AN ACYCLIC ANALOGUE TO HEAWOOD'S THEOREM

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(Received 25 April, 1977; revised 12 October, 1977)

1. Introduction. The concept of acyclic coloring was introduced by Grünbaum [**5**] and is a generalization of point arboricity.

A proper k-coloring of the vertices of a graph G is said to be *acyclic* if G contains no two-colored cycle. The *acyclic chromatic number* of a graph G, denoted by $a(G)$, is the minimum value of k for which G has an acyclic k-coloring. Let $a(n)$ denote the maximum value of the acyclic chromatic number among all graphs of genus n. In [**5**], Grünbaum conjectured that $a(0)=5$ and proved that $a(0)\leq 9$. The conjecture was proved by Borodin [**3**] after the upper bound was improved three times in [**7**], [**1**] and [**6**]. In [**2**], we proved that $a(1)\leq a(0)+3$. The purpose of this paper is to prove the following:

THEOREM. *Any graph of genus $n>0$ can be acyclically colored with $4n+4$ colors.*

It is not known for any $n>0$ whether $a(n)>H(n)$, the Heawood number [**8**].

2. Preliminaries. The proof of the theorem is by a double induction on n, the genus, and V, the number of vertices. Since the theorem is true for $n=1$ [**2**] and trivally true if $V\leq 4n+4$, the induction begins. Let G be a graph with V vertices which is 2-cell imbedded on S_n, the n-handle sphere, (i.e. every region is homeomorphic to a disc). The inductive hypotheses will be that any such graph of genus at most $n-1$ can be acyclically colored with $4n$ colors and that any graph of genus n with fewer than V vertices can be acyclically colored with $4n+4$ colors. The induction will proceed using the concept of reducibility.

A graph H is said to be *acyclically k-color reducible* if, whenever a graph J contains H as a subgraph, we can define a graph J' having fewer vertices than J and having the property that given any acyclic k-coloring of J' we can obtain an acyclic k-coloring of J. A graph H is said to be *reducible* if it is acyclically k-color reducible for every $k\geq 7$. Clearly, showing that G contains a reducible graph would suffice to prove the theorem. We restate a result proved in [**1**].

LEMMA 1. *Let C be a 4-cycle enclosing a planar region. Let L be the set of vertices interior to C. If H, the induced subgraph on the vertices of $L\cup C$, is a triangulation of the interior of C and if L has more than one vertex, then H contains a reducible graph.*

We call a cycle $C\subset G$ contractible (resp. non-contractible) if it is (resp. is not) homotopic to a point.

LEMMA 2. *Let $C=c_1, c_2, \ldots, c_r$ be a non-contractible cycle in G that does not separate S_n. Then $G-C$ has genus at most $n-1$.*

* Research supported in part by the Research Corporation through a Cottrell College Science Research Grant.

Proof. Define graph G^* by slitting C into two independent parallel cycles $C' = c'_1, \ldots, c'_r$ and $C^* = c^*_1, \ldots, c^*_r$ leaving all incident edges intact.

Now define graph G' by taking G^* and "cutting" the surface of S_n along a simple closed curve through the strip between C' and C^*. Then patch the surface by "pasting" discs into C' and C^*, creating two new faces bounded by C' and C^*, respectively. G' is 2-cell imbedded on the new surface.

If G has V vertices, E edges and F faces, then G' has $V+r$ vertices, $E+r$ edges and $F+2$ faces. Let n' be the genus of G'. By Euler's formula,

$$\begin{aligned} 2-2n' &= (V+r)-(E+r)+(F+2) \\ &= V-E+F+2 \\ &= 2-2n+2. \end{aligned}$$

Thus, $n' = n-1$. Since G' has genus $n-1$, $G-C = G'-(C' \cup C^*)$ has genus at most $n-1$.

LEMMA 3. *Let C be a non-contractible (simple) cycle in G that separates S_n. Then $G-C$ is the union of two graphs, each of genus at most $n-1$.*

Proof. When we perform the same "cut and paste" operation used in the proof of Lemma 2 the resulting graph G' is disconnected. Say $G' = G_1 \cup G_2$, where G_1 and G_2 are 2-cell imbedded on their respective surfaces with G_1 having V_1 vertices, E_1 edges, F_1 faces, and genus n_1; G_2 having V_2 vertices, E_2 edges, F_2 faces, and genus n_2.

As in the proof of Lemma 2,

$$V_1+V_2 = V+r, \quad E_1+E_2 = E+r \quad \text{and} \quad F_1+F_2 = F+2.$$

Then

$$\begin{aligned} 2-2n &= V-E+F \\ &= (V+r)-(E+r)+F \\ &= (V_1+V_2)-(E_1+E_2)+(F_1+F_2-2) \\ &= (V_1-E_1+F_1)+(V_2-E_2+F_2)-2 \\ &= (2-2n_1)+(2-2n_2)-2. \end{aligned}$$

So $n_1+n_2 = n$. Since C is non-contractible neither n_1 nor n_2 can be zero. Thus G' and, hence, $G-C = G'-(C' \cup C^*)$ is the union of two graphs each of genus at most $n-1$.

3. Proof of the Theorem. Let $C = c_1, \ldots, c_r$ be a non-contractible cycle of minimum length in G. Depending on whether C does or does not separate S_n, we apply Lemma 2 or Lemma 3 to show that $G-C$ either is the disjoint union of two graphs of genus at most $n-1$, or else is itself of genus at most $n-1$. In either case we can apply the inductive hypothesis to show that $G-C$ can be acyclically colored with $4n$ colors.

We show that if G contains no reducible subgraph then the vertices of C can be replaced and colored with four new colors in such a fashion that G will contain no two-colored cycle. This is done in three cases depending on r, the length of C.

(i) If $r \leq 4$, use a new color for each point of C. No two-colored cycle can be introduced as each of the new colors occurs only once in G.

For the next case we assume G is a triangulation. If G is not a triangulation the addition of edges to G cannot decrease $a(G)$. If edges cannot be added to G to make it a triangulation then G contains a pair of vertices, say x and y, such that $G-x-y$ has genus less than n. The proof would then proceed as in (i), using instead of C the subgraph $\{x, y\}$.

(ii) Assume G is a triangulation and $r=5$. Color $c_1, \ldots, c_5$ with four new colors a, b, a, c, d respectively. The only two-color cycle that can be introduced is a four-cycle of the form $C' = c_1, p, c_3, q$. C' must be contractible as we assumed C was a minimum length non-contractible cycle. Thus, there can be at most one vertex interior to C'. Otherwise Lemma 1 guarantees that G contains a reducible graph.

If C' has no interior vertices, then since G is a triangulation, either c_1 is adjacent to c_3 or p is adjacent to q. But in the first instance C was not the shortest non-contractible cycle and in the second instance the acyclic coloring of G' was not proper. Thus we may assume there is exactly one point, say x, inside C'.

By the minimality of C, both c_1, c_2, c_3, p and c_1, c_2, c_3, q must be contractible. But then either c_1, c_2, c_3, p contains x and q in its interior, or else c_1, c_2, c_3, q contains x and p in its interior. We invoke Lemma 1 to show that G contains a reducible graph.

(iii) If $r \geq 6$, color the vertices of C according to the following prescription. Construct the graph C^2 whose vertices are the vertices of C, with edges joining two vertices of C^2 if the vertices are of distance one or two in the graph induced on the vertices of C. Since the latter graph is regular of degree two (C can have no diagonals), C^2 is regular of degree four. Since C^2 has maximum degree four and does not contain K_5, Brooks' Theorem [**4**] implies that C^2 can be properly four-colored.

When replacing C into G, color it with four new colors according to the proper four-coloring of C^2. The only two-color cycle which can be introduced is of the form $\ldots c_i, p, c_j, \ldots$, where we assume $i<j$.

Since c_i and c_j are colored the same they cannot be adjacent in C^2; thus they must have at least two vertices between them along C. Now consider the cycles $c_1, \ldots, c_i, p, c_j, \ldots, c_r$ and $c_i, c_{i+1}, \ldots, c_j, p$. Both have length less than r, and at least one is non-contractible since their mod 2 sum is C. Thus no two-color cycles can be introduced and the theorem is proved.

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