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COMMUTING RINGS OF SIMPLE $A(k)$-MODULES

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Abstract

For the Weyl algebra $A(k)$ and each finite dimensional division ring D over k, there exists a simple $A(k)$-module whose commuting ring is D.

It has been known for some time that if $A(k)$ denotes the Weyl algebra over a field k of characteristic zero, the commuting ring of a simple $A(k)$-module is a division algebra finite dimensional over k (see the introduction of [1]). Which division algebras actually appear? Quebbemann [1] showed that if D is a finite dimensional division algebra whose center is k, then it occurs as a commuting ring. We complete this circle of ideas by showing that any D appears: *a division algebra over k appears as the commuting ring of a simple $A(k)$-module if and only if it is finite dimensional over k.*

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The construction

In what follows k is a field of characteristic zero and D is any division algebra finite dimensional over k. The Weyl algebra $A(k)$ is $k[x,y]$ subject to $yx - xy = 1$ and $A(D)$ denotes $D \otimes_k A(k)$.

We review Quebbemann's construction [1]. A polynomial $p \in D[x]$ is fixed and an action of $A(D)$ is defined on $D[x]$ where x, as well as elements of D, act by left multiplication and

$$y \cdot \pi = \pi' + \pi p \quad \text{for } \pi \in D[x].$$

Quebbemann proves that $D[x]$ is a simple $A(D)$-module.

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The crux of this note is to calculate C, the centralizer of the restricted $A(k)$-action on $D[x]$, for a carefully chosen p. Here we modify a trick of Quebbemann. The center K of D is a finite field extension of k; choose a primitive element so that $K = k(\theta)$. Take any basis $c_1, \ldots, c_n$ for D over k with $c_n = \theta$ and set $p = \sum_{i=1}^{n} c_i x^i$.

We begin with a general ring theoretic lemma which is undoubtedly well known.

LEMMA 1. *Let D be a division ring finite dimensional over its center K. Suppose S is a ring containing D and containing a nonzero additive subgroup L such that $DLD \subseteq L$ and $K \subseteq C_S(L)$. Then $C_L(D) \neq 0$ where $C_S(L)$ and $C_L(D)$ denote the centralizer of L in S and D respectively.*

PROOF. By assumption L is a $D \otimes_K D^{op}$-module. Since $D \otimes_K D^{op}$ is a central simple K-algebra, it has a unique simple module up to isomorphism–namely D. Inside D,

$$(d \otimes 1 - 1 \otimes d)(1) = 0.$$

All $D \otimes_K D^{op}$-modules are semisimple, so L contains a copy of D. Consequently there is an element $g \in L$ with $dg - gd = 0$.

The centralizer C consists of those members of $\mathrm{End}_k(D[x])$ which commute with the actions by x and y. The $k[x]$-module endomorphisms of $D[x]$ can be identified with $E[x]$ where $E = \mathrm{End}_k D$. Thus C is the centralizer of the y action in $E[x]$. Notice that the map sending $\pi \in D[x]$ to πp is the element $\bar{p} = \sum_{i=1}^{n} \bar{c}_i x^i \in E[x]$, where $\bar{c}_j$ denotes right multiplication by c_j on D.

LEMMA 2. *Suppose $f = f_T x^T + \sum_{j<T} f_j x^j \in E[x]$.*

(i) *$f \in C$ if and only if $f' = f\bar{p} - \bar{p}f$.*

(ii) *If $f \in C$ then f_T commutes with multiplication by elements in K, the center of D.*

PROOF. $f \in C$ means $(yf - fy) \cdot \pi = 0$ for all $\pi \in D[x]$. Expanding,

$$(f(\pi))' + f(\pi) \cdot p - f(\pi') - f(\pi p) = 0.$$

But $(f(\pi))' = f'(\pi) + f(\pi')$. Hence

$$f'(\pi) = f(\pi p) - f(\pi)p.$$

We immediately obtain (i).

Look at the coefficient of x^{n+T} in equation (i). On the left it is zero and on the right it is $f_T\bar{\theta} - \bar{\theta}f_T$. The lemma follows because $K = k(\theta)$.

Lemma 3. $C = D$. (*The centralizer consists of left multiplications by elements in* D.)

Proof. Since left and right multiplication by elements of D are commuting maps, Lemma 2(i) yields $D \subseteq C$. Consequently, if we set

$$L_T = \left\{ f_T \in E \,|\, f_T x^T + \sum_{j<T} f_j x^j \in C \right\}$$

then $DL_T D \subseteq L_T$. By Lemma 2(ii), elements of K centralize L_T. Lemma 1 now applies: if $L_T \neq 0$ there exists a nonzero $g \in L_T$ with $dg = gd$ for all multiplications $d \in D$. However, the members of $\mathrm{End}_K D$ which centralize all such left multiplications are precisely the right multiplications by elements of D. We summarize:

$L_T \neq 0$ implies L_T contains a nonzero right multiplication.

We next claim that C is algebraic over k. One way to see this is to observe that Lemma 2(i) implies that nonzero elements of C have nonzero constant terms. (Don't forget that char $k = 0$.) Thus the map sending a polynomial in C to its constant term in E is an injective ring homomorphism. Since E is finite dimensional over k, so is C.

Putting the last two paragraphs together, we see that if $L_T \neq 0$ there is a polynomial in $E[x]$ of degree T which is algebraic and has as its leading coefficient "right multiplication" by a nonzero element in the division ring D. But a *nonconstant* algebraic polynomial has a leading coefficient which is nilpotent. Therefore $C \subseteq E$.

Now if $h \in C$ then Lemma 2(i) yields

$$0 = \sum_{i=1}^{n} (\bar{c}_i h - h\bar{c}_i) x^i.$$

Hence $\bar{c}_i h = h\bar{c}_i$ for $i = 1, \ldots, n$. Evaluate these k-endomorphisms on $1 \in D$.

$$h(1)c_i = h(c_i) \quad \text{for } i = 1, \ldots, n.$$

Since the c_i span D over k,

$$h(1)d = h(d) \quad \text{for all } d \in D.$$

As required, we have shown that h is left multiplication by an element of D.

Theorem. $D[x]$ *is a simple* $A(k)$*-module with commuting ring* D.

Proof. The simplicity argument can be found in [1]. We sketch an alternate proof.

Since $D[x]$ is a simple $A(D)$-module, $D[x] = A(D) \cdot \pi$ for some π. Hence $D[x] = \sum_{i=1}^{n} c_i A(k)\pi$; $D[x]$ is a noetherian $A(k)$-module. If V is a maximal submodule then $\bigcap c_i^{-1} V$ is an $A(D)$-submodule and so is zero.

Thus $D[x]$ contains a simple $A(k)$-module W. By simplicity, $D[x] = DW$ which, in turn, is a direct sum of copies of W as an $A(k)$-module. Since Lemma 3 states that the commuting ring of $D[x]$ as an $A(k)$-module is a division ring, there is only one copy of W in that sum.

References

[1] H. G. Quebbemann, 'Schiefkörper als Endomorphismenringe einfacher Moduln über einer Weyl-Algebra', *J. of Alg.* **59** (1979), 311–312.

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