KUROSH'S CHAINS OF ASSOCIATIVE RINGS

by R. R. ANDRUSZKIEWICZ and E. R. PUCZYŁOWSKI

(Received 1 October, 1988)

Let N be a homomorphically closed class of associative rings. Put $N_1 = N^1 = N$ and, for ordinals $\alpha \geqslant 2$, define N_α (N^α) to be the class of all associative rings R such that every non-zero homomorphic image of R contains a non-zero ideal (left ideal) in N_β for some $\beta < \alpha$. In this way we obtain a chain $\{N_\alpha\}$ ($\{N^\alpha\}$), the union of which is equal to the lower radical class lN (lower left strong radical class lsN) determined by N. The chain $\{N_\alpha\}$ is called *Kurosh's chain* of N. Suliński, Anderson and Divinsky proved [**7**] that $lN = \bigcup_{i=1}^{\infty} N_i$. Heinicke [**3**] constructed an example of N for which $lN \neq N_k$ for $k = 1, 2, \ldots$. In [**1**] Beidar solved the main problem in the area showing that for every natural number $n \geqslant 1$ there exists a class N such that $lN = N_{n+1} \neq N_n$. Some results concerning the termination of the chain $\{N^\alpha\}$ were obtained in [**2, 4**]. In this paper we present some classes N with $N_\alpha = N^\alpha$ for all α. Using this and Beidar's example we prove that for every natural number $n \geqslant 1$ there exists an N such that $N_\alpha = N^\alpha$ for all α and $N_n \neq N_{n+1} = N_{n+2}$. This in particular answers Question 6 of [**4**].

All rings in the paper are associative and N is a homomorphically closed class of such rings. To denote that I is an ideal (left ideal) of a ring R we write $I \lhd R$ ($I < R$). A subring A of a ring R is called *accessible* (*left accessible*) if there is a chain of subrings $A = A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_0 = R$ such that $A_i \lhd A_{i-1}$ ($A_i < A_{i-1}$) for $i = 1, 2, \ldots, n$.

$\mathbb{Z}$ is used to denote the ring of integers, $\mathbb{Q}$ the field of rational numbers and $\mathbb{Q}(i)$ the field of complex numbers $a + bi$, where $a, b \in \mathbb{Q}$.

The fundamental definitions and properties of radicals may be found in [**8**] and those of strong radicals in [**2**]. The following proposition collects some well known properties of classes N_α and N^α.

Proposition 1. (i) *$N_\alpha \subseteq N^\alpha$ for every ordinal α;*

(ii) *classes N_α and N^α are homomorphically closed for all α;*

(iii) ([**2**]) *if $0 \neq R \in lsN$ then R contains a non-zero left accessible subring in N;*

(iv) ([**7**]) *$R \in lN$ if and only if every non-zero homomorphic image of R contains a non-zero accessible subring in N;*

(v) ([**2**]) *if $0 \neq R \in N^{n+1}$, where n is an integer $\geqslant 1$, then there are subrings $0 \neq L_n < \ldots < L_0 = R$ of R such that $L_n \in N$;*

(vi) [**7**] *$R \in N_{n+1}$, where n is an integer $\geqslant 1$, if and only if every non-zero homomorphic image R' of R contains subrings $0 \neq I_n \lhd \ldots \lhd I_0 = R'$ such that $I_n \in N$.*

Recall that a radical class S is called *left stable* if for every $L < R$, $S(L) \subseteq S(R)$. An example of a left stable radical class is the generalized nil radical N_g; this is the upper radical determined by the class of reduced rings i.e. rings without non-zero nilpotent elements.

Theorem 1. *If S is a left stable radical class containing N_g then for $N = S \cup P$, where P is a homomorphically closed class of commutative rings, $N_\alpha = N^\alpha$ for every ordinal α.*

https://doi.org/10.1017/S001708950000906X Published online by Cambridge University Press

Proof. In view of Proposition 1 it suffices to prove that if $0 \neq L_n < \ldots < L_0 = R$ and $L_n \in N$ then there are $0 \neq I_n \lhd \ldots \lhd I_0 = R$ with $I_n \in N$. It is so if $S(R) \neq 0$. Hence, since $N_g \subseteq S$ and S is left stable, we can assume that the ring R is reduced and $L_n \in P$. Let k be a minimal number such that L_n is contained in the centre of L_k. Suppose that $k \geq 1$. Then there are $l \in L_n$ and $l' \in L_{k-1}$ such that $ll' - l'l \neq 0$. Now $(ll' - l'l)^2 = (ll')^2 - l(l')^2 l - l' l^2 l' + (l'l)^2$. Since $l'l$, $(l')^2 l \in L_k$ and L_n is contained in the centre of L_k, we have $(ll')^2 = l' l^2 l'$, $l(l')^2 l = (l')^2 l^2$ and $(l'l)^2 = (l')^2 l^2$. Thus $(ll' - l'l)^2 = 0$ and, since the ring R is reduced, $ll' - l'l = 0$. This contradiction shows that L_n is contained in the centre of R. On the other hand $RL_n^n \subseteq L_n$. Hence $L_n = L_n + RL_n^n \lhd L_n + RL_n^{n-1} \lhd \ldots \lhd L_n + RL_n \lhd R$ and the result follows.

Let p be a prime of the form $4m + 3$ and, for $n \geq 0$, let A_n be the subring of $\mathbb{Q}(i)$ generated by p and ip^n. The following properties of the rings A_n were established by Beidar in [**1**, Lemma 1].

Proposition 2. (i) $A_n \lhd A_m$ *if and only if* $n = m$ *or* $n = m + 1$;
(ii) *proper homomorphic images of* A_n *are finite*;
(iii) *the only subring of* $\mathbb{Q}(\mathrm{i})$ *isomorphic to* A_n *is the ring* A_n *itself*;
(iv) *if* B *is a subring of* $\mathbb{Q}(\mathrm{i})$ *and* $A_n \lhd B$ *then* $1 \in B$ *or* $B = A_n$ *or* $B = A_{n-1}$.

Now we prove the following theorem.

Theorem 2. *If* $N = N_g \cup T \cup \{A_n\}$, *where* T *is the class of rings whose additive groups are torsion and* $\{A_n\}$ *is the class of all isomorphic images of* A_n *for an* $n \geq 1$, *then* $N_1 = N^1 \subsetneq N_2 = N^2 \subsetneq \ldots \subsetneq N_{n+1} = N^{n+1} = N_{n+2} = N^{n+2}$.

Proof. As an immediate consequence of Theorem 1 and Proposition 2(ii) one obtains that, for every ordinal α, $N_\alpha = N^\alpha$. It follows from Proposition 1 (vi) and Proposition 2 that $A_0 \in N_{n+1} \backslash N_n$. Hence $N_1 \subsetneq N_2 \subsetneq \ldots \subsetneq N_{n+1}$. It remains to prove that $lN = N_{n+1}$ or, equivalently, that every non-zero ring $R \in lN$ contains a non-zero ideal in N_n. Obviously we can assume that R is semiprime and the additive group of R is torsion-free. Then by Propositions 1 (iv) and 2 (ii), R contains an accessible subring isomorphic to A_n. Let t be the minimal integer ≥ 0 such that there are $I_t \lhd I_{t-1} \lhd \ldots \lhd I_0 = R$ with I_t isomorphic to A_k for some $0 < k \leq n$. We claim that $t \leq 1$. For, suppose that $t \geq 2$ and take I the ideal of I_{t-2} generated by I_t. By Andrunakievich's lemma, $I^3 \subseteq I_t$. This, semiprimeness of R and properties of A_n imply that I is a prime ring without 1. Since the additive group of I is torsion-free, we can form the quotient ring $\mathbb{Z}^{-1}I$. Now $\mathbb{Z}^{-1}I_t \lhd \mathbb{Z}^{-1}I$ and $\mathbb{Z}^{-1}I_t$, being isomorphic to $\mathbb{Q}(\mathrm{i})$, contains 1. However the ring $\mathbb{Z}^{-1}I$ is prime, so $\mathbb{Z}^{-1}I_t = \mathbb{Z}^{-1}I$. Thus I is a ring without 1 isomorphic to a subring of $\mathbb{Q}(\mathrm{i})$. By Proposition 2 (iii) and (iv), I is isomorphic to A_n or A_{n-1}. Moreover, if I is isomorphic to A_{n-1} then, since I is a ring without 1, $n - 1 > 0$. This and the fact that the sequence $I \lhd I_{t-2} \lhd I_{t-3} \lhd \ldots \lhd I_0 = R$ is shorter than $I_t \lhd I_{t-1} \lhd \ldots \lhd I_0 = R$ prove the claim. Thus R contains a non-zero ideal isomorphic to A_k for some $0 < k \leq n$. It is clear from Proposition 2 that $A_k \in N_n$. The result follows.

Remark. Let $N = N_g \cup T \cup \{A_2 n : n = 1, 2, \ldots\}$. One can easily check using Propositions 1 and 2 that for every $0 < i < 2^n$, $n = 1, 2, \ldots, A_2 n_{+i} \in N_2 n_{-i+1} \backslash N_2 n_{-i}$. Hence by Theorem 1, $N_k = N^k \subsetneq N_{k+1} = N^{k+1}$ for $k = 1, 2, \ldots$ and $lN = lsN = \bigcup_{k=1}^{\infty} N^k$.

In [4] it was proved that if the class N is hereditary (hereditary and contains nilpotent rings) then $lsN = N^4$ ($lsN = N^3$) i.e. in those cases the chain $\{N^\alpha\}$ terminates one step further than $\{N_\alpha\}$. In [6] Stewart proved that if N is a class of zero rings then $lN = N_3$. The following theorem shows in particular that in this case $lsN = lN = N^3$. It can also be regarded as a generalization of the fact that the prime radical is strong.

THEOREM 3. *If N is a class of zero rings then $S = lN$ is a left strong radical class.*

Proof. Suppose that $L < R$ and $L \in S$. Let $U = \{x \in L : Lx = 0\}$. If $U = L$ then $L^2 = 0$ and it is easy to check that $LR^1 \in S$, where R^1 is the ring R with an unity adjoined. Suppose $U \neq L$. Then $0 \neq L/U \in S$, so L/U contains a non-zero accessible subring $A/U \in N$. Hence there are subrings $A_0, \ldots, A_n$ of L such that $A \lhd A_n \lhd \ldots \lhd A_0 = L$. Now $0 \neq \mathrm{LAR}^1 \lhd R$. Since $(\mathrm{LAR}^1)^m \subseteq (\mathrm{LA})^m R^1$ for $m = 1, 2, \ldots$ and the ideal of L generated by A is nilpotent, LAR^1 is a nilpotent ideal of R. Suppose that $(\mathrm{LAR}^1)^{k+1} = 0$, $(\mathrm{LAR}^1)^k \neq 0$ and take $t \in (\mathrm{LAR}^1)^{k-1}$. Obviously $\mathrm{LAR}^1 tR^1 \lhd R$. For every $l \in L$, $x, y \in R^1$, the map $f : A \to lAxty$ given by $f(a) = laxty$ is a ring epimorphism and $lAxty \lhd LAR^1tR^1$. Hence $(\mathrm{LAR}^1)^k \in S$. Therefore $S(R) \neq 0$ and the result follows.

Let us observe that if N is a class of N-nilpotent rings [5] and N_0 is the class of zero lN-radical rings then $lN_0 = l(N_0 \cap N)$. Applying [5, Theorem 4] to $\alpha = lN$ one obtains that $R^0 \in N$, where R^0 is the zero ring on the additive group of a ring $R \in N$. The same theorem applied to $\alpha = lN_0$ implies $N \subseteq lN_0$. Hence $lN = LN_0$ and Theorem 3 gives

COROLLARY. *If N is a class of M-nilpotent rings then $lN = lsN = N_3 = N^3$.*

REFERENCES

1. K. I. Beidar, A chain of Kurosh may have an arbitrary finite length, *Czech. Math. J.* **32** (1982), 418–422.
2. N. Divinsky, J. Krempa and A. Suliński, Strong radical properties of alternative and associative rings, *J. Algebra* **17** (1971), 369–381.
3. A. Heinicke, A note on lower radical constructions for associative rings, *Canad. Math. Bull.* **11** (1968), 23–30.
4. E. R. Puczyłowski, On questions concerning strong radicals of associative rings, *Quaestiones Math.* **10** (1987), 321–338.
5. A. D. Sands, On M-nilpotent rings, *Proc. Royal Soc. Edinburgh Sect. A* **93** (1982), 63–70.
6. P. N. Stewart, On the lower radical construction, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 31–32.
7. A. Suliński, T. Anderson and N. Divinsky, Lower radical properties for associative and alternative rings, *J. London Math. Soc* **41** (1966), 417–424.
8. R. Wiegandt, *Radical and semisimple classes of rings*, Queen's University, Kingston, Ontario, 1974.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW, BIAŁYSTOK DIVISION
AKADEMICKA 2, 15-267 BIAŁYSTOK
POLAND

INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
PKIN, 00-901 WARSAW
POLAND