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On Deformations of Nodal Hypersurfaces

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Abstract. We extend the infinitesimal Torelli theorem for smooth hypersurfaces to nodal hypersurfaces.

1 Introduction

Deformations of smooth hypersurfaces provide examples of great interest and importance in the theory of variation of Hodge structures, especially because of the generic Torelli theorem; see [13, Chapter 6]. In a recent thesis, Y. Zhao [15] considers deformations of nodal surfaces in the 3-dimensional complex projective space $\mathbb{P}^3$ and shows that the infinitesimal Torelli theorem still holds.

Let $S = \mathbb{C}[x_0, \dots, x_n] = \bigoplus_{d=0}^{\infty} S_d$ be the graded ring of polynomials and let $f \in S_d$ be a homogeneous polynomial of degree d. Denote by $X_f : f = 0$ the hypersurface in $\mathbb{P}^n$ defined by f. Moreover, let

$$J(f) = \Big(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}\Big)$$

be the graded ideal generated by the first derivatives of f, also called the *Jacobian ideal* of f. We consider the map

$$\begin{aligned} \varphi\colon \big(S/J(f)\big)_d &\longrightarrow \operatorname{Hom}\big((S/J(f))_{d-n-1}, (S/J(f))_{2d-n-1}\big), \\ [P] &\longmapsto \big([Q] \longmapsto [PQ]\big). \end{aligned} \tag{1.1}$$

As a matter of fact, Y. Zhao [15] proves the infinitesimal Torelli theorem by showing that the map φ is injective when $n = 3$ and X_f is a nodal surface. This result can be extended to higher dimensional cases.

Theorem 1.1 *Assume $n \geq 3$ is an integer and $d \geq n+1$. Let $f \in S$ be a homogeneous polynomial of degree d such that $X_f : f = 0$ is a nodal hypersurface in $\mathbb{P}^n$. Then the map φ is injective.*

As was proved in [15, Chapter 3, Example 3.1.3], $(S/J(f))_d$ parameterizes the equivalence classes of deformations of the pair $(\mathbb{P}^n, X_f)$. Alternatively, let $GL = GL(n+1, \mathbb{C})$ be the general linear group of rank $n+1$. Then GL acts on S_d by coordinate transformations and for any $f \in S_d$, the tangent space at f of the orbit $GL \cdot f$ is given by $J(f)_d$; see [1, Chapter 4, Formula (4.16)]. It follows that $(S/J(f))_d$ can be seen as the set of directions in S_d that are transversal to the orbit $GL \cdot f$ at f. In

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addition, any smooth analytic subset $\mathcal{U} \subseteq S_d$ can be seen as a family of hypersurfaces in $\mathbb{P}^n$. If $f \in \mathcal{U}$ and $T_f\mathcal{U} \cap J(f)_d = \{0\}$, then we call $\mathcal{U}$ an *effective* deformation of f. From this point of view, $(S/J(f))_d$ is the maximal set of effective deformations of f.

Now let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ and let $n(f)$ be the number of nodes in X_f. Then we have a moduli space, denoted by $\mathfrak{B}_f \subseteq S_d$, parameterizing all nodal hypersurfaces in $\mathbb{P}^n$ having exactly $n(f)$ nodes. By the discussion following [2, Chapter 1, Corollary 3.8], we have that $\mathfrak{B}_f$ is a constructible subvariety of S_d and the topological type of $(\mathbb{P}^n, X_g)$ is locally trivial for $g \in \mathfrak{B}_f$. Moreover, for any g lying in the connected component of $\mathfrak{B}_f$ containing f, $(\mathbb{P}^n, X_g)$ is topologically equivalent to $(\mathbb{P}^n, X_f)$.

Now assume that $\mathcal{U} \subseteq \mathfrak{B}_f$ is a connected smooth subvariety and $f \in \mathcal{U}$. For any $g \in \mathcal{U}$, X_g is homeomorphic to X_f by the local topological triviality of the pair $(\mathbb{P}^n, X_g)$. So there is a natural identification $H_0^{n-1}(X_g) \cong H_0^{n-1}(X_f)$, where $H_0^{n-1}(X_g)$ is the primitive cohomology of X_g defined by $H_0^{n-1}(X_g) = \operatorname{Coker}(H^{n-1}(\mathbb{P}^n) \to H^{n-1}(X_g))$. In particular, $\dim H_0^{n-1}(X_g)$ is constant for $g \in \mathcal{U}$.

Moreover, $H_0^{n-1}(X_g)$ has a natural mixed Hodge structure, since X_g is a singular algebraic variety, see [11, Part II, Chapter 5]. It turns out that $\dim F^{n-1}H_0^{n-1}(X_g)$ and $\dim F^{n-2}H_0^{n-1}(X_g)$ are also constant for $g \in \mathcal{U}$ (in most cases); see Corollary 3.4. Thus, we have the well-defined map

$$\mathcal{P}: \mathcal{U} \ni g \longmapsto \left(F^{n-1}H_0^{n-1}(X_g), F^{n-2}H_0^{n-1}(X_g)\right) \in \mathcal{F}, \tag{1.2}$$

where $\mathcal{F}$ is the corresponding flag manifold of subspaces of $H_0^{n-1}(X_f)$.

By relating the primitive cohomology with the graded pieces of the algebra $S/J(f)$ and applying Theorem 1.1, we prove the following theorem, as a generalization of [15, Chapter 3].

Theorem 1.2 *Assume $n \geq 3$ is odd or $n \geq 6$ is even. Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$ and let $\mathcal{U} \subseteq \mathfrak{B}_f$ be a smooth subvariety of $\mathfrak{B}_f$ that gives an effective deformation of X_f. Then the map $\mathcal{P}$ above is well defined, and the differential $d\mathcal{P}$ is injective at f.*

Thus, loosely speaking, the infinitesimal Torelli theorem also holds for nodal hypersurfaces.

Note that for smooth hypersurfaces, the generic Torelli theorem holds; see [13, Part II, Chapter 6, Section 6.3.2], and it remains an interesting question whether this is also the case for nodal hypersurfaces. Recall that in the proof of the generic Torelli theorem for smooth hypersurfaces, the essential part is to show that a generic homogeneous polynomial can be reconstructed from its Jacobian ideal, which also holds for nodal hypersurfaces by [14, Theorem 1.1], because a generic f of degree $d > 3$ with the associated hypersurface X_f having a fixed number of nodes is not of Sabastiani–Thom type, which is the only exception for f not to be reconstructed from $J(f)$; another key ingredient in the smooth case is the symmetriser lemma, which is still open for nodal hypersurfaces.

2 Syzygies of the Jacobian Ideal

Let $K^\bullet(f)$ be the Koszul complex of $\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n}$ with the natural grading $\deg(x_j) = 1$ and $\deg(dx_j) = 1$:

$$K^\bullet(f)\colon 0 \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \cdots \longrightarrow \Omega^{n+1} \longrightarrow 0,$$

where $\Omega^1 = \sum_{i=0}^n S dx_i$ and $\Omega^p = \bigwedge^p \Omega^1$, and the differentials are given by the wedge product with $df = \sum_{i=0}^n \frac{\partial f}{\partial x_i} dx_i$.

The homogeneous component of the cohomology group $H^n(K^\bullet(f))_{n+r}$ describes the syzygies

$$\sum_{j=0}^n a_j \frac{\partial f}{\partial x_j} = 0$$

with $a_j \in S_r$ modulo the trivial syzygies generated by

$$\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} + \Big(-\frac{\partial f}{\partial x_i}\Big)\Big(\frac{\partial f}{\partial x_j}\Big) = 0, \qquad i < j.$$

We can restate the main result in [5] or in [6, Theorem 9] in the following form.

Lemma 2.1 *Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d > 2$ and $n \geq 3$, then $H^n(K^\bullet(f))_m = 0$ for any*

$$m \leq \frac{nd-1}{2}.$$

Let $f_s \in S_d$ be such that $X_{f_s} : f_s = 0$ is a smooth hypersurface. It is well known that $\dim(S/J(f_s))_k$ depends only on n, d, and k; see [1, Chapter 7, Proposition 7.22]. In the introduction part of [4], the following two notions are given:

$$\operatorname{ct}(X_f) = \max\big\{q : \dim(S/J(f))_k = \dim(S/J(f_s))_k \text{ for all } k \leq q\big\},$$
$$\operatorname{mdr}(X_f) = \min\big\{q : H^n(K^\bullet(f))_{q+n} \neq 0\big\}.$$

They have the following relation

$$\operatorname{ct}(X_f) = \operatorname{mdr}(X_f) + d - 2;$$

see loc. cit.. We have the following lemma.

Lemma 2.2 *Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$ and $n \geq 3$, then*

$$\dim\big(S/J(f)\big)_k = \dim\big(S/J(f_s)\big)_k, \quad k = d-n-1, 2d-n-1.$$

In particular, $\dim(S/J(f))_k$ *does not depend on the concrete equation of the polynomial f for $k = d-n-1, 2d-n-1$.*

Proof We only need to check that $2d - n - 1 \leq \operatorname{ct}(X_f)$. Indeed, by Lemma 2.1, we immediately have

$$\operatorname{ct}(X_f) = \operatorname{mdr}(X_f) + d - 2 \geq \Big(\frac{nd-1}{2} - n\Big) + d - 2 > 2d - n - 1,$$

where the last inequality follows from $n \geq 3$ and $d \geq n+1$. ∎

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2.1 Proof of Theorem 1.1

To prove Theorem 1.1, we first need the following lemma.

Lemma 2.3 *Assume $X_f : f = 0$ is a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$ and $n \geq 3$. Let $G \in S_t$ such that $t < 2d - n - 1$ and $Gx_j \in J(f)$ for all $j = 0, \dots, n$; then $G \in J(f)$.*

Proof Assume

$$Gx_i = \sum_{k=0}^{n} H_{ik} \frac{\partial f}{\partial x_k}, \quad i = 0, \dots, n,$$

with $H_{ik} \in S_{t+2-d}$, $i, k = 0, \dots, n$, then

$$0 = x_i(x_j G) - x_j(x_i G) = \sum_{k=0}^{n} (x_i H_{jk} - x_j H_{ik}) \frac{\partial f}{\partial x_k}.$$

Note that

$$t + 3 - d + n \leq (2d - n - 2) + 3 + n - d = d + 1 \leq \frac{nd - 1}{2},$$

so by Lemma 2.1, we get $x_i H_{jk} - x_j H_{ik} \in J(f)$ for all $i, j, k = 0, \dots, n$ while all these polynomials have degree $t + 3 - d < (2d - n - 1) + 3 - d = d - n + 2 \leq d - 1$, so they must all vanish identically; in particular,

$$x_i H_{jk} - x_j H_{ik} = 0, \quad i \neq j,$$

thus, $x_i | H_{ik}$. It follows that $G \in J(f)$ as desired. ■

Proof of Theorem 1.1 We first remark that Theorem 1.1 holds when $d = n+1$. In fact, in this case, $J(f)_{d-n-1} = J(f)_0 = 0$ and $(S/J(f))_{d-n-1} = S_0 = \mathbb{C}$ consists of constants. Since $1 \in (S/J(f))_{d-n-1}$ and $\varphi([P])(1) = [P]$, one sees easily that φ is injective.

Thus, in the sequel of the proof, we will focus on the case $d > n+1$.

Aiming at a contradiction, we assume that there exists $P \in S_d \setminus J(f)_d$ such that $\varphi([P]) = 0$.

Then there exists a $Q \in S_l$, $0 \leq l < d-n-1$ such that $PQ \notin J(f)$ and l is chosen to be maximal. By the maximality of l, we have $(PQ)x_j \in J(f)$ for all $j = 0, \dots, n$. Note that $PQ \in S_{l+d}$ and $l + d < 2d - n - 1$, hence by Lemma 2.3, $PQ \in J(f)$, contradiction. ■

3 Hodge Theory for Nodal Hypersurfaces

Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$ and $n \geq 3$. The cohomology groups under consideration below all have $\mathbb{C}$ as coefficients unless otherwise explicitly pointed out.

By the Lefschetz hyperplane theorem for singular varieties (see [9]), we have

$$H^i(X_f) = H^i(\mathbb{P}^n), \qquad i < n-1,$$

and

$$H^{n-1}(\mathbb{P}^n) \longrightarrow H^{n-1}(X_f)$$

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is injective. Let

$$H_0^{n-1}(X_f) = \operatorname{Coker}(H^{n-1}(\mathbb{P}^n) \to H^{n-1}(X_f)),$$

be the primitive cohomology of X_f. Then $H_0^{n-1}(X_f)$ admits a mixed Hodge structure. Moreover, let $U_f = \mathbb{P}^n \smallsetminus X_f$ be the complement of $X = X_f$; then $H^n(U_f)$ also admits a mixed Hodge structure and $H^n(U_f)$ and $H_0^{n-1}(X_f)$ are closely related.

3.1 Relation Between $H^*(U_f)$ and $H^*(X_f)$

Let X_f^* be the smooth locus of X_f and let

$$H_0^{n-1}(X_f^*) = \operatorname{Coker}\left(H^{n-1}(\mathbb{P}^n) \longrightarrow H^{n-1}(X_f^*)\right).$$

Then $H_0^{n-1}(X_f^*)$ has a natural mixed Hodge structure. Moreover, as is shown in [2, Chapter 6, Corollary 3.11], there is a natural residue isomorphism

$$\overline{R}_f\colon H^n(U_f) \xrightarrow{\sim} H_0^{n-1}(X_f^*) \tag{3.1}$$

which is also an isomorphism of mixed Hodge structures of type $(-1,-1)$.

Let $i\colon X_f^* \to X_f$ be the inclusion. We have the naturally induced homomorphisms in cohomology

$$i^*\colon H^{n-1}(X_f) \longrightarrow H^{n-1}(X_f^*),$$
$$i_0^*\colon H_0^{n-1}(X_f) \longrightarrow H_0^{n-1}(X_f^*).$$

Moreover, i^*, i_0^* are also morphisms of mixed Hodge structures. Our discussion will be divided into two cases.

3.1.1 Case 1: n is odd

When n is odd, the variety X_f is a $\mathbb{Q}$-homology manifold, *i.e.,* for any point $x \in X_f$, $H^i(X_f, X_f \smallsetminus \{x\}, \mathbb{Q}) = \mathbb{Q}$ if $i = 2(n-1)$ and 0 otherwise. Moreover, we have the following claim.

Claim 3.1 *i^* and i_0^* are both isomorphisms.*

Proof Indeed, we have a long exact sequence of mixed Hodge structures with respect to the pair (X_f, X_f^*):

$$\longrightarrow H^{n-1}(X_f, X_f^*) \longrightarrow H^{n-1}(X_f) \xrightarrow{i^*} H^{n-1}(X_f^*) \longrightarrow H^n(X_f, X_f^*) \tag{3.2}$$

Let x_i, $i = 1, \dots, r$ be all the nodes in X_f; then $X_f^* = X_f \smallsetminus \{x_1, \dots, x_r\}$, and furthermore, by the excision theorem,

$$H^{n-1}(X_f, X_f^*) = \bigoplus_{i=1}^{r} H^{n-1}\left(X_f, X_f \smallsetminus \{x_i\}\right) = 0,$$

since X_f is a $\mathbb{Q}$-homology manifold and $n-1 \neq 0, 2(n-1)$ for $n \geq 3$. Similarly, $H^n(X_f, X_f^*) = 0$. Thus, it follows from (3.2) that i^* and i_0^* are both isomorphisms. ∎

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Note that the weights of $H^{n-1}(X_f)$ are $\leq n-1$, since X_f is compact while the weights of $H^{n-1}(X_f^*)$ are $\geq n-1$ since X_f^* is smooth (see [11, p. 131, Table 5.1]), hence both $H^{n-1}(X_f^*)$ and $H^{n-1}(X_f)$ have pure Hodge structures of weight $n-1$ and it follows from the isomorphism (3.1) that $H^n(U_f)$ has a pure Hodge structure of weight $n+1$.

Let

$$R_f = (i_0^*)^{-1} \circ \overline{R}_f \colon H^n(U_f) \longrightarrow H_0^{n-1}(X_f).$$

Then R_f is an isomorphism of mixed Hodge structures of type $(-1,-1)$. It follows that we have isomorphisms

$$R_f \colon F^p H^n(U_f) \xrightarrow{\sim} F^{p-1} H_0^{n-1}(X_f)$$

for all p. In particular, there are isomorphisms

$$R_f \colon Gr_F^{p+1} H^n(U_f) \xrightarrow{\sim} Gr_F^p H_0^{n-1}(X_f), \quad p = n-1, n-2.$$

3.1.2 Case 2: n is even

When n is even, X_f is no longer a $\mathbb{Q}$-homology manifold. However, there is still an explicit description of the relations between $H^n(U_f)$ and $H_0^{n-1}(X_f)$. Note that in this case $H^{n-1}(\mathbb{P}^n) = 0$, and thus

$$H_0^{n-1}(X_f) = H^{n-1}(X_f), \quad H_0^{n-1}(X_f^*) = H^{n-1}(X_f^*), \quad \text{and} \quad i^* = i_0^*.$$

Moreover, there exists an exact sequence of mixed Hodge structures

(3.3)

$$\cdots \longrightarrow H^{n-1}(X_f, X_f^*) \longrightarrow H^{n-1}(X_f) \xrightarrow{i^*} H^{n-1}(X_f^*) \longrightarrow H^n(X_f, X_f^*) \longrightarrow \cdots.$$

To make use of this exact sequence, we first give the following claim.

Claim 3.2 *For $k = n-1, n$, $H^k(X_f, X_f^*)$ has a pure Hodge structure of type (ρ_k, ρ_k) for some $\rho_k \in \mathbb{N}$.*

Proof Let $a_1, \ldots, a_m$ be the nodes in X_f and let $B_i \ni a_i$, $i = 1, \ldots, m$ be a small ball in $\mathbb{P}^n$ around a_i such that $B_i \cap B_j = \varnothing$ for $i \neq j$.

By the excision theorem and conic structure theorem (see [2, Chapter 1, Theorem 5.1],

$$H^k(X_f, X_f^*) = \bigoplus_{i=1}^m H^k(B_i \cap X_f, B_i \cap X_f \smallsetminus \{a_i\}) \simeq \bigoplus_{i=1}^m H^{k-1}(K_i), \quad k = n-1, n,$$

where K_i is the link of X_f around a_i $(i = 1, \ldots, m)$.

For each i, K_i has the homotopy type of the unit sphere bundle of tangent bundle of S^{n-1}. Indeed, locally around a_i, X_f is defined as $z_1^2 + \cdots + z_n^2 = 0$, where $(z_1, \ldots, z_n)$ is the local coordinate system of $\mathbb{P}^n$ centered at a_i. Then K_i can be described as

$$K_i = \left\{ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n z_j^2 = 0, \text{ and } \sum_{j=1}^n |z_j|^2 = \epsilon^2 \right\},$$

where $\epsilon > 0$ is small. Let

$$z_j = \frac{\epsilon}{\sqrt{2}}(v_j + \sqrt{-1}w_j), \quad j = 1, \dots, n,$$
$$v = (v_1, \dots, v_n), \quad w = (w_1, \dots, w_n);$$

then

$$K_i = \left\{ (v, w) \in \mathbb{R}^n \times \mathbb{R}^n : |v|^2 = |w|^2 = 1 \text{ and } \langle v, w \rangle = 0 \right\},$$

which is the unit sphere bundle of tangent bundle of S^{n-1}.

It follows that

$$H^{k-1}(K_i) = \mathbb{C}, \quad k = n-1, n.$$

Note also that $H^{k-1}(K_i) = H^k(B_i \cap X_f, B_i \cap X_f \smallsetminus \{a_i\}) = H^k(X_f, X_f \smallsetminus \{a_i\})$ admits a natural mixed Hodge structure. In particular,

$$1 = \dim H^{k-1}(K_i) = \sum_{w \in \mathbb{N}} \dim Gr_w^W H^{k-1}(K_i) = \sum_{w \in \mathbb{N}} \sum_{p+q=w} \dim(Gr_w^W H^{k-1}(K_i))^{p,q},$$

where $Gr_w^W H^{k-1}(K_i)$ is a pure Hodge structure of weight w and

$$Gr_w^W H^{k-1}(K_i) = \bigoplus_{p+q=w} \left(Gr_w^W H^{k-1}(K_i) \right)^{p,q}$$

is the Hodge decomposition. By the Hodge symmetry, we have

$$\left(Gr_w^W H^{k-1}(K_i) \right)^{p,q} = \overline{(Gr_w^W H^{k-1}(K_i))^{q,p}}.$$

It follows that there exists $\rho_{k,i} \in \mathbb{N}$ such that

$$Gr_w^W H^{k-1}(K_i) = 0, \qquad \text{if } w \neq 2\rho_{k,i},$$
$$(Gr_{2\rho_{k,i}}^W H^{k-1}(K_i))^{p,q} = 0, \qquad \text{if } p \neq q,$$
$$\dim \left(Gr_{2\rho_{k,i}}^W H^{k-1}(K_i) \right)^{\rho_{k,i}, \rho_{k,i}} = 1.$$

In particular, $H^{k-1}(K_i)$ is pure of type $(\rho_{k,i}, \rho_{k,i})$.

Note that the mixed Hodge structure on $H^{k-1}(K_i)$ depends only on the local structure of X_f around a_i (see [8, Theorem 3.4]). Since all the a_i's are nodes, $H^{k-1}(K_i)$ is naturally isomorphic to $H^{k-1}(K_j)$ as mixed Hodge structures for any i, j, hence there exists $\rho_k \in \mathbb{N}$ such that

$$\rho_{k,1} = \rho_{k,2} = \cdots = \rho_{k,m} = \rho_k,$$

and thus $H^k(X_f, X_f^*)$ is pure of type (ρ_k, ρ_k) for $k = n-1, n$. ∎

By [2, Proposition (C28), Appendix C] (see also [8, Proposition 3.8]), it follows that $2\rho_{n-1} \leq n-2$. Thus,

$$Gr_F^p H^{n-1}(X_f, X_f^*) = 0, \quad \text{for } p = n-2, n-1.$$

Moreover, by the discussions above [2, Chapter 6, Example 3.18], $H^{n-1}(K_i)$ has weight n, namely, $2\rho_n = n$, and thus, for $n \geq 6$,

$$Gr_F^p H^n(X_f, X_f^*) = 0, \quad \text{for } p = n-2, n-1.$$

Therefore, it follows from (3.3) that we have an isomorphism

$$i_0^* : Gr_F^{n-1} H_0^{n-1}(X_f) = F^{n-1} H_0^{n-1}(X_f) \xrightarrow{\sim} Gr_F^{n-1} H_0^{n-1}(X_f^*) = F^{n-1} H_0^{n-1}(X_f^*)$$

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for $n \geq 4$. Furthermore, we have isomorphisms

$$i_0^*: Gr_F^{n-2} H_0^{n-1}(X_f) \xrightarrow{\sim} Gr_F^{n-2} H_0^{n-1}(X_f^*),$$
$$i_0^*: F^{n-2} H_0^{n-1}(X_f) \xrightarrow{\sim} F^{n-2} H_0^{n-1}(X_f^*)$$

for $n \geq 6$; but for $n = 4$, we only have injections

$$i_0^*: Gr_F^{n-2} H_0^{n-1}(X_f) \hookrightarrow Gr_F^{n-2} H_0^{n-1}(X_f^*),$$
$$i_0^*: F^{n-2} H_0^{n-1}(X_f) \hookrightarrow F^{n-2} H_0^{n-1}(X_f^*).$$

Using the residue isomorphism (3.1), we denote

$$F^{n-1}(U_f, X_f) = \overline{R}_f^{-1}\big(i_0^*(F^{n-2} H_0^{n-1}(X_f))\big) \subseteq F^{n-1} H^n(U_f)$$

for $n \geq 4$ (and n is even). Then clearly, $F^{n-1}(U_f, X_f) = F^{n-1} H^n(U_f)$ for $n \geq 6$.

We still denote by $\overline{R}_f$ its restriction to $F^{n-1}(U_f, X_f)$. Then

$$i_0^*: F^{n-2} H_0^{n-1}(X_f) \longrightarrow \overline{R}_f\big(F^{n-1}(U_f, X_f)\big)$$

is an isomorphism, and we have an isomorphism

$$R_f = (i_0^*)^{-1} \circ \overline{R}_f: F^{n-1}(U_f, X_f) \xrightarrow{\sim} F^{n-2} H_0^{n-1}(X_f).$$

3.1.3 Conclusion

In conclusion, no matter whether n is even or odd, we always have isomorphisms

(3.4) $$R_f: Gr_F^n H^n(U_f) \xrightarrow{\sim} Gr_F^{n-1} H_0^{n-1}(X_f),$$

(3.5) $$R_f: F^{n-1}(U_f, X_f)/F^n H^n(U_f) \xrightarrow{\sim} Gr_F^{n-2} H_0^{n-1}(X_f)$$

where $F^{n-1}(U_f, X_f) = \overline{R}_f^{-1}(i_0^*(F^{n-2} H_0^{n-1}(X_f)))$ is a subspace of $F^{n-1} H^n(U_f)$ containing $F^n H^n(U_f)$, and $R_f = (i_0^*)^{-1} \circ \overline{R}_f$.

3.2 Cohomology of X_f

Denote by

$$\Omega = \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \cdots \wedge dx_{i-1} \wedge \widehat{dx_i} \wedge dx_{i+1} \wedge \cdots \wedge dx_n,$$

where $\widehat{(\cdot)}$ means that the term is omitted. As is shown in [2, Chapter 6], any cohomology class in $F^p H^n(U_f)$ can be represented by a form

$$\omega(h) = \frac{h\Omega}{f^{n-p+1}}$$

with $h \in S_{(n-p+1)d-n-1}$. Hence, by (3.4), we see that any element in $Gr_F^{n-1} H_0^{n-1}(X_f)$ can be represented by

$$R_f\Big(\Big[\frac{h_1\Omega}{f}\Big]\Big)$$

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with $h_1 \in S_{d-n-1}$, and similarly, by (3.5), any element in $Gr_F^{n-2} H_0^{n-1}(X_f)$ can be represented by

$$R_f\Big(\Big[\frac{h_2\Omega}{f^2}\Big]\Big)$$

with $h_2 \in S_{2d-n-1}$.

Such results agree with [7, Theorem 2.2], where the following formulae are given for $n > 3$:

$$Gr_F^n H^n(U_f) = (S/J(f))_{d-n-1}, \quad Gr_F^{n-1} H^n(U_f) = (S/J(f))_{2d-n-1},$$

and for $n = 3$,

$$Gr_F^n H^n(U_f) = (S/J(f))_{d-n-1}, \quad Gr_F^{n-1} H^n(U_f) = (I(f)/J(f))_{2d-n-1},$$

where $I(f)$ is the saturation of $J(f)$, which is also equal to the radical of $J(f)$ for a nodal hypersurface (see [4, Remark 2.2]).

Putting together all the discussions above in this section, we obtain the following proposition.

Proposition 3.3 *Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$.*

(i) *When $n \geq 3$, there is an isomorphism*

$$\Lambda_f\colon (S/J(f))_{d-n-1} \longrightarrow Gr_F^{n-1} H_0^{n-1}(X_f), \quad \Lambda_f(h_1) = R_f\Big(\Big[\frac{h_1\Omega}{f}\Big]\Big).$$

(ii) *When $n > 4$, there is an isomorphism*

$$\Lambda_f\colon (S/J(f))_{2d-n-1} \longrightarrow Gr_F^{n-2} H_0^{n-1}(X_f), \quad \Lambda_f(h_2) = R_f\Big(\Big[\frac{h_2\Omega}{f^2}\Big]\Big).$$

(iii) *When $n = 3$, there is an isomorphism*

$$\Lambda_f\colon (I(f)/J(f))_{2d-n-1} \longrightarrow Gr_F^{n-2} H_0^{n-1}(X_f), \quad \Lambda_f(h_2) = R_f\Big(\Big[\frac{h_2\Omega}{f^2}\Big]\Big).$$

(iv) *When $n = 4$, there is an isomorphism*

$$\Lambda_f\colon S'/J(f)_{2d-n-1} \longrightarrow Gr_F^{n-2} H_0^{n-1}(X_f), \quad \Lambda_f(h_2) = R_f\Big(\Big[\frac{h_2\Omega}{f^2}\Big]\Big),$$

where $S' \subseteq S_{2d-n-1}$ is a vector subspace containing $J(f)_{2d-n-1}$ obtained via

$$S'/J(f)_{2d-n-1} = \omega^{-1}\big(F^{n-1}(U_f, X_f)/F^n H^n(U_f)\big),$$

where ω is the isomorphism

$$\omega\colon (S/J(f))_{2d-n-1} \longrightarrow Gr_F^{n-1} H^n(U_f), \qquad \omega(h_2) = \Big[\frac{h_2\Omega}{f^2}\Big]$$

established in [7, Theorem 2.2], *and $F^{n-1}(U_f, X_f)$ is obtained in* (3.5).

In all the formulae above, R_f denotes the residue map.

As a corollary, we have the following.

Corollary 3.4 *Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$.*

(i) *If $n \geq 3$, the dimension* $\dim F^{n-1} H_0^{n-1}(X_f)$ *depends only on n and d.*

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(ii) *If $n \geq 3$ is odd or $n \geq 6$ is even, the dimension* $\dim F^{n-2}H_0^{n-1}(X_f)$ *depends only on n and d and possibly the number of nodes in X_f.*

Proof Note that

$$\dim F^{n-1}H_0^{n-1}(X_f) = \dim Gr_F^{n-1}H_0^{n-1}(X_f),$$
$$\dim F^{n-2}H_0^{n-1}(X_f) = \dim Gr_F^{n-1}H_0^{n-1}(X_f) + \dim Gr_F^{n-2}H_0^{n-1}(X_f).$$

If $n > 4$, the results follow from Proposition 3.3 and Lemma 2.2, and the dimensions depend only on n and d. When $n = 3$, X_f is a $\mathbb{Q}$-homology manifold and the Hodge numbers of X_f depend only on n, d, and the number of nodes in X_f; see also [3]. ∎

4 Variations of Mixed Hodge Structures

Let $X_f : f = 0$ be a nodal hypersurface in $\mathbb{P}^n$ of degree $d \geq n+1$. When n is odd, assume that $n \geq 3$, while when n is even, assume that $n \geq 6$.

4.1 Topological Triviality

Recall that $\mathfrak{B}_f \subseteq S_d$ parameterizes all nodal hypersurfaces with the same number of nodes as X_f. Let $\mathcal{U} \subseteq \mathfrak{B}_f$ be a contractible smooth subvariety containing f such that it gives an effective deformation for X_f. Set

$$\mathfrak{X}_{\mathcal{U}} = \{(x,g) \in \mathbb{P}^n \times \mathcal{U} : x \in X_g\},$$

which can be seen as the union of all nodal hypersurfaces parameterized by $\mathcal{U}$.

Then by the First Thom Isotopy Lemma (see [2, Chapter 1, Section 3]), there is a homeomorphism Φ satisfying the commutative diagram

$$\begin{tikzcd}
(\mathbb{P}^n \times \mathcal{U}, \mathfrak{X}_{\mathcal{U}}) \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & (\mathbb{P}^n, X_f) \times \mathcal{U} \arrow[dl, "p_2"] \\
& \mathcal{U}, &
\end{tikzcd}$$

where p_1, p_2 are natural projections. In fact, Φ can be obtained by integrating some well-controlled stratified vector field; for a proof, see [10]. From now on, we fix such a homeomorphism.

In particular, for any $g \in \mathcal{U}$, there is a canonical homeomorphism $\Phi_g : \mathbb{P}^n \to \mathbb{P}^n$, which induces homeomorphisms $\Phi_{g,X} : X_f \to X_g$ and $\Phi_{g,U} : U_f \to U_g$ with $\Phi_f = \mathrm{Id}$.

Moreover, we have an induced isomorphism of groups

$$\Phi_{g,X}^* : H_0^{n-1}(X_g) \xrightarrow{\sim} H_0^{n-1}(X_f).$$

Hence, $\dim H_0^{n-1}(X_g)$ is constant for $g \in \mathcal{U}$.

In addition, by Corollary 3.4, under our assumption on n, the dimensions

$$\dim F^{n-1}H_0^{n-1}(X_g) \quad \text{and} \quad \dim F^{n-2}H_0^{n-1}(X_g)$$

are constant with respect to $g \in \mathcal{U}$. Via the identification

$$\Phi_{g,X}^* : H_0^{n-1}(X_g) \xrightarrow{\sim} H_0^{n-1}(X_f),$$

it follows that $(F^{n-1}H_0^{n-1}(X_g), F^{n-2}H_0^{n-1}(X_g))$ can be identified with

$$\big(\Phi_{g,X}^* F^{n-1}H_0^{n-1}(X_g), \Phi_{g,X}^* F^{n-2}H_0^{n-1}(X_g)\big),$$

which are two subspaces of $H_0^{n-1}(X_f)$ of fixed dimension. Therefore, we have the well-defined map as in (1.2):

$$\mathcal{P}: \mathcal{U} \ni g \longmapsto \big(\Phi_{g,X}^* F^{n-1}H_0^{n-1}(X_g), \Phi_{g,X}^* F^{n-2}H_0^{n-1}(X_g)\big) \in \mathcal{F},$$

where $\mathcal{F}$ is the following flag manifold

$$\mathcal{F} = \Big\{ (E_1, E_2) : E_1 \subseteq E_2 \text{ are vector subspaces of } H_0^{n-1}(X_f) \text{ and } \\ \dim E_1 = \dim F^{n-1}H_0^{n-1}(X_f) \text{ and } \dim E_2 = \dim F^{n-2}H_0^{n-1}(X_f) \Big\}.$$

When n is odd, all the Hodge numbers of X_g are constant for $g \in \mathcal{U}$, and $\mathcal{P}$ is just two components of the period map in the theory of variation of Hodge structures; see [12, Part III, Chapter 10].

4.2 Infinitesimal Deformation

Now we consider the differential of $\mathcal{P}$. Note that a component of $d\mathcal{P}_f$ is the map

$$d\mathcal{P}_f: T_f\mathcal{U} \longrightarrow \operatorname{Hom}\big(F^{n-1}H_0^{n-1}(X_f), H_0^{n-1}(X_f)/F^{n-1}H_0^{n-1}(X_f)\big);$$

for the properties of tangent spaces of flag manifolds, we refer to [12, Part III, Chapter 10], and for analogous treatments for smooth hypersurfaces, see [13, Part II, Chapter 6]. Recall that Proposition 3.3 implies that any element in $F^{n-1}H_0^{n-1}(X_f)$ is of the form

$$\omega(h_1) = R_f\Big(\Big[\frac{h_1\Omega}{f}\Big]\Big).$$

The following lemma holds.

Lemma 4.1 *For $h \in T_f\mathcal{U} \subseteq S_d$, we have*

$$d\mathcal{P}_f(h)\big(\omega(h_1)\big) = d\mathcal{P}_f(h)\Big(R_f\Big(\Big[\frac{h_1\Omega}{f}\Big]\Big)\Big) = R_f\Big(\Big[-\frac{hh_1\Omega}{f^2}\Big]\Big) \bmod F^{n-1}H_0^{n-1}(X_f).$$

Its proof is a little lengthy, so we postpone it to the end of this section; instead, we first derive Theorem 1.2 from Lemma 4.1.

4.3 Proof of Theorem 1.2

From Lemma 4.1 and Proposition 3.3, the image of $d\mathcal{P}_f$ is contained in

$$\operatorname{Hom}\big(F^{n-1}H_0^{n-1}(X_f), F^{n-2}H_0^{n-1}(X_f)/F^{n-1}H_0^{n-1}(X_f)\big) = \\ \operatorname{Hom}\big(Gr_F^{n-1}H_0^{n-1}(X_f), Gr_F^{n-2}H_0^{n-1}(X_f)\big).$$

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Moreover, we get the following commutative diagram:

$$
\begin{tikzcd}
T_f\mathcal{U} \arrow[r, "d\mathcal{P}_f"] \arrow[d, "i_1"] & \mathrm{Hom}(Gr_F^{n-1}H_0^{n-1}(X_f), Gr_F^{n-2}H_0^{n-1}(X_f)) \arrow[d, "i_2"] \\
(S/J(f))_d \arrow[r, "\varphi"] & \mathrm{Hom}((S/J(f))_{d-n-1}, (S/J(f))_{2d-n-1}),
\end{tikzcd}
\tag{4.1}
$$

where φ is given in (1.1). i_1 is the composite $T_f\mathcal{U} \subseteq S_d \to S_d/J(f)_d$, which is injective, since $\mathcal{U}$ is an effective deformation. i_2 is defined as follows: for

$$\eta \in \mathrm{Hom}\left(Gr_F^{n-1}H_0^{n-1}(X_f), Gr_F^{n-2}H_0^{n-1}(X_f)\right)$$

and $h_1 \in (S/J(f))_{d-n-1}$,

$$i_2(\eta)(h_1) = -\Lambda_f^{-1}\left(\eta\left(\Lambda_f(h_1)\right)\right),$$

where Λ_f is the isomorphism given in Proposition 3.3.

By Theorem 1.1, φ is injective, hence $\varphi \circ i_1$ is injective. Thus, it follows from (4.1) that $d\mathcal{P}_f$ is injective, hence Theorem 1.2 follows.

Remark 4.2 The result is probably also true for $n = 4$. We exclude this case, because we do not know whether the dimension $\dim F^{n-2}H_0^{n-1}(X_g)$ or equivalently $\dim Gr_F^{n-2}H_0^{n-1}(X_g)$ is constant for $g \in \mathcal{U}$ in this case.

4.4 Proof of Lemma 4.1

The proof is almost the same as that in [13, Part II, Chapter 6] where variations of smooth hypersurfaces are considered. However, to avoid any possible confusion, we give the details here.

From the topological triviality of the family $X_g, g \in \mathcal{U}$, it follows that there exists a small contractible neighbourhood $\mathcal{N} \ni f$ in $\mathcal{U}$, such that for any $g \in \mathcal{N}$, X_g is a deformation retract of

$$\mathcal{X}_{\mathcal{N}} := \bigcup_{g\in\mathcal{N}} X_g \subseteq \mathbb{P}^n.$$

Set $U_{\mathcal{N}} = \mathbb{P}^n \setminus X_{\mathcal{N}}$. Then $U_{\mathcal{N}}$ is a deformation retract of U_g for every $g \in \mathcal{N}$. For $g \in \mathcal{N}$, let $\tau_g\colon U_{\mathcal{N}} \hookrightarrow U_g$ be the natural inclusion; then the induced homomorphism in cohomology

$$\tau_g^*\colon H^n(U_g) \longrightarrow H^n(U_{\mathcal{N}})$$

is an isomorphism.

The differential $d\mathcal{P}_f$ can be computed as follows: for any $h \in T_f\mathcal{U} \subseteq S_d$, choose a curve $g(t) : (-\epsilon, \epsilon) \to \mathcal{N} \subseteq \mathcal{U}$ such that $g(0) = f$ and $\frac{dg}{dt}(0) = h$. For any element in $F^{n-1}H_0^{n-1}(X_f)$ of the form

$$\omega(h_1) = R_f\left(\left[\frac{h_1\Omega}{f}\right]\right),$$

let

$$\omega_t(h_1) = R_{g(t)}\left(\left[\frac{h_1\Omega}{g(t)}\right]\right)$$

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give an element of $F^{n-1}H_0^{n-1}(X_{g(t)})$. Then

$$d\mathcal{P}_f(h)(\omega(h_1)) = \frac{d}{dt}\Big|_{t=0} \Phi^*_{g(t),X}(\omega_t(h_1)) \bmod F^{n-1}H_0^{n-1}(X_f).$$

We have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Phi^*_{g(t),X}(\omega_t(h_1)) &= \frac{d}{dt}\Big|_{t=0} R_f\Big(\Phi^*_{g(t),U}\Big(\Big[\frac{h_1\Omega}{g(t)}\Big]\Big)\Big)\\ &= R_f\Big(\frac{d}{dt}\Big|_{t=0}\Phi^*_{g(t),U}\Big(\Big[\frac{h_1\Omega}{g(t)}\Big]\Big)\Big),\end{aligned}$$

where $\Phi^*_{g,U}$ is the homomorphism induced by the map $\Phi_{g,U}\colon U_f \to U_g$. Note that $\Phi^*_{g(t),U}\colon H^n(U_{g(t)}) \to H^n(U_f)$ is equal to the composition

$$H^n(U_{g(t)}) \xrightarrow{\tau^*_{g(t)}} H^n(U_{\mathcal{N}}) \xrightarrow{(\tau^*_f)^{-1}} H^n(U_f).$$

Hence,

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Phi^*_{g(t),X}(\omega_t(h_1)) &= R_f\Big(\frac{d}{dt}\Big|_{t=0}(\tau^*_f)^{-1}\tau^*_{g(t)}\Big[\frac{h_1\Omega}{g(t)}\Big]\Big)\\ &= R_f\Big((\tau^*_f)^{-1}\frac{d}{dt}\Big|_{t=0}\Big[\tau^*_{g(t)}\frac{h_1\Omega}{g(t)}\Big]\Big).\end{aligned}$$

Note that $\tau^*_{g(t)}$ acting on forms is a restriction map; it follows that

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Big[\tau^*_{g(t)}\frac{h_1\Omega}{g(t)}\Big] &= \frac{d}{dt}\Big|_{t=0}\Big[\frac{h_1\Omega}{g(t)}\Big|_{U_{\mathcal{N}}}\Big]\\ &= \Big[\frac{d}{dt}\Big|_{t=0}\frac{h_1\Omega}{g(t)}\Big|_{U_{\mathcal{N}}}\Big] = \Big[-\frac{hh_1\Omega}{f^2}\Big|_{U_{\mathcal{N}}}\Big].\end{aligned}$$

Therefore,

$$\frac{d}{dt}\Big|_{t=0}\Phi^*_{g(t),X}(\omega_t(h_1)) = R_f\Big((\tau^*_f)^{-1}\Big[-\frac{hh_1\Omega}{f^2}\Big|_{U_{\mathcal{N}}}\Big]\Big) = R_f\Big(\Big[-\frac{hh_1\Omega}{f^2}\Big]\Big).$$

Now the proof of Lemma 4.1 is complete. ■

Remark 4.3 To prove Theorem 1.2, it is essential for us to obtain a diagram like (4.1). In fact, when Y. Zhao [15] proved the infinitesimal Torelli theorem for nodal surfaces, he used such a diagram implicitly; however, he did not give any proofs. We believe that a detailed proof is indeed needed, and this is why our discussions above always include the case $n = 3$.

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