5,4;3) > 16$	(7)	$N(6,4;3) > 21$	(15)
$N(5,5;3) > 32$	(18)	$N(11,5;3) > 268$	(40)
$N(7,6;3) > 186$	(21)	$N(18,6;3) > 5565$	(85)
$N(7,7;3) > 372$	(31)		

Robillard also gives the following results:

$$N(q+2,q+2;3) > q^2+q+1 \tag{5}$$

$$N(q^2+2,q+2;3) > q^3+q^2+q+1 \tag{6}$$

$$N(q+2,q+1,\ldots,5,4;3) > 2^q-1 \quad \text{for } q \geq 4. \tag{7}$$

One can easily verify that (4) improves substantially on (5) and (6). Also, since any $(q+2, q+1; 3)$ colouring is also a $(q+2, q+1, \ldots, 5, 4; 3)$ colouring, we have

$$N(q+2,q+1,\ldots,5,4;3) > N(q+2,q+1;3). \tag{8}$$

Now apply (4) to the right-hand side of (8). The result is an improvement on (7). Hence, in all cases considered by Robillard, (4) gives superior bounds.

5. **A general lower bound.** Choose $t>1$ and $n>q_1, q_2, \ldots, q_t \geq r$. The total number of ways to colour the r-subsets of an n-set with t colours is $t^{\binom{n}{r}}$. The number of colourings in which a specified q_i-set has colour i is $t^{\binom{n}{r}-\binom{q_i}{r}}$. Hence the number of colourings in which at least one q_i-set has colour i is less than $\binom{n}{q_i}t^{\binom{n}{r}-\binom{q_i}{r}}$. (Strict inequality is obtained because some colourings will be counted more than once.) The number of colourings in which, for some $i=1, 2, \ldots,$ or t, there is at least one q_i-set of colour i, is less than

$$\sum_{i=1}^{t}\binom{n}{q_i}t^{\binom{n}{r}-\binom{q_i}{r}}. \tag{9}$$

But, for $n \geq N(q_1, \ldots, q_t; r)$, every colouring produces a q_i-set of colour i for some i, so that (9) must exceed $t^{\binom{n}{r}}$. Hence

$$(10) \qquad n \geq N(q_1, \ldots, q_t; r) \text{ implies } \sum_{i=1}^{t} \binom{n}{q_i} / t^{\binom{q_i}{r}} > 1.$$

This result provides a general lower bound for $N(q_1, \ldots, q_t; r)$.

The above is a generalization of an argument given by Erdös [1] for $r=t=2$ and $q_1=q_2=q$. The generalization was previously given by Krieger [11] for general r and t but equal q's, in which case (10) becomes

$$(11) \qquad n \geq N_t(q; r) \text{ implies } t\binom{n}{q} > t^{\binom{q}{r}}.$$

We may compare the bounds provided by (4) with those given by (11) in the case $t=2$, $r=3$. We find that, for small values of q, (4) gives better bounds than (11). For example,

$$N(6,6;3) > 106 \text{ by (4)}; \qquad N(6,6;3) > 29 \text{ by (11)}.$$
$$N(7,7;3) > 372 \text{ by (4)}; \qquad N(7,7;3) > 100 \text{ by (11)}.$$

The reverse is true for large q. A straightforward application of Stirling's formula shows that, for large q, the bound for $N(q, q; 3)$ obtained from (4) is less than 2^{2q-4}, while the bound from (11) exceeds $2^{(q-1)(q-2)/6}$.

For further results on the estimation of $N(p, q; r)$ with large p and q, see [11] and [3].

References

1. P. Erdös, *Some remarks on the theory of graphs*, Bull. Amer. Math. Soc. **53** (1947), 292–294.

2. ———, *Graph theory and probability II*, Canad. J. Math. **13** (1961), 346–352.

3. P. Erdös, A. Hajnal, and R. Rado, *Partition relations for cardinal numbers*, Acta. Math. Acad. Sci. Hungar. **16** (1965), 93–196.

4. Guy Giraud, *Sur les nombres de Ramsey ternaires-bicolores de la diagonale*, C. R. Acad. Sc. Paris, Sér. A, **268** (1969), 85–87.

5. J. E. Graver and J. Yackel, *Some graph theoretic results associated with Ramsey's theorem*, J. Comb. Theory **4** (1968), 125–175.

6. R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math. **7** (1955), 1–7.

7. John R. Isbell, $N(4, 4; 3) \geq 13$, J. Comb. Theory **6** (1969), p. 210.

8. J. G. Kalbfleisch, *Chromatic graphs and Ramsey's theorem*, Ph.D. thesis, Univ. of Waterloo, 1966.

9. ———, *On the Ramsey number* $N(4, 4; 3)$, *Recent progress in combinatorics*, (edited by W. T. Tutte), Academic Press, New York, 1969, 273–282.

10. G. Kéry, *Ramsey egy gráfélmeleti tételéröl*, Mat. Lapok **15** (1964), 202–224.

11. Michael M. Krieger, *An inequality for higher Ramsey numbers*, Notices Amer. Math. Soc. **15** (1968), p. 1035, Abstract No. 662–18.

12. F. P. Ramsey, *On a problem in formal logic*, Proc. London Math. Soc. (2) **30** (1930), 264–286.

13. Pierre Robillard, *Lower bounds for the Ramsey numbers*, Canad. Math. Bull. **13** (1970), 227–229.

14. K. Walker, *Dichromatic graphs and Ramsey numbers*, J. Comb. Theory **5** (1968), 238–243.

UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO