Free Product C*-algebras Associated with Graphs, Free Differentials, and Laws of Loops

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Abstract. We study a canonical C^*-algebra, $\mathcal{S}(\Gamma, \mu)$, that arises from a weighted graph (Γ, μ), specific cases of which were previously studied in the context of planar algebras. We discuss necessary and sufficient conditions of the weighting that ensure simplicity and uniqueness of trace of $\mathcal{S}(\Gamma, \mu)$, and study the structure of its positive cone. We then study the $*$-algebra, $\mathcal{A}$, generated by the generators of $\mathcal{S}(\Gamma, \mu)$, and use a free differential calculus and techniques of Charlesworth and Shlyakhtenko as well as Mai, Speicher, and Weber to show that certain "loop" elements have no atoms in their spectral measure. After modifying techniques of Shlyakhtenko and Skoufranis to show that self adjoint elements $x \in M_n(\mathcal{A})$ have algebraic Cauchy transform, we explore some applications to eigenvalues of polynomials in Wishart matrices and to diagrammatic elements in von Neumann algebras initially considered by Guionnet, Jones, and Shlyakhtenko.

1 Introduction

The study of graphs has been a prevailing force in the development of many areas of operator algebras. Cuntz and Krieger [CK80] initiated the study of graph C^*-algebras, which are canonical C^*-algebras that are formed from directed graphs. The principal graph of a planar algebra or subfactor [Jon83, Pop95, Bis97] has played an important role in the development and study of subfactor theory.

Given a standard invariant of a subfactor $\mathcal{P}_\bullet$, Popa reconstructed a Jones tower (see [Jon83] for a definition) of II$_1$ factors whose standard invariant is $\mathcal{P}_\bullet$ [Pop95]. More recently, Guionnet, Jones, and Shlyakhtenko (GJS) gave a diagrammatic construction of a Jones' tower $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ of II$_1$ factors whose standard invariant is $\mathcal{P}_\bullet$ [GJS10]. In subsequent work, GJS constructed a (semi)finite factor by applying Shlyakhtenko's operator-valued generalization [Shl99] of Voiculescu's free Gaussian functor [VDN92] to obtain the isomorphism class of the $\mathcal{M}_k$ in the case where $\mathcal{P}_\bullet$ is finite-depth [GJS11]. More specifically, if $\mathcal{P}_\bullet$ is finite-depth with finite principal graph Γ, GJS constructed a finite von Neumann algebra $\mathcal{M}(\Gamma)$ by assigning to each edge e of Γ, an $\ell^\infty(V(\Gamma))$-valued semicircular element X_e. The factor $\mathcal{M}_0$ was realized as a corner of $\mathcal{M}(\Gamma)$ that was shown to be an interpolated free group factor.

In later work [Har13, BHP12], the author developed a canonical free-product von Neumann algebra, $\mathcal{M}(\Gamma, \mu)$, that one can associate with an arbitrary undirected, weighted, graph with weighting $\mu: V(\Gamma) \to \mathbb{R}^+$. The key observation in [Har13] is

that if (Γ, μ) is a subgraph of (Γ', μ), then there is a canonical, possibly nonunital, inclusion $\mathcal{M}(\Gamma, \mu) \to \mathcal{M}(\Gamma', \mu)$, compressions of which are a standard embedding in the sense of [Dyk93]. This observation showed that if a planar algebra $\mathcal{P}_\bullet$ is infinite depth, then in the construction in [GJS10], $\mathcal{M}_k \cong L(\mathbb{F}_\infty)$ for all k.

In an effort to study the diagrammatic C*-algebras $(\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots)$ that arise in the construction in [GJS10], the author and Penneys defined the C*-algebra analogue of $\mathcal{M}(\Gamma, \mu)$, called $\mathcal{S}(\Gamma, \mu)$ [HP14a, HP14b]. Although this algebra was defined for an arbitrary weighting μ, most properties of $\mathcal{S}(\Gamma, \mu)$ were studied only in the case of Γ being the principal graph of a planar algebra with μ the associated weighting. In this case, the authors proved that $\mathcal{S}(\Gamma, \mu)$ is simple, has unique trace, and established KK-equivalence between $C_0(V(\Gamma))$ and $\mathcal{S}(\Gamma, \mu)$. Utilizing a Morita equivalence between $\mathcal{A}_k$ and $\mathcal{S}(\Gamma, \mu)$, the same properties were deduced for the algebras $\mathcal{A}_k$.

In Section 2 we will study $\mathcal{S}(\Gamma, \mu)$ in the setting of an arbitrary connected undirected graph Γ and arbitrary weighting μ on $V(\Gamma)$. This C*-algebra is generated by $C_0(V(\Gamma))$ (spanned by the indicator functions $\{p_\alpha \mid \alpha \in V(\Gamma)\}$) and $C_0(V(\Gamma))$-valued semicircular elements X_e associated with each edge e. In the von Neumann algebra case, $\mathcal{M}(\Gamma, \mu)$, the following theorem was proved in [Har13].

Theorem ([Har13]) *Suppose Γ is a finite, connected, unoriented graph with at least two edges. If $\alpha, \beta \in V(\Gamma)$, then write $\alpha \sim \beta$ if α and β are joined by at least one edge, and let $n_{\alpha, \beta}$ be the number of edges joining with α and β as endpoints. Finally, let $V_>$ be the set of vertices, β satisfying $\mu(\beta) > \sum_{\alpha \sim \beta} n_{\alpha, \beta} \mu(\alpha)$. We have*

$$\mathcal{M}(\Gamma, \mu) \cong L(\mathbb{F}_t) \oplus \bigoplus_{\gamma \in V_>} \overset{r_\gamma}{\mathbb{C}},$$

where $r_\gamma \leq p_\gamma$ and $\operatorname{tr}(r_\gamma) = \mu(\gamma) - \sum_{\alpha \sim \gamma} n_{\alpha, \beta} \mu(\alpha)$ (see Notation 2.10 for the meaning of the direct sum notation). Moreover, the parameter, t, can be computed using Dykema's "free dimension" formulas [Dyk93, DR13]. In particular, $\mathcal{M}(\Gamma, \mu)$ is a factor if and only if $V_>$ is non-empty.

We prove the C*-algebra analogue.

Theorem A *Let Γ and $V_>$ be as in the statement of the previous theorem . Let $V_=$ be the set of vertices, β satisfying $\mu(\beta) = \sum_{\alpha \sim \beta} n_{\alpha, \beta} \mu(\alpha)$ and $V_\geq = V_> \cup V_=$. Let I be the norm-closed ideal generated by some p_α with $\alpha \in V(\Gamma) \smallsetminus V_\geq$. Then I contains $\{p_\beta \mid \beta \in V(\Gamma) \smallsetminus V_\geq\}$ and does not intersect $\{p_\gamma \mid \gamma \in V_\geq\}$. In addition, I is generated by $\{X_e \mid e \in E(\Gamma)\}$. Furthermore, the following hold:*

(i)   *I is simple, has unique tracial state, and has stable rank 1.*

(ii)   *I is unital if and only if $V_=$ is empty. If $V_=$ is empty, then*

$$\mathcal{S}(\Gamma, \mu) = I \oplus \bigoplus_{\gamma \in V_>} \overset{r_\gamma}{\mathbb{C}}$$

with $r_\gamma \leq p_\gamma$ and $\operatorname{tr}(r_\gamma) = \mu(\gamma) - \sum_{\alpha \sim \gamma} n_{\alpha, \beta} \mu(\alpha)$. If $V_=$ is not empty, then

$$\mathcal{S}(\Gamma, \mu) = \mathcal{I} \oplus \bigoplus_{\gamma \in V_>} \overset{r_\gamma}{\mathbb{C}}$$

where $\mathfrak{I}$ is unital, the strong operator closures of I and $\mathfrak{I}$ coincide in $L^2(\mathfrak{S}(\Gamma,\mu),\mathrm{Tr})$, and $\mathfrak{I}/I \cong \bigoplus_{\beta \in V_=} \mathbb{C}$.

(ii) $K_0(I) \cong \mathbb{Z}\left\{[p_\beta] \mid \beta \in V \smallsetminus V_\geq\right\}$ and $K_1(I) = \{0\}$ where the first group is the free abelian group on the classes of projections $[p_\beta]$. Furthermore,

$$K_0(I)^+ = \left\{x \in K_0(I) \mid \mathrm{Tr}(x) > 0\right\} \cup \{0\}.$$

In particular, $\mathfrak{S}(\Gamma,\mu)$ is simple with unique tracial state if and only if $V_\geq$ is empty.

A version of this theorem for infinite Γ is given in Corollary 2.30.

Section 3 switches gears and studies elements of $\mathcal{A}$, the $*$-algebra generated by $\{p_\alpha \mid \alpha \in V(\Gamma)\} \cup \{X_e \mid e \in E(\Gamma)\}$. We develop a free differential calculus along the lines of Voiculescu [Voi93]. Using modifications of the methods in [CS15, MSW14], we are able to prove the following theorem.

Theorem B   Let $Q \in \mathcal{A}$ be nonzero and satisfying

$$Qp_\alpha = Q \quad and \quad \mu(\alpha) = \min\left\{\mu(\beta) \mid \beta \in V(\Gamma)\right\}.$$

If $a = a^* \in \mathcal{M}(\Gamma,\mu)$, $ap_\alpha = a$, and $Qa = 0$, then $a = 0$. In particular, if $Q = Q^*$, then the law of Q with respect to Tr has no atoms.

Section 3 also uses the techniques of [SS15] to prove the following theorem.

Theorem C   Let $x \in M_n(\mathcal{A})$ be self-adjoint. Then the Cauchy transform of x is algebraic.

As a result, further deductions can be made about the laws of self-adjoint elements in $M_n(\mathcal{A})$. See Corollary 3.17 and Theorem 3.18.

Section 4 is devoted to applications of the previous sections to the limiting laws of polynomials in Wishart matrices and to diagrammatic elements in the construction in [GJS10]. Much like the observations about Gaussian unitary ensembles in [SS15], it is shown that eigenvalues in self-adjoint polynomials of Wishhart matrices also experience a weak repulsion.

2   The Free Graph Algebra

2.1   Constructing the Algebra

We will start with countable, weighted, connected, undirected, locally finite graph, $\Gamma = (\Gamma, V, E, \mu)$ where V is the vertex set of Γ, E is the edge set, and $\mu: V \to \mathbb{R}_{>0}$ is a strictly positive function on the vertices of Γ. We will now form the directed version of Γ as in [HP14a].

Definition 2.1   If Γ is an undirected graph, we will form the directed version of Γ, $\vec{\Gamma} = (\vec{\Gamma}, \vec{V}, \vec{E}, \vec{\mu}, s, t)$ as the weighted directed graph with the following properties.

(i) $\vec{V} = V$ and $\vec{\mu} = \mu$.

(ii) $\vec{E}$ is constructed as follows: For each $e \in E(\Gamma)$ having $\alpha \neq \beta$ as its two endpoints, there are two edges ϵ and ϵ^{op} satisfying

$$s(\epsilon) = t(\epsilon^{op}) = \alpha \quad \text{and} \quad t(\epsilon) = s(\epsilon^{op}) = \beta.$$

For each $e \in E(\Gamma)$ that is a loop at a vertex, γ, there is one loop (which we will denote as ϵ) in $E(\vec{\Gamma})$ at γ.

Notice that there is an involution $\epsilon \mapsto \epsilon^{op}$ on $\vec{E}$ provided that $\epsilon = \epsilon^{op}$ whenever ϵ is a loop. Let $X(\vec{\Gamma})$ be the complex vector space with basis $\vec{E}$, and $C_{\text{fin}}(V)$ be the finitely supported complex-valued functions on V. The algebra $C_{\text{fin}}(V)$ is spanned by the elements p_α for $\alpha \in \Gamma$ satisfying $p_\alpha(\beta) = \delta_{\alpha,\beta}$.

There is a natural $C_{\text{fin}}(V)$ bimodule structure on $X(\vec{\Gamma})$ that is given by

$$p_\alpha \epsilon = \delta_{s(\epsilon),\alpha} \epsilon \quad \text{and} \quad \epsilon p_\alpha = \delta_{t(\epsilon),\alpha} \epsilon.$$

With these actions in mind, $X(\Gamma)$ exhibits a $C_{\text{fin}}(V)$-valued inner product $\langle \cdot | \cdot \rangle$ which is given by

$$\langle \epsilon' | \epsilon \rangle = \delta_{\epsilon,\epsilon'} p_{t(\epsilon)}$$

with linearity in the second variable and conjugate-linearity in the first. Let $\mathcal{C}$ denote $C_0(V)$. Using this inner product, $X(\vec{\Gamma})$ completes to a $\mathcal{C} - \mathcal{C}$ bimodule $\mathcal{X}(\vec{\Gamma})$ with inner product $\langle \cdot | \cdot \rangle_{\mathcal{C}}$. We will denote this Hilbert bimodule as $\mathcal{X}$ when the context is clear.

We now form the *full Fock space* of $\vec{\Gamma}$, $\mathcal{F}(\vec{\Gamma})$, which is given by

$$\mathcal{F}(\vec{\Gamma}) = \mathcal{C} \oplus \bigoplus_{n \geq 1} \mathcal{X}^{\otimes_{\mathcal{C}}^n},$$

where $\otimes_{\mathcal{C}}$ denotes the internal tensor product of $\mathcal{C} - \mathcal{C}$ Hilbert bimodules. $\mathcal{C}$, together with elements of the form $\epsilon_1 \otimes \epsilon_2 \otimes \cdots \otimes \epsilon_n$ span a dense subset of $\mathcal{F}(\vec{\Gamma})$. Note that the elementary tensor $\epsilon_1 \otimes \epsilon_2 \otimes \cdots \otimes \epsilon_n$ is zero unless the edges form a path in $\vec{\Gamma}$ *i.e.*, $t(\epsilon_k) = s(\epsilon_{k+1})$ for $1 \leq k < n$.

We have the usual creation operator $\ell(\xi)$ for $\xi \in \mathcal{X}$ which is defined by

$$\ell(\xi) p_\alpha = \xi \cdot p_\alpha \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

The creation operator $\ell(\xi)$ is bounded and adjointable with adjoint $\ell(\xi)^*$ given by

$$\ell(\xi)^* p_\alpha = 0 \quad \text{and} \quad \ell(\xi)^*(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \langle \xi | \xi_1 \rangle_{\mathcal{C}} \xi_2 \otimes \cdots \otimes \xi_n.$$

We define the *Pimsner-Topelitz algebra of* $\vec{\Gamma}$, $\mathcal{T}(\vec{\Gamma})$ [Pim97] to be the C*-algebra generated by $\mathcal{C}$ and $\{\ell(\xi) | \xi \in \mathcal{X}\}$.

We will now construct our finite algebra associated with Γ with weighting μ.

Definition 2.2 Let $\Gamma = (\Gamma, V, E, \mu)$ with $\vec{\Gamma}$ as in Definition 2.1.

(i) If $e \in E$ is a loop in Γ, and ϵ is the associated loop in $\vec{E}$, we define $X_e = \ell(\epsilon) + \ell(\epsilon)^*$.

(ii) If $e \in E$ has $\alpha \neq \beta$ as its endpoints, let ϵ and ϵ^{op} be the two associated edges as in Definition 2.1, with $s(\epsilon) = t(\epsilon^{op}) = \alpha$ and $t(\epsilon) = s(\epsilon^{op}) = \beta$. Also set

$$a_\epsilon = \sqrt[4]{\frac{\mu(\alpha)}{\mu(\beta)}}.$$

Then we define

$$X_e = a_\epsilon \ell(\epsilon) + a_\epsilon^{-1}\ell(\epsilon^{op})^* + a_\epsilon^{-1}\ell(\epsilon^{op}) + a_\epsilon \ell(\epsilon)^*.$$

We define the *free graph algebra associated with* $\Gamma, \mu, \mathcal{S}(\Gamma, \mu)$, to be the subalgebra of $\mathcal{T}(\Gamma)$ generated by $\mathcal{C}$ and the elements $(X_e)_{e \in E}$.

Remark 2.3 The elements X_e are always self adjoint. Note that if e has $\alpha \neq \beta$ as endpoints with $\alpha = s(\epsilon)$ and $\beta = t(\epsilon)$, then

$$p_\alpha X_e p_\beta = a_\epsilon \ell(\epsilon) + a_\epsilon^{-1}\ell(\epsilon^{op})^* \quad \text{and} \quad p_\beta X_e p_\alpha = a_\epsilon^{-1}\ell(\epsilon^{op}) + a_\epsilon \ell(\epsilon)^*$$

We will set $X_\epsilon = p_\alpha X_e p_\beta$ and $X_{\epsilon^{op}} = p_\beta X_e p_\alpha$.

As in [Shl99, HP14a] there is a conditional expectation $E \colon S(\Gamma, \mu) \to \mathcal{C}$ that is given by $E(x) = PxP = \sum_{\alpha \in V} \langle p_\alpha | x p_\alpha \rangle_{\mathcal{C}}$ with $P \colon \mathcal{F}(\vec{\Gamma}) \to \mathcal{C}$ the orthogonal projection. We have the following theorem.

Theorem 2.4 ([HP14a])

(i) *The expectation E is faithful on $\mathcal{S}(\Gamma, \mu)$. Moreover, if we let tr be the (semi)-finite trace on $\mathcal{C}$ determined by $\mathrm{tr}(p_v) = \mu(v)$, then $\mathrm{Tr} = \mathrm{tr} \circ E$ is a (semi)-finite tracial weight on $\mathcal{S}(\Gamma, \mu)$.*

(ii) *The elements $(X_e)_{e \in E(\Gamma)}$ are free with amalgamation over $\mathcal{C}$ with respect to E.*

(iii) *If ϵ has $s(\epsilon) = \alpha$ and $t(\epsilon) = \beta$ and X_ϵ is as in Remark 2.3 then the law of $X_\epsilon^* X_\epsilon$ in $p_\beta \mathcal{S}(\Gamma, \mu) \epsilon p_\beta$ with respect to $\mathrm{Tr}(p_\beta \cdot p_\beta)$ is a free Poisson with the following distribution, where $a = a_\epsilon$:*

$$\mu(\beta)\frac{\sqrt{4a^2x - (a^4 - 1 - a^2 x)^2}}{2\pi x}\mathbf{1}_{[a^2 + a^{-2} - 2, a^2 + a^{-2} + 2]}\, dx \quad \text{if } \mu(\beta) \leq \mu(\alpha),$$

$$(\mu(\beta) - \mu(\alpha))\delta_0 + \mu(\alpha)\frac{\sqrt{4a^2x - (a^4 - 1 - a^2 x)^2}}{2\pi x}\mathbf{1}_{[a^2 + a^{-2} - 2, a^2 + a^{-2} + 2]}\, dx \quad \text{if } \mu(\beta) \geq \mu(\alpha),$$

Remark 2.5 If $\mu(\alpha) \neq \mu(\beta)$, then $a_\epsilon^2 + a_\epsilon^{-2} - 2 > 0$ so we conclude that the polar part of X_ϵ is in $\mathcal{S}(\Gamma, \mu)$. Therefore, if $\mu(\alpha) < \mu(\beta)$ then p_α is equivalent to a subprojection of p_β in $\mathcal{S}(\Gamma, \mu)$. It is important to note that the polar part of X_ϵ will not be in $\mathcal{S}(\Gamma, \mu)$ if $\mu(\alpha) = \mu(\beta)$. This observation is consistent with Corollary 2.9 below. Notice that $\mathrm{Tr}(X_\epsilon^* X_\epsilon) = \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}$.

We will also need the following lemma, to be used in Section 3.3.

Lemma 2.6 *Suppose that $\epsilon_1, \ldots, \epsilon_n$ is a loop in in $\vec{\Gamma}$ of length at least 3. Then*

$$\mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_n}) = \frac{1}{\sqrt{\mu(s(\epsilon_n))\mu(t(\epsilon_n))}} \sum_{\substack{\epsilon_j = \epsilon_n^{op} \\ j \notin \{1, n-1\}}} \mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) \cdot \mathrm{Tr}(X_{\epsilon_{j+1}} \cdots X_{\epsilon_{n-1}})$$

$$+ \delta_{\epsilon_{n-1}, \epsilon_n^{op}} \sqrt{\frac{\mu(s(\epsilon_n))}{\mu(t(\epsilon_n))}} \mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{n-2}}) + \delta_{\epsilon_1, \epsilon_n^{op}} \sqrt{\frac{\mu(t(\epsilon_n))}{\mu(s(\epsilon_n))}} \mathrm{Tr}(X_{\epsilon_2} \cdots X_{\epsilon_{n-1}})$$

Proof Noting that $\mathrm{Tr} = \mathrm{tr} \circ E$, we will first compute the $\mathcal{C}$-valued inner product $\langle \widehat{p_{t(\epsilon_n)}} | X_{\epsilon_1} \cdots X_{\epsilon_n} \widehat{p_{t(\epsilon_n)}} \rangle$. To this end, we note that

$$X_{\epsilon_1} \cdots X_{\epsilon_n} \widehat{p_{t(\epsilon_n)}} = a_{\epsilon_n} X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \epsilon_n.$$

Clearly for $X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \epsilon_n$ to have a $\widehat{p_{t(\epsilon_n)}}$ component, it must be the case that at least one element in $\{\epsilon_1, \dots, \epsilon_{n-1}\}$ must be ϵ_n^{op}. By expanding each $X_{\epsilon_j} = a_{\epsilon_j} \ell(\epsilon_j) + a_{\epsilon_j}^{-1} \ell(\epsilon_j)^*$, we will count how many terms in $a_{\epsilon_n} X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \epsilon_n$ have a $\widehat{p_{t(\epsilon_n)}}$ component.

Assuming that there is at least one $j \in \{1, 2, \dots, n-1\}$ satisfying $\epsilon_j = \epsilon_n^{\mathrm{op}}$, we see that

$$
\begin{aligned}
&\langle \widehat{p_{t(\epsilon_n)}} | a_{\epsilon_n} X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \epsilon_n \rangle \\
&= \sum_{\epsilon_j = \epsilon_n^{\mathrm{op}}} \sum_{\substack{T_k \in \{ a_{\epsilon_k} \ell(\epsilon_k), a_{\epsilon_k}^{-1} \ell(\epsilon_k^{\mathrm{op}})^* \} \\ T_k T_{k+1} \cdots T_{n-1} \epsilon_n \notin \mathbb{C} \widehat{p_{t(\epsilon_n)}}}} E(a_{\epsilon_n} X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) \langle \widehat{p_{t(\epsilon_n)}} | a_{\epsilon_n} \ell(e_j^{\mathrm{op}})^* T_{j+1} \cdots T_{n-1} \epsilon_n \rangle \\
&= \sum_{\epsilon_j = \epsilon_n^{\mathrm{op}}} a_{\epsilon_n}^2 E(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) \ell(\epsilon_n)^* E(X_{j+1} \cdots X_{n-1}) \epsilon_n.
\end{aligned}
$$

Note that by the definition of the trace,

$$E(X_{j+1} \cdots X_{n-1}) = \frac{\mathrm{Tr}(X_{j+1} \cdots X_{n-1})}{\mathrm{Tr}(p_{s(\epsilon_n)})} \widehat{p_{s(\epsilon_n)}},$$

$$E(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) = \frac{\mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}})}{\mathrm{Tr}(p_{s(\epsilon_1)})} \widehat{p_{s(\epsilon_1)}}.$$

Since $s(\epsilon_1) = t(\epsilon_n)$, we have

$$
\begin{aligned}
\langle \widehat{p_{t(\epsilon_n)}} | a_{\epsilon_n} X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \epsilon_n \rangle &= \sum_{\substack{\epsilon_j = \epsilon_n^{\mathrm{op}} \\ j \notin \{1, n-1\}}} a_{\epsilon_n}^2 \frac{\mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}})}{\mathrm{Tr}(p_{t(\epsilon_n)})} \cdot \frac{\mathrm{Tr}(X_{j+1} \cdots X_{n-1})}{\mathrm{Tr}(p_{s(\epsilon_n)})} \widehat{p_{t(\epsilon_n)}} \\
&\quad + \delta_{\epsilon_{n-1}, \epsilon_n^{\mathrm{op}}} a_{\epsilon_n}^2 \frac{\mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{n-2}})}{\mathrm{Tr}(p_{t(\epsilon_n)})} \widehat{p_{t(\epsilon_n)}} + \delta_{\epsilon_1, \epsilon_n^{\mathrm{op}}} a_{\epsilon_n}^2 \frac{\mathrm{Tr}(X_{\epsilon_2} \cdots X_{\epsilon_{n-1}})}{\mathrm{Tr}(p_{s(\epsilon_n)})} \widehat{p_{t(\epsilon_n)}}.
\end{aligned}
$$

Taking the trace of both sides gives the desired formula. ∎

2.2 KK**-groups**

We will begin by computing the KK-groups of $\mathcal{S}(\Gamma, \mu)$. This computation was done in [HP14a], but for completeness, we will present the argument here. If we let $\mathcal{B}$ be a separable C*-algebra and $\mathcal{Y}$ a countably generated Hilbert $\mathcal{B} - \mathcal{B}$ bimodule, then we form the Fock space of $\mathcal{Y}$,

$$\mathcal{F}(\mathcal{Y}) = \mathcal{B} \oplus \bigoplus_{n \geq 1} \mathcal{Y}^{\otimes_{\mathcal{B}}^n}.$$

We define the Pimsner–Toeplitz algebra of $\mathcal{Y}$, $\mathcal{T}(\mathcal{Y})$ to be C*-algebra generated by the operators $\mathcal{B}$ and $\ell(\xi)$ for $\xi \in \mathcal{Y}$ where

$$\ell(\xi) b = \xi \cdot b \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

for all $b \in \mathcal{B}$ and $\xi_1, \dots, \xi_n \in \mathcal{Y}$. A result of Pimsner says the following.

Theorem 2.7 ([Pim97]) *Assume that $\mathcal{B}$ acts on $\mathcal{Y}$ by compact operators. Then the inclusion $\mathcal{B} \hookrightarrow \mathcal{T}(\mathcal{Y})$ is a KK-equivalence.*

In [Ger], one chooses a closed real subspace $\mathcal{Y}_{\mathbb{R}}$ of $\mathcal{Y}$ with the property that $\overline{\mathcal{B}\mathcal{Y}_{\mathbb{R}}\mathcal{B}} = \mathcal{Y}$, and defines the algebra $\mathcal{S}(\mathcal{Y}_{\mathbb{R}})$ to be the C*-algebra generated by $\mathcal{B}$ and $\{\ell(\xi) + \ell(\xi)^* | \xi \in \mathcal{Y}_{\mathbb{R}}\}$. By observing that the homotopy in [Pim97] leaves the subspace $\mathcal{F}(\mathcal{Y}) \otimes_{\mathcal{B}} \mathcal{S}(\mathcal{Y}_{\mathbb{R}})$ of $\mathcal{F}(\mathcal{Y}) \otimes_{\mathcal{B}} \mathcal{T}(\mathcal{Y})$ invariant, the following theorem was proved.

Theorem 2.8 ([Ger]) *Assume that $\mathcal{B}$ acts on $\mathcal{Y}$ by compact operators. The inclusions $\mathcal{B} \hookrightarrow \mathcal{S}(\mathcal{Y}_{\mathbb{R}}) \hookrightarrow \mathcal{T}(\mathcal{Y})$ are KK-equivalences.*

We will let $\mathcal{X}_{\mathbb{R},\mu}$ be the closure of the real subspace spanned by the elements $a_{\epsilon}\epsilon + a_{\epsilon}^{-1}\epsilon^{op}$. We observe that $\mathcal{S}(\Gamma, \mu)$ is generated by $\mathcal{C}$ and the elements $\ell(\xi) + \ell(\xi)^*$ for $\xi \in \mathcal{X}_{\mathbb{R},\mu}$. Using this realization, Theorem 2.8 gives the following corollary.

Corollary 2.9 *Since Γ is locally finite, $\mathcal{C}$ acts on $\mathcal{X}$ by compact operators. Therefore, the inclusions $\mathcal{C} \hookrightarrow \mathcal{S}(\Gamma, \mu) \hookrightarrow \mathcal{T}(\vec{\Gamma})$ are KK-equivalences. As a consequence,*

$$K_0(\mathcal{S}(\Gamma, \mu)) = \mathbb{Z}\{[p_{\alpha}] | \alpha \in V\} \quad and \quad K_1(\mathcal{S}(\Gamma, \mu)) = \{0\}$$

where the first group is the free abelian group on the vertices of Γ.

In particular, this shows that the *KK*-groups are independent of the weighting μ, but the positive cone $K_0^+(\mathcal{S}(\Gamma, \mu))$ will depend on μ; see Subsection 2.3.

2.3 Ideal Structure

In this section we fix a finite, connected graph, (Γ, μ) with at least two undirected edges. We will give an explicit description of the closed ideals of $\mathcal{S}(\Gamma, \mu)$. To begin, we will define two sets $V_>(\Gamma, \mu)$ and $V_=(\Gamma, \mu)$ as follows:

$$V_>(\Gamma, \mu) = \left\{ \alpha \in V \mid \mu(\alpha) > \sum_{\beta \sim \alpha} n_{\alpha,\beta}\mu(\beta) \right\},$$

$$V_=(\Gamma, \mu) = \left\{ \alpha \in V \mid \mu(\alpha) = \sum_{\beta \sim \alpha} n_{\alpha,\beta}\mu(\beta) \right\},$$

where we write $\beta \sim \alpha$ to indicate that β and α are the endpoints of at least one edge of Γ, and we let $n_{\alpha,\beta}$ be the number of unoriented edges that have α and β as endpoints. When Γ and μ are understood, we will write $V_>$ instead of $V_>(\Gamma, \mu)$, and $V_=$ for $V_=(\Gamma, \mu)$. We will also write $V_{\geq}$ to denote the set $V_{\geq} = V_= \cup V_>$.

We will show that there is an aesthetically pleasing description of the ideal structure of $\mathcal{S}(\Gamma, \mu)$ in terms of these vertex sets. To begin, we will state some results on the simplicity and stable rank of certain reduced (amalgamated) free products.

Notation 2.10 (i) Let A_1 and A_2 be two unital C*-algebras with faithful tracial states tr_1 and tr_2, respectively. We will write $A_1 * A_2$ as the *reduced* free product with respect to the tracial states tr_1 and tr_2.

(ii) More generally, if A_1 and A_2 are two C*-algebras with faithful tracial states tr_1 and tr_2 respectively and with trace preserving conditional expectations $E_i: A_i \to D$

onto a common subalgebra D, then $A_1 \underset{D}{*} A_2$ will denote the *reduced* amalgamated free product with respect to the expectations E_1 and E_2.

(iii) Suppose A is a unital C^* algebra with a finite faithful (not necessarily normalized) trace tr. Furthermore, assume that $A = B_1 \oplus \cdots \oplus B_n$ for unital C^*-algebras $B_1, \ldots, B_n$ with identities $p_1, \ldots, p_n$, respectively, satisfying $\text{tr}(p_k) = \mu_k$. When this happens, we will write

$$A = \overset{p_1}{\underset{\mu_1}{B_1}} \oplus \cdots \oplus \overset{p_n}{\underset{\mu_n}{B_n}}.$$

With this notation in hand, we will now state several free product results that will be used in our analysis of $\mathcal{S}(\Gamma, \mu)$.

Lemma 2.11 ([Dyk99])   *Let $A = A_1 * A_2$ and let $p \in A_1$, $q \in A_2$ be projections.*

(i)   *If $\text{tr}(p + q) > \text{tr}(1)$, then $p + q$ is invertible.*

(ii)   *If $\text{tr}(p + q) < \text{tr}(1)$, then $\{0\}$ is an isolated point in the spectrum of $p + q$, and if μ is the law of $p + q$ according to tr, then $\mu\{0\} = \text{tr}(1) - \text{tr}(p + q)$.*

(iii) *If $\text{tr}(p + q) = \text{tr}(1)$, the support projection of $p + q$ is 1; however, $p + q$ is not invertible, and $\{0\}$ is not isolated in the spectrum of $p + q$.*

A positivity argument and induction lead to the following corollary.

Corollary 2.12   *Assume $n \geq 2$, set $A = A_1 * A_2 * \cdots * A_n$, and let $p_i \in A_i$ be a projection for each i.*

(i)   *If $\sum_{i=1}^{n} \text{tr}(p_i) > \text{tr}(1)$, then $\sum_{i=1}^{n} p_n$ is invertible.*

(ii)   *If $\sum_{i=1}^{n} \text{tr}(p_i) < \text{tr}(1)$, then $\{0\}$ is an isolated point in the spectrum of $\sum_{i=1}^{n} p_n$, and if μ is the law of $\sum_{i=1}^{n} p_n$ according to tr, then $\mu\{0\} = \text{tr}(1) - \sum_{i=1}^{n} \text{tr}(p_n)$.*

(iii) *If $\sum_{i=1}^{n} \text{tr}(p_i) = \text{tr}(1)$, the support projection of $\sum_{i=1}^{n} p_n$ is 1; however, $\sum_{i=1}^{n} p_n$ is not invertible, and $\{0\}$ is not isolated in the spectrum of $\sum_{i=1}^{n} p_n$.*

Definition 2.13

(i)   If A is a C^*-algebra, then

$$K_0(A)^+ = \{x \in K_0(A) \mid x = [p] \text{ for some projection } p \in M_n(A)\}.$$

(ii)   If A is a unital C^*-algebra, we say that A has stable rank one if the set of invertible elements in A is norm-dense in A. If A is nonunital, A is said to have stable rank one if its unitization does.

Theorem 2.14   *Assume A_1 and A_2 are unital C^*-algebras with faithful tracial states tr_1 and tr_2 respectively. Suppose that A_1 contains a unitary u, and that A_2 contains unitary elements v and w satisfying*

$$\text{tr}_1(u) = \text{tr}_2(v) = \text{tr}_2(w) = \text{tr}_2(vw^*) = 0.$$

Let $A = A_1 * A_2$.

(i)   [Avi82] *A is simple with unique tracial state,* tr.

(ii)   [DHR97] *A has stable rank one.*

(iii) [DR98] *Let $j_i: A_i \to A$ be the canonical inclusions, and let $j_i^*: K_0(A_i) \to K_0(A)$ be the induced map on K_0. If G is the subgroup of $K_0(A)$ generated by $j_1^*(A_1)$ and $j_2^*(A_2)$, then*

$$G \cap K_0(A)^+ = \{x \in G \mid \operatorname{tr}(x) > 0\} \cup \{0\}.$$

Definition 2.15   Let A be a unital, abelian C^*-algebra with faithful state ϕ. Identify $A \cong C(X)$ with X a compact Hausdorff space, and note that $\phi(f) = \int_X f(x)d\mu(x)$ for some probability measure μ on X. We say that (A, ϕ) is *diffuse* if μ has no atoms. This is equivalent to A containing a *Haar unitary*, *i.e.*, a unitary element u satisfying $\phi(u^k) = 0$ for all $k \in \mathbb{Z} \smallsetminus \{0\}$ [DHR97].

Theorem 2.16 ([Dyk99])

(i)   *Suppose $A = A_1 * A_2$ where A_1 contains a diffuse abelian C^*-subalgebra and $A_2 \neq \mathbb{C}$. Then A is simple, has stable rank one, and has unique tracial state* tr.

(ii)   *Suppose $A = A_1 * \cdots * A_n$, where each A_i has the form*

$$A_i = \underset{\mu_i}{\mathcal{D}} \oplus \mathbb{C},$$

where $\mathcal{D}$ contains a diffuse abelian subalgebra. Then A is simple and has unique trace if and only if $\sum_{i=1}^n \mu_i > 1$. A always has stable rank one, regardless of the weighting.

Lemma 2.17 ([Dyk93, Dyk99, HP14b])

(i) *Suppose that A, B, and C are tracial C^*-algebras with*

$$\mathcal{D} = \left(\overset{p}{A} \oplus B\right) * C,$$

and $\mathcal{D}$ is endowed with the canonical free product trace. Then

$$p\mathcal{D}p = A * p\left(\left(\overset{p}{\mathbb{C}} \oplus B\right) * C\right)p.$$

(ii) *Suppose there are two unital, tracial, C^*-algebras $\overset{p+r}{B} \oplus \overset{r'}{\mathbb{C}}$ and C that both contain $D = \overset{p}{\mathbb{C}} \oplus \overset{q}{\mathbb{C}}$ as a unital C^*-subalgebra with $q = r + r'$. Assume that the algebras are equipped with conditional expectations E_D^1 and E_D^2 onto D, respectively as well as traces tr_1 and tr_2 so that $\operatorname{tr}_i = \operatorname{tr}_i \circ E_D^i$ for $i = 1, 2$, and the restrictions of tr_1 and tr_2 to D coincide. Form the reduced amalgamated free product*

$$\mathcal{D} = \left(\overset{p+r}{B} \oplus \overset{r'}{\mathbb{C}}\right) \underset{D}{*} C.$$

Then

$$(p+r)\mathcal{D}(p+r) = (B, E_{D'}^1) \underset{D'}{*} \left((p+r)\left(\left(\overset{p}{\mathbb{C}} \oplus \overset{r}{\mathbb{C}} \oplus \overset{r'}{\mathbb{C}}\right) \underset{D}{*} C, E_D^2\right)(p+r), E_{D'}^2\right)$$

where $D' = \overset{p}{\mathbb{C}} \oplus \overset{r}{\mathbb{C}}$, the conditional expectations $E_{D'}^i$ onto D' are the trace preserving ones, and the free product is reduced.

Lemma 2.18 ([Iva11]) *Let C_1 and C_2 be unital C^* algebras containing the unital C^* subalgebra D unitally. Suppose each C_i is equipped with a trace tr_i such that tr_1 and tr_2 coincide on D and that there exist trace preserving conditional expectations E_D^i of C_i onto D. Consider the reduced amalgamated free product with conditional expectation E*

$$(C, E) = (C_1, E_D^1) \underset{D}{*} (C_2, E_D^2).$$

Then C is simple and has unique trace $\mathrm{tr} = \mathrm{tr}_1 \circ E_D = \mathrm{tr}_2 \circ E_D$ provided the following conditions hold:

(i) *There exist unitaries $u_1 \in C_1$ and u_2, u_2' in C_2 such that $E_D(u_1) = 0 = E_D(u_2) = E_D(u_2') = E(u_2^* u_2')$.*

(ii) *For every $a_1, \dots, a_n \in D$ with zero trace, there exists a unitary $u \in C_2$ with expectation 0 such that $E_D(u a_i u^*) = 0$ for each i.*

(iii) *There are unitaries $w, v \in C_2$ with expectation 0 such that $E_D(wav) = 0$ for all $a \in D$.*

Furthermore, let G be the subgroup of $K_0(C)$ that is generated by $j_1^*(K_0(C_1))$ and $j_2^*(K_0(C_2))$ with $j_i \colon C_i \to C$ the canonical inclusion. If the above three conditions hold, then

$$K_0(C)^+ \cap G = \{x \in G : \mathrm{tr}(x) > 0\} \cup \{0\}.$$

Lemma 2.19 ([HP14b]) *Suppose that B_1 and B_2 are unital separable C^*-algebras both unitally containing $D = \mathbb{C}^2$ as a subalgebra. Assume that B_1 and B_2 are equipped with faithful traces tr_1 and tr_2, respectively such that tr_1 and tr_2 agree on D. Let E_D^i be the trace preserving conditional expectation from B_i to D for $i \in \{1, 2\}$. Form the reduced amalgamated free product*

$$(B, E) = (B_1, E_D^1) \underset{D}{*} (B_2, E_D^2).$$

Let p and q be the two minimal projections in D and assume $\mathrm{tr}_i(p) = \mathrm{tr}_i(q)$. Suppose B_1 contains a unitary u_1 and B_2 contains unitaries u_2 and u_2' with $v = pvp + qvq$ for $v \in \{u_1, u_2, u_2'\}$, which also satisfy

$$E(u_1) = 0 = E(u_2) = E(u_2') = E(u_2^* u_2').$$

Then pBp and qBq both have stable rank 1.

The key to unlocking the ideal structure of $\mathcal{S}(\Gamma, \mu)$ is showing the existence of a minimal ideal that "avoids" the sets $V_>(\Gamma, \mu)$ and $V_=(\Gamma, \mu)$. To begin, we fix a vertex, $\alpha \in V \smallsetminus V_{\geq}$.

Lemma 2.20 *Suppose $\beta \sim \alpha$.*

(i) *If $\beta \notin V_{\geq}$, then p_β is in the ideal I_α generated by p_α.*

(ii) *If $\beta \in V_{\geq}$, then β is not in the ideal J_β that is generated by $\{p_\delta \mid \delta \in V \smallsetminus \{\beta\}\}$. If $\gamma \sim \beta$, then $p_\gamma \in I_\alpha$.*

Proof (i) Suppose that $\beta \notin V_{\geq}(\Gamma, \mu)$. Let $e_1, \dots, e_n$ denote the edges in Γ that have β as an endpoint, and let $\epsilon_1, \dots, \epsilon_n$ be the corresponding oriented edges in $\vec{\Gamma}$ having $t(\epsilon_i) = \beta$ for all i, and $s(\epsilon_1) = \alpha$.

If $\mu(\beta) < \mu(\alpha)$, then $X_{\epsilon_1}^* X_{\epsilon_1} (= X_{\epsilon_1}^* p_\alpha X_{\epsilon_1})$ is invertible in $p_\beta \mathcal{S}(\Gamma, \mu) p_\beta$, so it follows that $p_\beta \in I_\alpha$.

If $\mu(\beta) \geq \mu(\alpha)$, then we necessarily have $n \geq 2$. Using Theorem 2.4, the traces of the support projections of the elements $X_{\epsilon_i}^* X_{\epsilon_i}$ add up to $\sum_{\gamma \sim \beta} n_{\gamma, \beta} \mu(\beta)$, which by hypothesis exceeds $\mu(\beta)$. It follows from Theorem 2.16 that the C*-algebra generated by the elements $X_{\epsilon_i}^* X_{\epsilon_i}$ is simple, which means that p_β is in the ideal generated by these elements. Noting that $X_{\epsilon_1}^* X_{\epsilon_1} = X_{\epsilon_1}^* p_\alpha X_{\epsilon_1}$, this shows that $p_\beta \in I_\alpha$.

(ii) Suppose that $\beta \in V_\geq$ and let $e_1, \ldots, e_n$ and $\epsilon_1, \ldots, \epsilon_n$ be as in the previous part. Every element in I which is supported under p_β must be a norm limit of elements of the form

$$\sum_{i,j} X_{\epsilon_i}^* y_{i,j} X_{\epsilon_j}$$

for $y_{i,j} \in \mathcal{S}(\Gamma, \mu)$.

If $n = 1$, then for any $y \in \mathcal{S}(\Gamma, \mu)$, $X_{\epsilon_1}^* y X_{\epsilon_1}$ is not invertible in $p_\beta \mathcal{S}(\Gamma, \mu) p_\beta$. It follows that $p_\beta \notin J_\beta$.

If $n \geq 2$, then the support projections p_{ϵ_i} of $X_{\epsilon_i}^* X_{\epsilon_i}$ are in $\mathcal{S}(\Gamma, \mu)$ and are free with respect to $\mathrm{Tr}(p_\beta \cdot p_\beta)$. We have $\sum_{i=1}^n \mathrm{Tr}(p_{\epsilon_i}) \leq \mathrm{Tr}(p_\beta)$. Furthermore, if $x \in p_\beta J_\beta p_\beta$, then x must be in the hereditary C*-algebra $(\sum_{i=1}^n p_{\epsilon_i}) \mathcal{S}(\Gamma, \mu) (\sum_{i=1}^n p_{\epsilon_i})$. By Corollary 2.12, $\sum_{i=1}^n p_{\epsilon_i}$ is not invertible in $p_\beta \mathcal{S}(\Gamma, \mu) p_\beta$. This implies that $p_\beta \notin J_\beta$. The unital C*-subalgebra in $p_\beta \mathcal{S}(\Gamma, \mu) p_\beta$ generated by the elements $X_{\epsilon_i}^* X_{\epsilon_i}$ is the reduced free product

$$B = \mathop{\Large \ast}_{i=1}^n \Big(C^*(X_{\epsilon_i}^* X_{\epsilon_i}) \overset{p_{\epsilon_i}}{\oplus} \mathbb{C} \Big).$$

By Lemma 2.17 and Theorem 2.16, $p_{\epsilon_i} B p_{\epsilon_i}$ is simple for each i, and the element $p_{\epsilon_i} X_{\epsilon_1}^* X_{\epsilon_1} p_{\epsilon_i}$ is nonzero. This means that $p_{\epsilon_i} \in I$ for all i. Note if $\gamma \sim \beta$, then $\gamma \notin V_\geq$, and p_γ is equivalent to one of the p_{ϵ_i} (via the polar part of X_{ϵ_i}). This implies that $p_\gamma \in I_\alpha$.   ∎

Lemma 2.21   *Let I_α be the closed ideal in $\mathcal{S}(\Gamma, \mu)$ generated by p_α. Then I contains $\{p_\beta \mid \beta \notin V_\geq(\Gamma, \mu)\}$. Therefore, I_α does not depend on α, and we will denote it by I. Furthermore, $\mathcal{S}(\Gamma, \mu)/I \cong \bigoplus_{\gamma \in V_{\geq 0}} \mathbb{C}$. In addition, I is unital if and only if $V_=(\Gamma, \mu)$ is empty.*

Proof   Since Γ is connected, we can iterate Lemma 2.20 to conclude that I_α contains $\{p_\beta \mid \beta \notin V_\geq(\Gamma, \mu)\}$ and has trivial intersection with $\{p_\beta \mid \beta \in V_\geq(\Gamma, \mu)\}$. This means that I is generated as an ideal by $\{X_\epsilon \mid \epsilon \in \vec{E}\}$, since at least one vertex in $V \smallsetminus V_\geq$ is an endpoint of ϵ for each $\epsilon \in E$. If $\gamma \in V_\geq$, then every element in $p_\gamma \mathcal{S}(\Gamma, \mu) p_\gamma$ is a norm limit of elements of the form $c p_\gamma + p_\gamma x p_\gamma$ where x is a polynomial in the X_ϵ's. The arguments in Case (ii) show that p_γ is not a norm limit of expressions of the form $p_\gamma x p_\gamma$. This implies that $\mathcal{S}(\Gamma, \mu)/I \cong \bigoplus_{\gamma \in V_\geq} \mathbb{C}$.

Finally we verify the statement on the unitality of I. Let $\gamma \in V_\geq$ and let $\epsilon_1', \ldots, \epsilon_m'$ be all of the edges satisfying $t(\epsilon_i') = \gamma$. If $V_=$ is empty and $p_{\epsilon_i'}$ is the support projection of $X_{\epsilon_i}^* X_{\epsilon_i}$, then since $\sum_{i=1}^m \mathrm{Tr}(p_{\epsilon_i'}) < \mathrm{Tr}(p_\gamma)$, $\{0\}$ is open in the spectrum of $\sum_i p_i$ from Corollary 2.12, which implies that the support projection, q_γ of $\sum_{i=1}^n p_{\epsilon_i'}$ is in I. The element $\sum_{\gamma \notin V_\geq} p_\gamma + \sum_{\gamma \in V_\geq} q_\gamma$ is the unit for I.

If $V_=$ is not empty, $m \geq 2$, and $\gamma \in V_=$, then since $\sum_i \text{Tr}(p_{\epsilon_i}) = \text{Tr}(p_\gamma)$, p_γ is in the strong closure (in $L^2(\mathcal{S}(\Gamma, \mu), \text{Tr})$) of the C*-algebra generated by $\sum_i p_{\epsilon_i}$ [Har13], so it follows that p_γ is the support projection of $\sum_i \text{Tr}(p_{\epsilon_i})$. Therefore I is not unital. Finally, if $m = 1$ and $\gamma \in V_=$, then since the law of $X^*_{\epsilon_1'} X_{\epsilon_1'}$ in $p_\gamma(\mathcal{S}(\Gamma, \mu), \text{Tr})p_\gamma$ has no atoms, it follows that p_γ is in the strong closure of I, so I is not unital. ∎

Corollary 2.22 *If $V_=$ is empty, then*

$$\mathcal{S}(\Gamma, \mu) = I \oplus \bigoplus_{\gamma \in V_>} \overset{r_\gamma}{\mathbb{C}}$$

with $r_\gamma \leq p_\gamma$ and $\text{Tr}(r_\gamma) = \mu(\gamma) - \sum_{\beta \sim \gamma} n_{\alpha, \beta} \mu(\beta)$. If $V_=$ is not empty, then

$$\mathcal{S}(\Gamma, \mu) = \mathfrak{I} \oplus \bigoplus_{\gamma \in V_>} \overset{r_\gamma}{\mathbb{C}},$$

where $\mathfrak{I}$ is unital, the strong operator closures of I and $\mathfrak{I}$ coincide in $L^2(\mathcal{S}(\Gamma, \mu), \text{Tr})$, and $\mathfrak{I}/I \cong \bigoplus_{\beta \in V_=} \mathbb{C}$.

We will now show that I is minimal.

Lemma 2.23 *Let I be as in Lemma 2.21. Then I is simple, has unique trace, and has stable rank 1.*

Proof The arguments here mirror arguments used in [HP14b, Section 4].

Case 1: We first assume that there is a vertex α of minimal weight such that $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two connected subgraphs of Γ, each of which has a nonempty edge set, share no edges in common, and intersects only at the vertex α. We have $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha = p_\alpha \mathcal{S}(\Gamma_1, \mu) p_\alpha * p_\alpha \mathcal{S}(\Gamma_2, \mu) p_\alpha$. Let ϵ_1 and ϵ_2 be oriented edges with $t(\epsilon_i) = \alpha$ and $\epsilon_i \in \vec{E}(\Gamma_i)$. The elements $X^*_{\epsilon_i} X_{\epsilon_i}$ generate diffuse abelian C*-subalgebras of $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha$ with respect to $\text{Tr}(p_\alpha \cdot p_\alpha)$ from Theorem 2.4 and the assumption on α. It follows from Theorem 2.16 that $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha$ is simple, has unique trace, and has stable rank 1 from Theorem 2.14 or Theorem 2.16. Since p_α is full in I from Lemma 2.21, this implies that I is simple, has unique trace, and has stable rank 1.

Case 2: We assume that α is connected only to one other vertex β, and by only one edge e_1. Let $\epsilon_1 \in \vec{E}$ have $s(\epsilon_1) = \alpha$ and $t(\epsilon_1) = \beta$. Notice that the assumption on α implies that there is at least one other undirected edge e_2 with β as an endpoint, and $\mu(\beta) > \mu(\alpha)$ since equality implies that we are in Case 1. Assume that $t(\epsilon_2) = \beta$. Let $\widetilde{\Gamma}$ be the graph that is obtained by removing the edge e_1 from Γ. Let q_α be the support projection of $X^*_{\epsilon_1} X_{\epsilon_1}$ and note that $q_\alpha \leq p_\beta$ and q_α is equivalent to p_α. If B is the C*-algebra generated q_β and $\mathcal{S}(\widetilde{\Gamma}, \mu)$, it follows from Lemma 2.17 that

$$q_\alpha \mathcal{S}(\Gamma, \mu) q_\alpha = q_\alpha C^*(X^*_\epsilon X_\epsilon) q_\alpha * q_\alpha B q_\alpha.$$

The C*-algebra $q_\alpha C^*(X^*_\epsilon X_\epsilon) q_\alpha$ is diffuse and abelian. The algebra $q_\alpha B q_\alpha$ contains the element $q_\alpha X^*_{\epsilon_2} X_{\epsilon_2} q_\alpha$, which generates a diffuse von-Neumann algebra, since the support projection of $X^*_{\epsilon_2} X_{\epsilon_2}$ has trace at least as large as $\mu(\alpha)$. It follows that $q_\alpha B q_\alpha$ has a unitary of trace zero. Therefore, from Theorem 2.14, $q_\alpha \mathcal{S}(\Gamma, \mu) q_\alpha$ is simple, has

unique trace, and has stable rank 1. Since p_α is full in I from Lemma 2.21, this implies that I is simple, has unique trace, and has stable rank 1.

Case 3: We assume that there are two distinct vertices, α and β, an edge e_1 joining α and β, and a path from α to β that avoids e_1. We can also assume that α is of minimal weight, and $\alpha \notin V_{\geq}$. Let $\epsilon_1 \in E(\vec{\Gamma})$ be the edge associated with e_1 that satisfies $t(\epsilon_1) = \beta$ and $s(\epsilon_1) = \alpha$, and set q_α to be the support projection of $X_{\epsilon_1}^* X_{\epsilon_1}$ and note that $q_\alpha \leq p_\beta$. Also note that $q_\alpha \in \mathcal{S}(\Gamma, \mu)$, since if $X_{\epsilon_1}^* X_{\epsilon_1}$ has connected spectrum, then $q_\alpha = p_\beta$.

Set $\mathcal{B} = (p_\alpha + q_\alpha)\mathcal{S}(\Gamma, \mu)(p_\alpha + q_\alpha)$. Let Γ' be the subgraph of Γ obtained by deleting the edge e_1, and let Γ'' be the subgraph of Γ whose vertices are α and β and whose edge set is $\{e_1\}$. Set $D = C^*(\{p_\alpha, p_\beta\})$ and $D' = C^*(\{p_\alpha, q_\alpha\})$. From Lemma 2.17, it follows that

$$\mathcal{B} = (p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha) \underset{D'}{*} (p_\alpha + q_\alpha)\mathcal{B}'(p_\alpha + q_\alpha),$$

where the conditional expectations are the trace preserving ones, and

$$\mathcal{B}' = \left(\overset{p_\alpha}{\mathbb{C}} \oplus \overset{q_\alpha}{\mathbb{C}} \oplus \overset{p_\beta - q_\alpha}{\mathbb{C}} \right) \underset{D}{*} (p_\alpha + p_\beta)\mathcal{S}(\Gamma', \mu)(p_\alpha + q_\beta).$$

We now show that $\mathcal{B}$ satisfies the conditions in Lemma 2.18.

Theorem 2.4 determines the structure of $(p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha)$. Explicitly, if $\mu(\alpha) < \mu(\beta)$, then

$$(p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha) \cong M_2(\mathbb{C}) \otimes C[0,1]$$

with trace $\mathrm{Tr}_{M_2}(\mathbb{C}) \otimes \int \cdot\, d\lambda$ with $d\lambda$ Lebesgue measure. In this isomorphism, we have

$$p_\alpha \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q_\alpha \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\mu(\alpha) = \mu(\beta)$, then

$$(p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha) \cong$$
$$\{f : [0,1] \to M_2(\mathbb{C}) \mid f \text{ is continuous and } f(0) \text{ is diagonal}\}$$

with the above trace and identifications for p_α and q_α. In either case, the unitary

$$U = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

lies in $(p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha)$. The traceless elements of D' are spanned by

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and it is easy to check that UxU^* has zero expectation, so Lemma 2.18(ii) is satisfied. Note that one has

$$E(u) = E(u^*) = E(u^2) = E(u(u^*)^*) = 0.$$

Choose edges e_2 and e_3 of Γ' whose oriented versions satisfy $t(\epsilon_2) = \alpha$ and $t(\epsilon_3) = \beta$. There is a Haar unitary v_1 in $p_\alpha \mathcal{S}(\Gamma', \mu)p_\alpha$ in the continuous functional calculus of $X_{\epsilon_2}^* X_{\epsilon_2}$. As in the previous case, $q_\alpha X_{\epsilon_3}^* X_{\epsilon_3} q_\alpha$ generates a diffuse von Neumann algebra, so it follows that there is a unitary, v_2, of trace zero in $q_\alpha \mathcal{S}(\Gamma', \mu)q_\alpha$. This

implies that $v_1 + v_2$ is an expectationless unitary in $(p_\alpha + q_\alpha)\mathcal{B}'(p_\alpha + q_\alpha)$ so Lemma 2.18(i) is satisfied.

Finally, under the matrix algebra identification of $(p_\alpha + q_\alpha)\mathcal{S}(\Gamma'', \mu)(p_\alpha + q_\alpha)$, set

$$V = \begin{pmatrix} e^{2i\pi t} & 0 \\ 0 & e^{2i\pi t} \end{pmatrix}.$$

It is easy to see that V is expectationless, and that $E(VyV) = 0$ for all $y \in D'$. This implies that Lemma 2.18(iii) is satisfied. This implies $\mathcal{B}$ is simple and has unique trace. Since p_α is full in I from Lemma 2.21, this implies that I is simple, has unique trace, and has stable rank 1.

The verification of this case shows that the conditions in Lemma 2.19 are satisfied, so $p_\alpha \mathcal{B} p_\alpha$ has stable rank 1. This implies that I has stable rank 1 as well. ∎

There are several immediate corollaries

Corollary 2.24 $\mathcal{S}(\Gamma, \mu)$ *is simple if and only if $V_\geq$ is empty.*

Corollary 2.25 *For each $\beta \in V_\geq$, the ideal J_β generated by $\{p_\gamma : \gamma \neq \beta\}$ is maximal and of co-dimension 1. Furthermore, every ideal in $\mathcal{S}(\Gamma, \mu)$ is an intersection of the ideals J_β, and*

$$I = \bigcap_{\beta \in V_\geq} J_\beta.$$

Corollary 2.26 $\mathcal{S}(\Gamma, \mu)$ *has stable rank 1.*

Proof $\mathcal{S}(\Gamma, \mu)/I$ is finite-dimensional. The result follows from [Rie83]. ∎

We now turn our attention to the K-groups and positive cone of I.

Lemma 2.27 *The K-groups of I are as follows:*

$$K_0(I) = \mathbb{Z}\left\{[p_\beta] \mid \beta \in V \smallsetminus V_\geq\right\} \quad \text{and} \quad K_1(I) = \{0\},$$

where $\mathbb{Z}\left\{[p_\beta] \mid \beta \in V \smallsetminus V_\geq\right\}$ is the free abelian group on the vertices of Γ that are not in $V_\geq$.

Proof Consider the six term exact sequence

$$
\begin{array}{ccccc}
K_0(I) & \xrightarrow{\iota_*^0} & K_0(\mathcal{S}(\Gamma, \mu)) & \xrightarrow{\pi_*^0} & K_0(\mathcal{S}(\Gamma, \mu)/I) \\
\partial_1 \uparrow & & & & \downarrow \partial_0 \\
K_1(\mathcal{S}(\Gamma, \mu)/I) & \xleftarrow{\pi_*^1} & K_1(\mathcal{S}(\Gamma, \mu)) & \xleftarrow{\iota_*^1} & K_1(I)
\end{array}
$$

where ι_*^i are the induced maps from the canonical inclusion $\iota: I \to \mathcal{S}(\Gamma, \mu)$, π_*^i are the induced maps from the canonical quotient map $\pi: \mathcal{S}(\Gamma, \mu) \to \mathcal{S}(\Gamma, \mu)/I$, and ∂_i are the connecting maps. Recall that

$$\mathcal{S}(\Gamma, \mu)/I \cong \bigoplus_{\beta \in V_\geq} \mathbb{C},$$

and the induced map $C^* \{ p_\beta \mid \beta \in V_\geq \} \to \mathcal{S}(\Gamma, \mu)/I$ is an isomorphism. Therefore, π^0_* is surjective, which implies that ∂_0 is the zero map, which implies that ι^1_* is injective. Since $K_1(\mathcal{S}(\Gamma, \mu)) = \{0\}$ it follows that $K_1(I) = \{0\}$.

The image of ι^0_* is $\mathbb{Z} \{ p_\beta \mid \beta \in V \smallsetminus V_\geq \}$. Since $\mathcal{S}(\Gamma, \mu)/I$ is finite-dimensional, $K_1(\mathcal{S}(\Gamma, \mu)/I) = \{0\}$, which implies ι^0_* is injective, hence

$$K_0(I) = \mathbb{Z} \{ [p_\beta] \mid \beta \in V \smallsetminus V_\geq \}. \qquad \blacksquare$$

Lemma 2.28 $\quad K_0(I)^+ = \{ x \in K_0(I) \mid \mathrm{Tr}(x) > 0 \} \cup \{0\}.$

Proof The proof of Lemma 2.23 shows that in the three cases where a hereditary subalgebra of I was expressed as a(n) (amalgamated) free product of A and B, the subgroup, $G \subset K_0(I)$, generated by $\iota^0_*(A)$ and $\iota^0_*(B)$, has the desired structure of the positive cone. We simply need to show that $G = K_0(I)$. This will be done by analyzing the three free product cases. We will now assume that α has minimal weight among all the vertices in Γ.

Case 1: If $\Gamma = \Gamma_1 \cup \Gamma_2$ as in the proof of Lemma 2.23, then from the description of Γ, if $\beta \neq \alpha$ and $\beta \in V(\Gamma) \smallsetminus V_\geq(\Gamma)$, then $\beta \in V(\Gamma_i) \smallsetminus V_\geq(\Gamma_i)$ for exactly one $i \in \{1, 2\}$. From the proof of Lemma 2.21, it follows that p_β is in the ideal generated by p_α in $\mathcal{S}(\Gamma_i)$. It follows that $K_0(I)$ is generated by $K_0(p_\alpha \mathcal{S}(\Gamma_j) p_\alpha)$ for $j \in \{1, 2\}$.

Case 2: Assume that Γ satisfies the conditions in Case 2 in Lemma 2.23. All notation here comes from Case 2 in the proof of Lemma 2.23. Let $\epsilon_2, \dots, \epsilon_n$ be all of the edges distinct from ϵ_1 that have target β. Let p_{ϵ_i} be the support projection of $X^*_{\epsilon_i} X_{\epsilon_i}$ for $i \in \{1, \dots, n\}$. If $i > 1$, we see that $p_{\epsilon_i} B p_{\epsilon_i}$ contains the algebra

$$p_{\epsilon_i} \big((\overset{q_\alpha}{\mathbb{C} \oplus \mathbb{C}}) * (C^*(X^*_{\epsilon_i} X_{\epsilon_i}) \oplus \overset{p_{\epsilon_i}}{\mathbb{C}} \overset{p_\beta - p_{\epsilon_i}}{\mathbb{C}}) \big) p_{\epsilon_i} =$$

$$C^*(X^*_{\epsilon_i} X_{\epsilon_i}) * p_{\epsilon_i} \big((\overset{q_\alpha}{\mathbb{C} \oplus \mathbb{C}}) * (\overset{p_{\epsilon_i}}{\mathbb{C}} \oplus \overset{p_\beta - p_{\epsilon_i}}{\mathbb{C}}) \big), p_{\epsilon_i}$$

which is simple, since $C^*(X^*_{\epsilon_i} X_{\epsilon_i})$ is diffuse. This implies that p_{ϵ_i} is in the ideal generated by p_{ϵ_1} in B.

If $\beta \in V \smallsetminus V_\geq$, then $\sum_{i=1}^n p_{\epsilon_i} \geq k p_\beta$ for some $k > 0$ (Corollary 2.12). Therefore, p_β is in the ideal generated by $p_{\epsilon_1} B p_{\epsilon_1}$ in B since each p_{ϵ_i} is in this ideal and ideals are hereditary. Using the iterative argument in Lemmas 2.20 and 2.21, we see that if $\gamma \in V \smallsetminus V_\geq$ and $\gamma \neq \alpha$, then γ is also in the ideal in B $p_{\epsilon_1} B p_{\epsilon_1}$. It follows from this that $K_0(I)$ is generated by $K_0(p_{\epsilon_1} C^*(X^*_{\epsilon_1} X_{\epsilon_1}) p_{\epsilon_1})$ and $K_0(p_{\epsilon_1} B p_{\epsilon_1})$.

If $\beta \in V_\geq$, then $\mathrm{tr}(p_{\epsilon_i}) < \mu(\beta)$ so it follows that p_{ϵ_i} is equivalent to $p_{s(\epsilon_i)}$, hence $p_{s(\epsilon_i)}$ is in the ideal in B generated by $p_{\epsilon_1} B p_{\epsilon_1}$. The inductive argument from Lemma 2.21 will conclude that $K_0(I)$ is generated by $K_0(p_{\epsilon_1} C^*(X^*_{\epsilon_1} X_{\epsilon_1}) p_{\epsilon_1})$ and $K_0(p_{\epsilon_1} B p_{\epsilon_1})$.

Case 3: By considering $\epsilon_2, \dots, \epsilon_n$ as in the previous case, the exact same proof shows that $K_0(I)$ is generated by

$$K_0\big((p_\alpha + q_\alpha) C^*(X^*_{\epsilon_1} X_{\epsilon_1})(p_\alpha + q_\alpha) \big) \quad \text{and} \quad K_0\big((p_\alpha + q_\alpha) \mathcal{B}'(p_\alpha + q_\alpha) \big). \qquad \blacksquare$$

Since I has stable rank one, we obtain the following corollary.

Corollary 2.29 *Suppose p and q are projections in $M_n(I)$ satisfying $\mathrm{Tr}_n(p) < \mathrm{Tr}_n(q)$, where Tr_n is the canonical trace on $M_n(I)$ induced from the trace on I. Then there is a $v \in M_n(I)$ satisfying $v^*v = p$ and $vv^* \leq q$.*

2.4 Extension to Infinite Graphs

Assume that Γ is a countably infinite connected graph with a countable edge set. We can write Γ as in increasing union of finite subgraphs Γ_n. Since simplicity, K-groups, stable rank 1, and unique trace are preserved under inductive limits, we have the following corollary from our work in the previous section.

Corollary 2.30 *Let I be the ideal in $\mathcal{S}(\Gamma, \mu)$ generated by $\left\{ X_\epsilon \mid \epsilon \in E(\vec{\Gamma}) \right\}$. Then the following statements hold.*

(i) *I is simple, contains the set $\left\{ p_\beta \mid \beta \in V \smallsetminus V_\geq \right\}$, and does not intersect*

$$\left\{ p_\gamma \mid \gamma \in V_\geq \right\}.$$

   Since Γ infinite and connected implies that $V \smallsetminus V_\geq$ is infinite, I is not unital.

(ii) *$\mathcal{S}(\Gamma, \mu)/I \cong \bigoplus_{\gamma \in V_\geq} \mathbb{C}$.*

(iii) *$K_0(I) \cong \mathbb{Z}\left\{ [p_\beta] \mid \beta \in V \smallsetminus V_\geq \right\}$, $K_1(I) = \{0\}$, and*

$$K_0(I)^+ = \{ x \in K_0(I) \mid \mathrm{Tr}(x) > 0 \} \cup \{0\}.$$

(iv) *I and $\mathcal{S}(\Gamma, \mu)$ have stable rank 1.*

(v) *I has a unique (up to scaling) lower semicontinuous tracial weight. This weight is finite if and only if the support projection of I in $\mathcal{M}(\Gamma, \mu)$, the von Neumann algebra generated by $\mathcal{S}(\Gamma, \mu)$, has finite trace.*

3 Free Differentials, Atomless Loops, and Algebraicity

For this section, we assume that Γ is finite with a fixed weighting, μ on $V(\Gamma)$. We let $\mathcal{A} = \mathbb{C}\langle (X_\epsilon)_{\epsilon \in E(\vec{\Gamma})}, (p_\beta)_{\beta \in V(\Gamma)} \rangle$. We aim to prove the following theorems, which are in the spirit of [SS15, MSW14, CS15].

Theorem 3.1 *Suppose $\alpha \in V$ and $Q \in \mathcal{A}$ are such that $\mu(\alpha) = \min\{\mu(\beta) \mid \beta \in V(\Gamma)\}$ and $Qp_\alpha = Q$. If $a = a^* = p_\alpha a p_\alpha \in W^*(\mathcal{S}(\Gamma, \mu), \mathrm{Tr})$ and $Qa = 0$, then either $Q = 0$ or $a = 0$.*

Theorem 3.2 *Let $x \in M_n(\mathcal{A})$ be self-adjoint. Then the law of x with respect to $\mathrm{Tr} \otimes \mathrm{tr}_n$ has algebraic Cauchy transform.*

Theorem 3.1 has the following corollary.

Corollary 3.3 *Suppose $Q = Q^* \in \mathcal{A}$, p_α is as in the statement of Theorem 3.1, and $Q = Qp_\alpha (= p_\alpha Q p_\alpha)$. If Q is not a scalar multiple of p_α, then the law of Q in $(p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha, \mathrm{Tr})$ has no atoms.*

Notice that any such Q described in Corollary 3.3 must be a linear combination of elements of the form $X_{\epsilon_1} \cdots X_{\epsilon_n}$, where $\epsilon_1 \cdots \epsilon_n$ is a loop in $\vec{\Gamma}$ based at α.

To develop the machinery to prove Theorem 3.1, we will develop a free differential calculus along the lines of [Voi93, CDS14].

3.1 Commutation with Finite Rank Operators

Let $\mathcal{H} = L^2(\mathcal{S}(\Gamma, \mu), \mathrm{Tr})$. We desire to place a more natural Euclidean structure on $\mathcal{H}$. First, some notation is in order.

Notation 3.4 Let Π denote the set of all paths in $\vec{\Gamma}$, and $\sigma = \epsilon_1 \cdots \epsilon_n \in \Pi$. Let $|\sigma| = n$ denote the length of σ. We set:

- $\sigma^{\mathrm{op}} = \epsilon_n^{\mathrm{op}} \cdots \epsilon_1^{\mathrm{op}}$
- $\ell(\sigma) = \ell(\epsilon_1) \cdots \ell(\epsilon_n)$.
- $a_\sigma = \sqrt[4]{\frac{\mu(s(\sigma))}{\mu(t(\sigma))}} = \sqrt[4]{\frac{\mu(s(\epsilon_1))}{\mu(t(\epsilon_n))}}$
- $X_\sigma = X_{\epsilon_1} \cdots X_{\epsilon_n}$.
- $Y_\sigma = \sum_{\sigma = \rho\tau} a_\rho a_\tau^{-1} \ell(\rho) \ell(\tau^{\mathrm{op}})^*$
- $\widehat{\sigma} = \epsilon_1 \otimes \cdots \otimes \epsilon_n$ (viewed as an element of $\mathcal{F}(\Gamma)$).

Proposition 3.5 (Change of basis) *Suppose σ is a path in $\vec{\Gamma}$ of length n.*

(i) $Y_\sigma \in \mathcal{S}(\Gamma, \mu)$.
(ii) $Y_\sigma = X_\sigma + Q$ *where Q is a linear combination of the $X_{\sigma'}$ with $|\sigma'| < n$.*
(iii) $X_\sigma = Y_\sigma + P$ *where P is a linear combination of the $Y_{\sigma'}$ with $|\sigma'| < n$.*
(iv) *Let $\widehat{1} \in \mathcal{F}(\vec{\Gamma})$ be defined by $\widehat{1} = \sum_{\alpha \in \Gamma} \widehat{p_\alpha}$. Set*

$$\widetilde{Y}_\sigma = \sqrt[4]{\frac{\mu(t(\sigma))}{\mu(s(\sigma))}} Y_\sigma.$$

Then $\widetilde{Y}_\sigma$ is the unique element x in $\mathcal{S}(\Gamma, \mu)$ satisfying $x \cdot \widehat{1} = \widehat{\sigma}$.

Proof We will prove this by induction on $|\sigma|$, the length of σ. If $|\sigma| = 1$, then $\sigma = \epsilon$ for some $\epsilon \in \vec{E}$ and it is apparent that $Y_\epsilon = X_\epsilon$.

Given, σ with $|\sigma| > 1$, write $\sigma = \epsilon\tau$ for $\epsilon \in \vec{E}$, and write $\tau = \epsilon'\tau'$. We see that

$$\begin{aligned}
X_\epsilon Y_\tau &= \left(a_\epsilon \ell(\epsilon) + a_{\epsilon^{\mathrm{op}}} \ell(\epsilon^{\mathrm{op}})^*\right) \sum_{\tau = \tau_1 \tau_2} a_{\tau_1} a_{\tau_2}^{-1} \ell(\tau_1) \ell(\tau_2^{\mathrm{op}})^* \\
&= \sum_{\tau = \tau_1 \tau_2} a_{\epsilon\tau_1} a_{\tau_2} \ell(\epsilon\tau_1) \ell(\tau_2^{\mathrm{op}})^* + \delta_{\epsilon^{\mathrm{op}}, \epsilon'} \sum_{\tau' = \tau_1' \tau_2'} a_{\tau_1'} a_{(\tau_2')^{\mathrm{op}}}^{-1} \ell(\tau_1') \ell((\tau_2')^{\mathrm{op}})^* \\
&\quad + a_{\epsilon^{\mathrm{op}}} a_\tau^{-1} \ell(\epsilon^{\mathrm{op}})^* \ell(\tau^{op})^* \\
&= Y_\sigma + \delta_{\epsilon^{\mathrm{op}}, \epsilon'} Y_{\tau'}
\end{aligned}$$

By induction, this proves (i) and (ii). (iii) follows directly from (ii).

For (iv), it is clear from the definition of $\widetilde{Y}_\sigma$ that $\widetilde{Y}_\sigma \cdot \widehat{1} = \widehat{\sigma}$. Furthermore, by the definition of the conditional expectation, E (Theorem 2.4), we have $E(x) = \langle \widehat{1} | x \widehat{1} \rangle$. Since E is faithful, $\widetilde{Y}_\sigma$ is the unique element, x, in $\mathcal{S}(\Gamma, \mu)$ satisfying $x \cdot \widehat{1} = \widehat{\sigma}$. ∎

Note that the Hilbert module $\mathcal{K} = \mathcal{X}(\Gamma) \otimes_{\mathfrak{C}} \ell^2(V(\Gamma), \mu)$ is a Hilbert space with inner product given by

$$\langle \widehat{\sigma_1} | \widehat{\sigma_2} \rangle_{\mathcal{K}} = \mathrm{Tr}\big(\langle \widehat{\sigma_1} | \widehat{\sigma_2} \rangle \big) = \delta_{\sigma_1, \sigma_2} \mu(t(\sigma_1))$$

for $\sigma_1, \sigma_2 \in \Pi$.

Lemma 3.6 *There is a $\mathcal{S}(\Gamma, \mu)$-linear unitary $U \colon \mathcal{H} \to \mathcal{K}$ given on finite sums by*

$$U\Big(\sum_{i=1}^{n} a_i \widetilde{Y}_\sigma \Big) = \sum_{i=1}^{n} a_i \widehat{\sigma}.$$

Proof It is apparent from the definition of the inner products on $\mathcal{H}$ and $\mathcal{K}$ that U is an isometry with dense range, hence U extends to be a unitary operator from $\mathcal{H}$ onto $\mathcal{K}$. Suppose $\tau \in \Pi$, with $\tau = \epsilon' \tau'$ for $\epsilon' \in \vec{E}$. The formula $X_\epsilon Y_\sigma = Y_{\epsilon\sigma} + \delta_{\epsilon', \epsilon^{\mathrm{op}}} Y_{\tau'}$ appeared in the proof of Proposition 3.5. It follows from this formula that $X_\epsilon \widetilde{Y}_\sigma = a_\epsilon \widetilde{Y}_{\epsilon\sigma} + \delta_{\epsilon', \epsilon^{\mathrm{op}}} a_{\epsilon^{\mathrm{op}}} \widetilde{Y}_{\tau'}$. Since $X_\epsilon \widehat{\sigma} = a_\epsilon \widehat{\epsilon\tau} + \delta_{\epsilon', \epsilon^{\mathrm{op}}} a_{\epsilon^{\mathrm{op}}} \widehat{\tau'}$, U is $\mathcal{S}(\Gamma, \mu)$-linear. ∎

We will canonically identify $\mathcal{H}$ with $\mathcal{K}$ and refer to this Hilbert space as $\mathcal{H}$. We will assume $\mathcal{H}$ has inner product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$, which is linear in the *right* variable. Let $J \colon \mathcal{H} \to \mathcal{H}$ be the modular conjugation, (*i.e.*, the isometric extension of $J(\widehat{x}) = \widehat{x^*}$ for $x \in \mathcal{S}(\Gamma, \mu)$). If we set $\mathcal{M} = \mathcal{M}(\Gamma, \mu)$ (the von-Neumann algebra generated by $\mathcal{S}(\Gamma, \mu)$ acting on $\mathcal{H}$), then $J\mathcal{M}J = \mathcal{M}'$, the commutant of $\mathcal{M}$.

Note that $\mathcal{H}$ is spanned by $\{ \widehat{p_\alpha} \mid \alpha \in V(\Gamma) \}$ and elementary tensors of the form $\epsilon_1 \otimes \cdots \otimes \epsilon_n$ where $e_1 \cdots e_n$ is a path in Γ. Therefore, we view $\mathcal{H}$ as being spanned by paths in $\vec{\Gamma}$ of finite length, with paths length 0 paths simply being vertices. According to the Euclidean structure above, we have

$$\| \epsilon_1 \cdots \epsilon_n \|_{\mathcal{H}} = \sqrt{\mu(t(\epsilon_n))} \quad \text{and} \quad J(\epsilon_1 \cdots \epsilon_n) = \sqrt{\frac{\mu(t(\epsilon_n))}{\mu(s(\epsilon_1))}} \epsilon_n^{\mathrm{op}} \cdots \epsilon_1^{\mathrm{op}}.$$

For $\xi, \eta \in \mathcal{H}$, consider the rank-one operator $|\xi\rangle\langle\eta|$ which is defined by

$$|\xi\rangle\langle\eta|(\zeta) = \xi \langle \eta | \zeta \rangle_{\mathcal{H}}.$$

We have the following lemma.

Lemma 3.7 $[\ell(\epsilon), JX_\epsilon J] = -\frac{1}{\sqrt[4]{\mu(s(\epsilon))^3 \cdot \mu(t(\epsilon))}} |\widehat{s(\epsilon)}\rangle\langle\widehat{t(\epsilon)}|$, *and* $[\ell(\epsilon), JX_{\epsilon'}J] = 0$ *if* $\epsilon \neq \epsilon'$.

Proof It is straightforward to see that $\ell(\epsilon)JX_\epsilon J(\epsilon_1 \cdots \epsilon_n) = JX_\epsilon J\ell(\epsilon)(\epsilon_1 \cdots \epsilon_n)$ if $n \geq 1$ and that $\ell(\epsilon)JX_\epsilon J(\widehat{\alpha}) = 0 = JX_\epsilon J\ell(\epsilon)(\widehat{\alpha})$ if $\alpha \neq t(\epsilon)$. Finally, we see that

$$\begin{aligned}
[\ell(\epsilon), JX_\epsilon J](\widehat{t(\epsilon)}) &= [\ell(\epsilon)JX_\epsilon J - JX_\epsilon J\ell(\epsilon)](\widehat{t(\epsilon)}) = \ell(\epsilon)[a_{\epsilon^{\mathrm{op}}} \epsilon^{\mathrm{op}}] - JX_\epsilon J(\epsilon) \\
&= a_{\epsilon^{\mathrm{op}}} \epsilon\epsilon^{\mathrm{op}} - JX_\epsilon (a_{\epsilon^{\mathrm{op}}}^2 \epsilon^{\mathrm{op}}) \\
&= a_{\epsilon^{\mathrm{op}}} \epsilon\epsilon^{\mathrm{op}} - \big(a_{\epsilon^{\mathrm{op}}} \epsilon\epsilon^{\mathrm{op}} + a_{\epsilon^{\mathrm{op}}}^3 \widehat{s(\epsilon)} \big) = -\Big(\frac{\mu(t(\epsilon))}{\mu(s(\epsilon))} \Big)^{3/4} \widehat{s(\epsilon)},
\end{aligned}$$

and

$$-\frac{1}{\sqrt[4]{\mu(s(\epsilon))^3 \cdot \mu(t(\epsilon))}}|\widehat{s(\epsilon)}\rangle\langle\widehat{t(\epsilon)}| \cdot \widehat{t(\epsilon)} = -\frac{\mu(t(\epsilon))}{\sqrt[4]{\mu(s(\epsilon))^3 \cdot \mu(t(\epsilon))}}\widehat{s(\epsilon)}$$

$$= -\Big(\frac{\mu(t(\epsilon))}{\mu(s(\epsilon))}\Big)^{3/4}\widehat{s(\epsilon)}.$$

The result follows.   ∎

We will now prove a commutation lemma that is central to our theorem. The idea of this proof was communicated to the author by Dima Shlyakhtenko, who has a similar proof for when each X_ϵ is a (scalar-valued) semicircular element.

Lemma 3.8   *For each $\epsilon \in \vec{E}$, let $R_\epsilon \in \mathcal{B}(\mathcal{H})$ be a finite-rank operator with*

$$R_\epsilon \colon Jp_{s(\epsilon)}J\mathcal{H} \longrightarrow Jp_{t(\epsilon)}J\mathcal{H}$$

(i.e., $\ker(R_\epsilon) \supset (Jp_{s(\epsilon)}J\mathcal{H})^\perp$ and $\operatorname{range}(R_\epsilon) \subset Jp_{t(\epsilon)}J\mathcal{H}$). Suppose further that

$$\sum_{\epsilon \in \vec{E}}[R_\epsilon, JX_\epsilon J] = 0.$$

Then $R_\epsilon = 0$ for all $\epsilon \in \vec{E}$.

Proof   Let $a, b \in \mathcal{M}$. Since $J\mathcal{M}J = \mathcal{M}'$, we have

$$0 = \sum_{\epsilon \in \vec{E}} a[R_\epsilon, JX_\epsilon J]b = \sum_{\epsilon \in \vec{E}}[aR_\epsilon b, JX_\epsilon J].$$

Fixing some $\epsilon' \in \vec{E}$, and letting $\operatorname{Tr}_{\mathcal{H}}$ be the trace on the finite rank operators, we have, with the aid of Lemma 3.7:

$$0 = \operatorname{Tr}_{\mathcal{H}}\Big(\sum_{\epsilon \in \vec{E}}\ell(\epsilon')[aR_\epsilon b, JX_\epsilon J]\Big)$$

$$= \sum_{\epsilon \in \vec{E}}\operatorname{Tr}_{\mathcal{H}}\big(\ell(\epsilon')aR_\epsilon bJX_\epsilon J - \ell(\epsilon')JX_\epsilon JaR_\epsilon b\big)$$

$$= \sum_{\epsilon \in \vec{E}}\operatorname{Tr}_{\mathcal{H}}\big(aR_\epsilon b[JX_\epsilon J\ell(\epsilon') - \ell(\epsilon')JX_\epsilon J]\big)$$

$$= \frac{1}{\sqrt[4]{\mu(s(\epsilon'))^3 \cdot \mu(t(\epsilon'))}}\operatorname{Tr}_{\mathcal{H}}(aR_{\epsilon'}b|s(\epsilon')\rangle\langle t(\epsilon')|)$$

$$= \frac{1}{\sqrt[4]{\mu(s(\epsilon'))^3 \cdot \mu(t(\epsilon'))^3}}\langle t(\epsilon')|aR_{\epsilon'}bs(\epsilon')\rangle_{\mathcal{H}}$$

$$= \frac{1}{\sqrt[4]{\mu(s(\epsilon'))^3 \cdot \mu(t(\epsilon'))^3}}\langle a^*t(\epsilon')|R_{\epsilon'}bs(\epsilon')\rangle_{\mathcal{H}}.$$

As this holds for all $a, b \in \mathcal{M}$, this implies $R_{\epsilon'} = 0$ since $R_{\epsilon'} \colon Jp_{s(\epsilon')}J\mathcal{H} \to Jp_{t(\epsilon')}J\mathcal{H}$.

∎

3.2 A Free Differential Calculus for $\mathcal{A}$

Before defining our free difference quotients, we claim that the elements X_ϵ are algebraically free with amalgamation over $C_0(V)$. From Theorem 2.4, the elements $(X_\epsilon)_{e\in E}$ are $*$-free with amalgamation over $C_0(V)$. Also, the spectral measure $X_\epsilon X_{\epsilon^{op}}$ has a nonzero diffuse part. These two facts imply that the elements X_ϵ are algebraically free with amalgamation over $C_0(V)$.

For each edge $\epsilon \in \vec{E}$, we define $\partial_\epsilon : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ to be the derivation that is the unique extension of

$$\partial_\epsilon(X_{\epsilon'}) = (\delta_{\epsilon,\epsilon'}) p_{s(\epsilon)} \otimes p_{t(\epsilon)}.$$

Also see [CDS14]. Notice that this means that if $\epsilon_1 \cdots \epsilon_n$ is a path in $\vec{\Gamma}$, then

$$\partial_\epsilon(X_{\epsilon_1} \cdots X_{\epsilon_n}) = \sum_{\epsilon_j = \epsilon} X_{\epsilon_1} \cdots X_{\epsilon_{j-1}} \otimes X_{\epsilon_{j+1}} \cdots X_{\epsilon_n},$$

with the understanding that if $\epsilon_1 = \epsilon$, then the term $p_{s(\epsilon_1)} \otimes X_{\epsilon_2} \cdots X_{\epsilon_n}$ appears, and if $\epsilon_n = \epsilon$, then the term $X_{\epsilon_1} \cdots X_{\epsilon_{n-1}} \otimes p_{t(\epsilon_n)}$ appears. Although the following lemma will not be needed in what follows, it demonstrates that the differential operators ∂_ϵ have a finite free Fisher information type property [Voi93].

Lemma 3.9 (i) *For any $P \in \mathcal{A}$, we have*

$$(\mathrm{Tr} \otimes \mathrm{Tr})(\partial_\epsilon P) = \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}\,\mathrm{Tr}(X_{\epsilon^{op}}P),$$

i.e., $\sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}X_{\epsilon^{op}}$ is a conjugate variable for ∂_e.
(ii) *If we define $\sigma \in \mathrm{Aut}(\mathcal{A} \otimes \mathcal{A})$ by linear extension of $\sigma(a \otimes b) = b \otimes a$, then for any $P \in \mathcal{A}$,*

$$\big(\partial_\epsilon(P)\big)^* = \sigma\big(\partial_{\epsilon^{op}}(P^*)\big).$$

(iii) *∂_ϵ has a densely defined adjoint, ∂_ϵ^*, which is given on individual tensors by*

$$\partial_\epsilon^*(Q \otimes R) = \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}\,QX_\epsilon R - (\mathrm{id} \otimes \mathrm{Tr})(\partial_{\epsilon^{op}}(Q))R$$
$$- Q(\mathrm{Tr} \otimes \mathrm{id})(\partial_{\epsilon^{op}}(R)).$$

Therefore, the operators ∂_ϵ are closable as operators from $\mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$.

Proof (i) Suppose $P = X_{\epsilon_1} \cdots X_{\epsilon_n}$. From Lemma 2.6, we have

$$\sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}\,\mathrm{Tr}(X_{\epsilon^{op}}P)$$
$$= \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}\,\mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_n} X_{\epsilon^{op}})$$
$$= \sum_{\substack{\epsilon_j = \epsilon \\ j \neq 1,n}} \mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) \cdot \mathrm{Tr}(X_{\epsilon_{j+1}} \cdots X_{\epsilon_n})$$
$$+ \delta_{\epsilon_n,\epsilon}\mu(s(\epsilon^{op}))\mathrm{Tr}(X_{\epsilon_1} \dots X_{\epsilon_{n-1}}) + \delta_{\epsilon_1,\epsilon}\mu(t(\epsilon^{op}))\mathrm{Tr}(X_{\epsilon_2} \dots X_{\epsilon_n})$$
$$= \sum_{\substack{\epsilon_j = \epsilon \\ j \neq 1,n}} \mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{j-1}}) \cdot \mathrm{Tr}(X_{\epsilon_{j+1}} \cdots X_{\epsilon_n})$$
$$+ \delta_{\epsilon_n,\epsilon}\mu(t(\epsilon))\mathrm{Tr}(X_{\epsilon_1} \dots X_{\epsilon_{n-1}}) + \delta_{\epsilon_1,\epsilon}\mu(s(\epsilon))\mathrm{Tr}(X_{\epsilon_2} \dots X_{\epsilon_n})$$

Note that

$$\partial_\epsilon(X_{\epsilon_1}\cdots X_{\epsilon_n}) = \sum_{\substack{\epsilon_j=\epsilon \\ j\notin\{1,n\}}} X_{\epsilon_1}\cdots X_{\epsilon_{j-1}} \otimes X_{\epsilon_{j+1}}\cdots X_{\epsilon_n}$$

$$+ \delta_{\epsilon_n,\epsilon} X_{\epsilon_1}\ldots X_{\epsilon_{n-1}} \otimes p_{t(\epsilon)} + \delta_{\epsilon_1,\epsilon} p_{s(\epsilon)} \otimes X_{\epsilon_2}\ldots X_{\epsilon_n}.$$

Taking $\mathrm{Tr}\otimes\mathrm{Tr}$ verifies the equation for monomials. The rest follows by linearity.

(ii) If P is a monomial, this immediately follows from the definition of σ and ∂_ϵ. The rest follows by linearity.

(iii) Set $\widetilde{X_\epsilon} = \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}X_\epsilon$ and $\widetilde{X}_{\epsilon^{\mathrm{op}}} = \sqrt{\mu(s(\epsilon))\mu(t(\epsilon))}X_{\epsilon^{\mathrm{op}}}$. For $P, Q, R \in \mathcal{A}$, we have

$$\left\langle \overline{Q\otimes R} \mid \overline{\partial_\epsilon(P)} \right\rangle_{\mathcal{H}\otimes\mathcal{H}} = \mathrm{Tr}\otimes\mathrm{Tr}((Q^*\otimes R^*)\partial_\epsilon(P))$$

$$= \mathrm{Tr}\otimes\mathrm{Tr}(Q^*\partial_\epsilon(P)R^*)$$

$$= \mathrm{Tr}\otimes\mathrm{Tr}((Q^*PR^*) - \partial_\epsilon(Q^*)PR^* - Q^*P\partial_\epsilon(R^*))$$

$$= \mathrm{Tr}(\widetilde{X}_{\epsilon^{\mathrm{op}}}Q^*PR^*) - \mathrm{Tr}((\mathrm{Tr}\otimes\mathrm{id})(\partial_\epsilon(Q^*))PR^*) - \mathrm{Tr}(Q^*P(\mathrm{id}\otimes\mathrm{Tr})(\partial_\epsilon(R^*)))$$

$$= \mathrm{Tr}(R^*\widetilde{X}_{\epsilon^{\mathrm{op}}}Q^*P) - \mathrm{Tr}(R^*(\mathrm{Tr}\otimes\mathrm{id})(\partial_\epsilon(Q^*))P) - \mathrm{Tr}((\mathrm{id}\otimes\mathrm{Tr})(\partial_\epsilon(R^*))Q^*P)$$

$$= \mathrm{Tr}((Q\widetilde{X_\epsilon}R)^*P) - \mathrm{Tr}([[(\mathrm{Tr}\otimes\mathrm{id})(\sigma(\partial_{\epsilon^{\mathrm{op}}}(Q))^*)^*R]^*P)$$

$$\quad - \mathrm{Tr}([Q(\mathrm{id}\otimes\mathrm{Tr})(\sigma(\partial_{\epsilon^{\mathrm{op}}}(R))^*)^*]^*P)$$

$$= \mathrm{Tr}((Q\widetilde{X_\epsilon}R)^*P) - \mathrm{Tr}([[(\mathrm{id}\otimes\mathrm{Tr})(\partial_{\epsilon^{\mathrm{op}}}(Q))R]^*P)$$

$$\quad - \mathrm{Tr}([Q(\mathrm{Tr}\otimes\mathrm{id})(\partial_{\epsilon^{\mathrm{op}}}(R))]^*P)$$

$$= \left\langle \overline{Q\widetilde{X_\epsilon}R - (\mathrm{id}\otimes\mathrm{Tr})(\partial_{\epsilon^{\mathrm{op}}}(Q))\widehat{R} - Q(\mathrm{Tr}\otimes\mathrm{id})(\overline{\partial_{\epsilon^{\mathrm{op}}}(R)})} \mid \widehat{P} \right\rangle_{\mathcal{H}}.$$

$\blacksquare$

Note that $\mathcal{M}\otimes\mathcal{M}$ has a canonical $\mathcal{M}-\mathcal{M}$ bimodule structure given by $a(x_1\otimes x_2)b = ax_1\otimes x_2 b$. Moreover, we can realize $\mathcal{M}\otimes_{\mathrm{alg}}\mathcal{M}$ as a subalgebra of the finite rank operators on $\mathcal{H}$ by $x\otimes y \mapsto |\widehat{x}\rangle\langle\widehat{y^*}|$. These two actions are compatible, *i.e.*, under the identification of $\mathcal{M}\otimes_{\mathrm{alg}}\mathcal{M}$ with finite rank operators on $\mathcal{H}$, we have

$$a(x\otimes y)b \mapsto |\widehat{ax}\rangle\langle\widehat{b^*y^*}| = a|\widehat{x}\rangle\langle\widehat{y^*}|b$$

where the last product is the product in $\mathcal{B}(\mathcal{H})$.

We will occasionally realize the $\mathcal{M}-\mathcal{M}$ bimodule action on $\mathcal{M}\otimes\mathcal{M}$ as an $\mathcal{M}\otimes\mathcal{M}$ action on $\mathcal{M}\otimes\mathcal{M}$ via the linear extension of $(a\otimes b)\#(x\otimes y) = ax\otimes yb$. We now prove the key inductive lemma which is along the lines of [MSW14].

Theorem 3.10 *Suppose $Q\in\mathcal{A}$ and there are $\alpha,\beta\in V$ with $p_\alpha Q = Q = Qp_\beta$. Suppose further that there are a and b in $\mathcal{M}$ with $aQ = 0 = Qb$. Then for each $\epsilon\in\vec{E}$, $a\partial_\epsilon(Q)b = 0$.*

Proof The assumptions in the problem imply that

$$a(Q\otimes p_\beta - p_\alpha\otimes Q)b = 0.$$

The identity

$$Q \otimes p_\beta - p_\alpha \otimes Q = \sum_{\epsilon \in \vec{E}} \partial_\epsilon(Q) \# (X_\epsilon \otimes 1 - 1 \otimes X_\epsilon)$$

can be observed by noting that if Q is a monomial $X_{\epsilon_1} \cdots X_{\epsilon_n}$ with $s(\epsilon_1) = \alpha$ and $t(\epsilon_n) = \beta$, then

$$\sum_{\epsilon \in \vec{E}} \partial_\epsilon(Q) \# (X_\epsilon \otimes 1) = Q \otimes p_\beta + \sum_{k=1}^{n} X_{\epsilon_1} \cdots X_{\epsilon_k} \otimes X_{\epsilon_{k+1}} \cdots X_{\epsilon_n} \text{ and}$$

$$\sum_{\epsilon \in \vec{E}} \partial_\epsilon(Q) \# (1 \otimes X_\epsilon) = p_\alpha \otimes Q + \sum_{k=1}^{n} X_{\epsilon_1} \cdots X_{\epsilon_k} \otimes X_{\epsilon_{k+1}} \cdots X_{\epsilon_n}$$

We therefore have

$$\sum_{e \in \vec{E}} (a \otimes b) \# \partial_\epsilon(Q) \# (X_\epsilon \otimes 1 - 1 \otimes X_\epsilon) = 0, \text{ which implies}$$

$$\sum_{e \in \vec{E}} [a \partial_\epsilon(Q) b, J X_{\epsilon^{\mathrm{op}}} J] = 0,$$

where in the last equation $a\partial_\epsilon(Q)b$ is viewed as a finite rank operator on $\mathcal{H}$. Note that under this identification, $a\partial_\epsilon(Q)b \colon Jp_{s(\epsilon^{\mathrm{op}})}J\mathcal{H} \to Jp_{t(\epsilon^{\mathrm{op}})}J\mathcal{H}$, so applying Lemma 3.8 (for ϵ^{op}) to $a\partial_\epsilon(Q)b$ gives $a\partial_\epsilon(Q)b = 0$ as desired. ∎

We now prove Theorem 3.1. The proof will be an amalgamated version of the proof of the corresponding theorem in [MSW14].

Proof of Theorem 3.1 Suppose p_α is in the statement of Theorem 3.1 with $Q \in \mathcal{A}$ nonzero and $Qp_\alpha = Q$. Suppose further that $Qa = 0$ for $a = a^*$ and $p_\alpha a p_\alpha = a$. Define the degree of Q to be the length of the longest monomial in the X_ϵ's with nonzero coefficient in the expansion of Q into monomials. If Q is a linear combination of the elements $\{p_\gamma \mid \gamma \in V\}$, then Q is said to have degree 0. We will prove the result by induction on the degree of Q. The result is trivial for Q of degree 0, so we will now assume that the degree of P is at least 1.

By expanding $Q = \sum_{\beta \in V} p_\beta Q$, we see that $Qa = 0$ if and only if $p_\beta Qa = 0$ for all β, so we assume that $Q = p_\beta Q$ for some $\beta \in V$. Let $\widetilde{q}$ be the projection onto the kernel of Q. Since $Qp_\alpha = Q$, $\mathrm{Tr}(\widetilde{q}) \geq \sum_{\gamma \in V \setminus \{\alpha\}} \mu(\gamma)$. Assume that $a \neq 0$. Then since a is supported under p_α, we have $\mathrm{Tr}(\widetilde{q}) > \sum_{\gamma \in V \setminus \{\alpha\}} \mu(\gamma)$. Let q be the projection onto the kernel of Q^* and note that $\mathrm{Tr}(q) = \mathrm{Tr}(\widetilde{q})$. Therefore, using minimality of $\mu(\alpha)$,

$$\mathrm{Tr}(\widetilde{q}) + \mathrm{Tr}(p_{\beta'}) > \Big(\sum_{\gamma \in V \setminus \{\alpha\}} \mu(\gamma) \Big) + \mu(\beta') \geq \Big(\sum_{\gamma \in V \setminus \{\alpha\}} \mu(\gamma) \Big) + \mu(\alpha) = \mathrm{Tr}(1).$$

It follows that $qp_{\beta'} \neq 0$ for all $\gamma \in V$. In particular, $qp_\beta \neq 0$.

Note that $qQ = 0$. By Theorem 3.10, $q\partial_\epsilon(Q)a = 0$. This implies that

$$0 = (\mathrm{Tr} \otimes \mathrm{id})(q\partial_\epsilon(Q)a) = (\mathrm{Tr} \otimes \mathrm{id})(q\partial_\epsilon(Q))a = 0$$

for all ϵ. Choose ϵ so that X_ϵ is the leftmost term in at least one monomial in the sum representing Q. We have $\mathrm{Tr}(qp_\beta) \neq 0$, so it follows that $(\mathrm{Tr} \otimes \mathrm{id})(q'\partial_\epsilon(Q)) \in \mathcal{A}$ is nonzero, has right support under p_α, and has degree strictly less than that of Q. Since $a \neq 0$, this contradicts the assumption that for any nonzero Q of strictly smaller

degree than Q with right support under p_α, $Qa = 0$ implies $a = 0$. Therefore, a had to already be zero in the previous paragraph. ■

3.3 Algebraicity

We will now turn toward proving Theorem 3.2. The majority of the discussion below comes from and is inspired by [SS15]. To begin, we need to set up some notation.

Definition 3.11   (i) If R is a ring, we define the ring of formal power series in the variables $z_1, \ldots, z_n$ (denoted as $R[[z_1, \ldots, z_n]]$) to be the set of formal sums of the form

$$P = \sum_{k_1, \ldots k_n \geq 0} P_{k_1, \ldots, k_n} z_1^{k_1} \cdots z_n^{k_n},$$

where $P_{k_1, \ldots, k_n} \in \mathbb{R}$. Addition of two power series is defined term-wise. Multiplication is defined by

$$(PQ)_{k_1, \ldots, k_n} = \sum_{j=1}^{n} \sum_{\ell_j = 0}^{k_j} P_{\ell_1, \ldots, \ell_n} Q_{k_1 - \ell_1, \ldots, k_n - \ell_n}.$$

We will let $R[z_1, \ldots, z_n]$ denote the polynomials in the variables $z_1, \ldots, z_n$. Note that $P \in R[z_1, \cdots, z_n]$ if and only if $P_{k_1, \ldots, k_n} = 0$ for all but finitely many $(k_1, \ldots, k_n)$.

     (ii) If R is an integral domain, we say that $P \in R[[z_1, \ldots, z_n]]$ is *algebraic* if there exist $Q_0, Q_1, \ldots, Q_n \in R[z_1, \ldots, z_n]$ not all zero satisfying

$$\sum_{j=0}^{n} Q_j P^j = 0.$$

The algebraic elements in $R[[z_1, \ldots, z_n]]$ form a ring. This ring will be denoted as $R_{\mathrm{alg}}[[z_1, \ldots, z_n]]$.

Given Neumann algebra, $\mathcal{N}$, a faithful positive linear functional ϕ on $\mathcal{N}$, and a self-adjoint $a \in \mathcal{N}$, there exists a unique positive measure μ_a, supported on the spectrum of a, which satisfies

$$\phi(a^n) = \int_{\mathbb{R}} x^n d\mu_a(x).$$

Recall that the *Cauchy Transform* of a measure, μ, is defined as

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}.$$

If μ is compactly supported, it is straightforward to see that $G_\mu(z)$ has a Laurent expansion about $z = 0$, and that $\lim_{z \to \infty} G_\mu(z) = 0$. Therefore, the Laurent series for $G_\mu(z)$ is an element of $\mathbb{C}[[\frac{1}{z}]]$, so it makes sense to ask if G_{μ_a} is algebraic for any $a \in \mathcal{A}$.

In order to answer this question, we will need a notion of rational power series.

Definition 3.12   (i) Let R be a unital subring of the unital ring, S. We say that R is rationally closed if whenever A is an $n \times n$ matrix with entries in R and invertible in $M_n(S)$, then $A^{-1} \in M_n(R)$. The rational closure of R is the smallest rationally closed subring of S containing R.

(ii) A power series $P \in R[[z_1, \ldots, z_n]]$ is rational if P is in the rational closure of $R[z_1, \ldots, z_n]$, viewed as a subring of $R[[z_1, \ldots, z_n]]$. The set of rational power series will be denoted as $R_{\mathrm{rat}}[[z_1, \ldots, z_n]]$

The following lemma from [SS15] will be helpful.

Lemma 3.13 ([SS15])   *Suppose that $\mathcal{N}$ is a von Neumann algebra with faithful trace τ. Given $x_1, x_2, \ldots, x_n \in \mathcal{N}$, and $\sum_{n=0}^{\infty} x_n z^n$, we define*

$$\mathrm{Tr}_{\mathcal{N}}\Big(\sum_{n=0}^{\infty} x_n z^n \Big) = \sum_{n=0}^{\infty} \tau(x_n) z^n \subset \mathbb{C}[[z]].$$

Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{N}$ and that

$$\mathrm{Tr}_{\mathcal{N}}\big(\mathcal{A}_{\mathrm{rat}}[[z]] \big) \in \mathbb{C}_{\mathrm{alg}}[[z]];$$

then for every self-adjoint matrix $A \in M_n(\mathcal{A})$, G_{μ_A} is algebraic.

As above, we assume that Γ is a finite graph, and we fix a weighting μ on $V(\Gamma)$. Let $\mathcal{M} = \mathcal{M}(\Gamma, \mu)$ and $\mathcal{A} = \mathbb{C}\langle \mathcal{C}, (X_\epsilon)_{\epsilon \in \vec{E}} \rangle$. In order to verify the hypotheses in Lemma 3.13 we need to make use of power series in non-commuting variables.

Definition 3.14   (i) Let R be a ring and let $X = \{x_1, \ldots, x_n\}$ be a finite set, often called an *alphabet*. A *word* in X is a finite string $x_{i_1}, \ldots, x_{i_k}$. The set of all words in X will be denoted as $W(X)$, and the empty word will be denoted **1**.

(ii) The non-commutative power series ring, $R\langle\!\langle X \rangle\!\rangle$ consists of formal sums of the form

$$P = \sum_{w \in W(X)} P_w w$$

for $P_w \in R$. Addition of elements in $R\langle\!\langle X \rangle\!\rangle$ is done coordinate wise. Multiplication of elements P and Q in $R\langle\!\langle X \rangle\!\rangle$ is defined as follows:

$$(PQ)_w = \sum_{\substack{u,v \in W(X) \\ w = uv}} P_u Q_v$$

The non-commutative polynomials in X, denoted by $R\langle X \rangle$, is the subring of $R\langle\!\langle X \rangle\!\rangle$ consisting of elements of the form

$$P = \sum_{w \in W(X)} P_w w$$

where $P_w = 0$ for all but finitely many $w \in W(X)$.

(iii) If $Z = \{z_1, \ldots, z_m\}$ is an alphabet disjoint from X, a proper algebraic system over R is a set of equations

$$z_i = p_i(x_1, \ldots, x_n, z_1, \ldots, z_m),$$

where $p_i \in R\langle X \cup Z \rangle$ has no constant term, nor any term of the form αz_j for $\alpha \in R$ and $j \in \{1, \ldots, m\}$. A *solution* to a proper algebraic system is $(P_1, \ldots, P_m) \in R\langle\!\langle X \rangle\!\rangle^m$ with $(P_i)_1 = 0$, satisfying

$$P_i = p_i(x_1, \ldots, x_n, P_1, \ldots, P_m)$$

for each each $i \in \{1, \ldots, m\}$.

(iv) We say that $P \in R\langle\!\langle X \rangle\!\rangle$ is *algebraic* if $P - P_1 \mathbf{1}$ is a component of a solution of a proper algebraic system. The set of the algebraic elements in $R\langle\!\langle X \rangle\!\rangle$ will be denoted by $R_{\text{alg}}\langle\!\langle X \rangle\!\rangle$. $R_{\text{alg}}\langle\!\langle X \rangle\!\rangle$ is a subring of $R\langle\!\langle X \rangle\!\rangle$.

(v) We say that $P \in R\langle\!\langle X \rangle\!\rangle$ is *rational* if P is in the rational closure of $R\langle X \rangle$ in $R\langle\!\langle X \rangle\!\rangle$. The set of rational elements in $R\langle\!\langle X \rangle\!\rangle$ will be denoted as $R_{\text{alg}}\langle\!\langle X \rangle\!\rangle$

Fix an alphabet $\mathscr{X} = \left\{ \epsilon \mid \epsilon \in \vec{E} \right\} \cup \left\{ \alpha \mid \alpha \in V \right\}$ and let

$$\mathcal{X} = \left\{ X_\epsilon \mid \epsilon \in \vec{E} \right\} \cup \left\{ p_\alpha \mid \alpha \in V \right\}$$

so that $\mathcal{A} = \mathbb{C}\langle \mathcal{X} \rangle$. If $w = \epsilon_1 \ldots \epsilon_n$, then set $w(\mathcal{X}) = X_{\epsilon_1} \cdots X_{\epsilon_n}$. For $\alpha \in V$, set $P^\alpha \in \mathbb{C}\langle\!\langle \mathscr{X} \rangle\!\rangle$ to be the following element:

$$P^\alpha = \sum_{w \in W(\mathscr{X})} P^\alpha_w w, \text{ where}$$

$$\begin{cases} P^\alpha_{\mathbf{1}} = 0, \\ P^\alpha_\gamma = \delta_{\gamma,\alpha}\mu(\alpha) & \text{if } \gamma \in V, \\ P^\alpha_w = 0 & \text{if } p_\alpha w(\mathcal{X}) p_\alpha = 0, \\ P^\alpha_w = 0 & \text{if } |w| \geq 2 \text{ and } w \text{ contains } \gamma \in V, \\ P^\alpha_w = \text{Tr}(w(\mathcal{X})) & \text{if } p_\alpha w(\mathcal{X}) p_\alpha = w(\mathcal{X}) \neq 0 \text{ and } w = \epsilon_1 \ldots \epsilon_n. \end{cases}$$

Let $L(\alpha)$ denote the loops in $\vec{\Gamma}$ that are based at α. A key point in our analysis is the following lemma.

Lemma 3.15 $P^\alpha - \mu(\alpha)\alpha$ *is algebraic for each* $\alpha \in V$.

Proof We note that

$$P^\alpha - \mu(\alpha)\alpha = \sum_{\epsilon_1 \ldots \epsilon_n \in L(\alpha)} \text{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_n})\epsilon_1 \cdots \epsilon_n.$$

Using Lemma 2.6, this series can be rewritten as:

$$\sum_{\substack{t_{\epsilon_n} = \alpha \\ \epsilon_1 \cdots \epsilon_{k-1} \in L(\alpha) \\ \epsilon_{k+1} \cdots \epsilon_{n-1} \in L(s(\epsilon_{n-1}))}} \frac{1}{\sqrt{\mu(s(\epsilon_n))\mu(\alpha)}} \left[\text{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{k-1}})\epsilon_1 \cdots \epsilon_{k-1} \right] \epsilon_n^{\text{op}}$$

$$\times \left[\text{Tr}(X_{\epsilon_{k+1}} \cdots X_{\epsilon_{n-1}})\epsilon_{k+1} \cdots \epsilon_{n-1} \right] \epsilon_n$$

$$+ \sum_{\epsilon_1 \ldots \epsilon_{n-2} \in L(\alpha)} \sqrt{\frac{\mu(\alpha)}{\mu(s(\epsilon_n))}} \left[\text{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_{n-2}})\epsilon_1 \cdots \epsilon_{n-2} \right] \epsilon_n^{\text{op}} \epsilon_n$$

$$+ \sum_{\epsilon_2 \cdots \epsilon_{n-1} \in L(s(\epsilon_n))} \sqrt{\frac{\mu(s(\epsilon_n))}{\mu(\alpha)}} \epsilon_n^{\text{op}} \left[\text{Tr}(X_{\epsilon_2} \cdots X_{\epsilon_{n-1}})\epsilon_2 \cdots \epsilon_{n-1} \right] \epsilon_n$$

$$+ \sqrt{\mu(\alpha)\mu(s(\epsilon))} \epsilon_n^{\text{op}} \epsilon_n$$

This means that $(P^\alpha - \mu(\alpha)\alpha)_{\alpha \in V}$ is a solution to the proper algebraic system

$$z_\alpha = \sum_{\beta \sim \alpha} \sum_{\substack{t(\epsilon)=\alpha \\ s(\epsilon)=\beta}} \sqrt{\mu(\alpha) \cdot \mu(\beta)} \epsilon^{\mathrm{op}} \epsilon + \sqrt{\frac{\mu(\alpha)}{\mu(\beta)}} z_\alpha \epsilon^{\mathrm{op}} \epsilon$$

$$+ \frac{1}{\sqrt{\mu(\beta)\mu(\alpha)}} z_\alpha \epsilon^{\mathrm{op}} z_\beta \epsilon + \sqrt{\frac{\mu(\beta)}{\mu(\alpha)}} \epsilon^{\mathrm{op}} z_\beta \epsilon,$$

where α ranges through all of V. ∎

Corollary 3.16 P^α *is algebraic for all* $\alpha \in V$.

We will now let $P \in \mathcal{A}_{\mathrm{rat}}[[z]]$, and we set $P^\Gamma = \sum_{\alpha \in V} P^\alpha$ and note that $P^\Gamma \in \mathbb{C}_{\mathrm{alg}}\langle\!\langle X \rangle\!\rangle$. There is a homomorphism $\pi : \mathbb{C}\langle \mathcal{X} \rangle \to \mathcal{A}$ that is uniquely determined by $\pi(\epsilon) = X_\epsilon$ for all $\epsilon \in \vec{E}$ and $\pi(\alpha) = p_\alpha$ for all $\alpha \in V$. This map extends to a map (also denoted π) from $\mathbb{C}\langle \mathcal{X} \rangle[[z]] \to \mathcal{A}[[z]]$. We can find $\overline{P} \in \mathbb{C}\langle \mathcal{X} \rangle_{\mathrm{rat}}[[z]]$ such that $\pi(\overline{P}) = P$.

In [Sau03] there is a canonical way to realize $\mathbb{C}\langle \mathcal{X} \rangle_{\mathrm{rat}}[[z]] \subset \mathbb{C}(z)_{\mathrm{rat}}\langle\!\langle \mathcal{X} \rangle\!\rangle$, where $\mathbb{C}(z)$ is the quotient field of $\mathbb{C}[z]$. Note that we can realize $P^\Gamma \in \mathbb{C}(z)_{\mathrm{alg}}\langle\!\langle \mathcal{X} \rangle\!\rangle$. If we consider the Hadamard product

$$\overline{P} \odot P^\Gamma = \sum_{w \in X} \overline{P}_w \cdot P^\Gamma_w \cdot w,$$

then $\overline{P} \odot P^\Gamma \in \mathbb{C}(z)_{\mathrm{alg}}\langle\!\langle X \rangle\!\rangle$ by [Sch62].

Observe that if we write $P = \sum_{m=0}^\infty [p_{m,\vec{E}}((X_\epsilon)_{\epsilon \in \vec{E}}) + p_{m,V}((p_\alpha)_{\alpha \in V})] z^m$ for polynomials $p_{m,\vec{E}}$ and $p_{m,V}$, then

$$\overline{P} = \sum_{m=0}^\infty \Big[p_{m,\vec{E}}((\epsilon)_{\epsilon \in \vec{E}}) + p_{m,V}((\alpha)_{\alpha \in V}) + q_m((\epsilon)_{\epsilon \in \vec{E}}, (\alpha)_{\alpha \in V}) \Big] z^m,$$

where $q_m((X_\epsilon)_{\epsilon \in \vec{E}}, (p_\alpha)_{\alpha \in V}) = 0$. If we let $\mathrm{coeff}(p, w)$ be the coefficient of w in the expansion of p into monomials, we see that

$$\overline{P} \odot P^\Gamma = \sum_{\alpha \in V} \mu(\alpha) \Big(\sum_{m=0}^\infty [\mathrm{coeff}(p_{m,V}, \alpha) + \mathrm{coeff}(q_m, \alpha)] z^m \Big) \alpha$$

$$+ \sum_{\alpha \in V} \sum_{\epsilon_1 \ldots \epsilon_n \in L(\alpha)} \mathrm{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_n})$$

$$\times \Big(\sum_{m=0}^\infty [\mathrm{coeff}(p_{m,\vec{E}}, \epsilon_1 \cdots \epsilon_n) + \mathrm{coeff}(q_m, \epsilon_1 \cdots \epsilon_n)] z^m \Big) \epsilon_1 \ldots \epsilon_n.$$

If one inserts 1 for each ϵ and α then we get

$$\sum_{\alpha \in V} \mu(\alpha) \Big(\sum_{m=0}^{\infty} [\operatorname{coeff}(p_{m,V}, \alpha) + \operatorname{coeff}(q_m, \alpha)] z^m \Big)$$

$$+ \sum_{\alpha \in V} \sum_{\epsilon_1 \ldots \epsilon_n \in L(\alpha)} \operatorname{Tr}(X_{\epsilon_1} \cdots X_{\epsilon_n})$$

$$\times \Big(\sum_{m=0}^{\infty} [\operatorname{coeff}(p_{m,\vec{E}}, \epsilon_1 \cdots \epsilon_n) + \operatorname{coeff}(q_m, \epsilon_1 \cdots \epsilon_n)] z^m \Big)$$

$$= \sum_{m=0}^{\infty} \operatorname{Tr}(p_{m,\vec{E}}((X_\epsilon)_{\epsilon \in \vec{E}}) + p_{m,V}((p_\alpha)_{\alpha \in V}) + q_m((X_\epsilon)_{\epsilon \in \vec{E}}, (p_\alpha)_{\alpha \in V})) z^m$$

$$= \sum_{m=0}^{\infty} \operatorname{Tr}(p_{m,\vec{E}}((X_\epsilon)_{\epsilon \in \vec{E}}) + p_{m,V}((p_\alpha)_{\alpha \in V}))) z^m$$

$$= \operatorname{Tr}_{\mathcal{M}}(P)$$

We see that $\operatorname{Tr}_{\mathcal{M}}(P)$ is algebraic from [Sau03]. This discussion proves Theorem 3.2.

Due to Corollary 3.3, if $Q \in \mathcal{A}$ is self-adjoint with $p_\alpha Q p_\alpha = Q$ and Q not a scalar multiple of p_α, then the law, μ_Q of Q in $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha$ has no atoms. Since the Cauchy transform of Q is algebraic, it follows that μ_Q is absolutely continuous with respect to Lebesgue measure and that the support of μ_Q is a finite union of closed intervals. Finally, since $K_0(\mathcal{S}(\Gamma, \mu))$ is generated by the classes $[p_\beta]$, we have the following corollary.

Corollary 3.17 *Let $Q \in \mathcal{A}$ be self-adjoint with $p_\alpha Q p_\alpha = Q$ and Q not a scalar multiple of p_α. If $\mu(\alpha) = \min\{\mu(\beta) \mid \beta \in V\}$, then the law of Q in $(p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha, \operatorname{Tr})$ is absolutely continuous with respect to Lebesgue measure. Moreover, the spectrum of Q in $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha$ is a finite union of disjoint closed intervals, each of which has measure in $(0, \mu(\alpha)] \cap \mathbb{Z}[\{\mu(\beta) \mid \beta \in V\}]$.*

Since the Cauchy transform G_{μ_P} is algebraic for any self-adjoint $P \in M_n(\mathcal{A})$, we can deduce results stating that the law of any positive $P \in M_n(\mathcal{A})$ cannot have a significant portion of mass near 0 (provided μ_P has no atom at 0). The approach is exactly the same as [SS15, Theorem 5.17]. Given a finite measure, μ on $\mathbb{R}$, the spectral density function F_μ is defined by $F_\mu(t) = \mu((-\infty, t])$.

Theorem 3.18 *Let $P \in M_n(\mathcal{A})$ be positive, and let $\alpha \in V$ satisfy*

$$\mu(\alpha) = \min\{\mu(\beta) \mid \beta \in V\}.$$

Then the following hold:

(i)
$$\lim_{\delta \to 0^+} \int_\delta^{\|P\|} \frac{1}{t} (F_{\mu_P}(t) - F_{\mu_P}(0)) < \infty,$$

(ii)
$$\lim_{\delta \to 0^+} \int_\delta^{\|P\|} \log(t) d\mu_P(t) > -\infty.$$

In particular, if $P \in \mathcal{A}$, and $p_\alpha P p_\alpha = P$, and μ_P' is a the law of P in $p_\alpha \mathcal{S}(\Gamma, \mu) p_\alpha$, then

$$\int_0^{\|P\|} \log(t) d\mu_P'(t) > -\infty.$$

4   Applications

4.1   An Application to Wishart Matrices

Recall that a Wishart matrix A is an $M \times N$ random matrix with independent complex Gaussian entries a_{ij}. Moreover, the entries have the following covariances

$$E(a_{ij}\overline{a_{kl}}) = \frac{1}{\sqrt{MN}}\delta_{ik}\delta_{jl} \quad \text{and} \quad E(a_{ij}a_{kl}) = 0.$$

Notation 4.1   For each integer, n, fix positive integers $M_1(n), \dots, M_k(n)$ with $M_1(n) = n$ for all n and $M_i(n) \geq n$ for each $i \in \{1, \dots, k\}$. Assume further that $\lim_{n\to\infty} M_i(n)/M_1(n) = \gamma_i$. Let $\{A_{n_{ij}} \mid 1 \leq i, j \leq k\}$ be a family of $M_i(n) \times M_j(n)$ Wishart matrices and assume further that the entries of $A_{n_{ij}}$ are independent from the entries of $A_{n_{kl}}$ provided that $i \neq k$ or $j \neq l$.

We are ready to state an application of our work to this family of Wishart matrices.

Theorem 4.2   *Let Q be a noncommutative, nonconstant polynomial in the variables $(X_{ij})_{1 \leq i,j \leq n} \cup (X_{ij}^*)_{1 \leq i,j \leq n}$. Assume that Q makes sense as a random matrix, Q_n, when $A_{n_{ij}}$ and $A_{n_{ij}}^*$ are inserted for X_{ij} and X_{ij}^*, respectively. Assume further that $Q_n = Q_n^*$ and is of size $n \times n$.*

(i)   *There is a unique compactly supported probability measure μ_Q so that if tr_n is the (normalized) trace on the $n \times n$ complex matrices, then for each $m \in \mathbb{N}$,*

$$\mathrm{tr}_n \otimes E(Q_n^m) \to \int_{\mathbb{R}} x^m \, d\mu_Q(x) \quad \text{as } n \to \infty.$$

(ii)   *The measure μ_Q has no atoms. Moreover, the support of μ_Q is a finite union of intervals where each interval has measure in the set $\mathbb{Z}[\{\gamma_i \mid 1 \leq i \leq k\}] \cap (0,1]$. In particular, the eigenvalues of Q do not cluster around any point.*

Let (Y, μ) be a probability space, set $M(n) = \sum_{m=1}^k M_m(n)$, and let

$$\mathcal{B}_n = M_{M(n)}(\mathbb{C}) \otimes L^\infty(Y, \mu)$$

be the algebra of $M(n) \times M(n)$ complex random matrices. If Tr_n is the (non-normalized) trace on the algebra of $M(n) \times M(n)$ matrices (satisfying $\mathrm{Tr}(I_{M(n)}) = M(n)$), then we define $\phi_n \colon \mathcal{B}_n \to \mathbb{C}$ by

$$\phi_n(A) = \frac{1}{n}(\mathrm{Tr}_n \otimes E)(A).$$

We will view elements in $\mathcal{B}_n$ as $k \times k$ block matrices where the ij block is of size $M_i(n) \times M_j(n)$. If we are given the Wishart matrices $A_{n_{ij}}$ as above, then we let $\widetilde{A}_{n_{ij}} \in \mathcal{B}_n$ be the matrix whose ij block is $A_{n_{ij}}$ and whose other blocks are all zero. Notice that if P is as in the statement of Theorem 4.2 so that $P((A_{n_{ij}})_{1 \leq i,j, \leq k}, (A_{n_{ij}}^*)_{1 \leq i,j, \leq k})$ is an $n \times n$ random matrix, then

$$P\big((\widetilde{A}_{n_{ij}})_{1 \leq i,j, \leq k}, (\widetilde{A}_{n_{ij}}^*)_{1 \leq i,j, \leq k}\big) \in \mathcal{B}_n$$

has its $1,1$ block equal to $P((A_{n_{ij}})_{1 \leq i,j, \leq k}, (A_{n_{ij}}^*)_{1 \leq i,j, \leq k})$ and all other blocks zero.

To be able to prove Theorem 4.2, we let (Γ, V, E, μ) be the complete graph on k vertices with two unoriented edges, e_{ij} and e_{ji} having the distinct vertices i and j as endpoints. Let $\mu\colon V \to \mathbb{R}_+$ given by $\mu(n) = \gamma_n$, and let $\vec{\Gamma}$ be the directed version of Γ. The following theorem was proved in [GJS10].

Theorem 4.3 ([GJS10, BG05]) *The elements $\widetilde{A}_{n_{ij}}$ converge in $*$-distribution, under ϕ_n, to the elements $X_{\epsilon_{ij}}$. More specifically, if Q is any non-commutative polynomial in the variables $(X_{ij})_{1\le i,j\le k}$ and $(X_{ij}^*)_{1\le i,j\le k}$, then*

$$\lim_{n\to\infty} \phi_n\Big(Q\big((\widetilde{A}_{n_{ij}})_{1\le i,j,\le k}, (\widetilde{A}_{n_{ij}}^*)_{1\le i,j,\le k} \big) \Big) =$$
$$\mathrm{Tr}_{\mathcal{S}(\Gamma,\mu)}\Big(Q\big((X_{\epsilon_{ij}})_{1\le i,j,\le k}, (X_{\epsilon_{ij}^{\mathrm{op}}})_{1\le i,j,\le k} \big) \Big).$$

Using this theorem, we can prove Theorem 4.2

Proof of Theorem 4.2 In the algebra $\mathcal{B}_n$, let $P_{1,n}$ be the matrix whose $1,1$ block is the identity and all other blocks are zero and Q is in the statement of Theorem 4.2. In $\mathcal{B}_n$, if $\widetilde{Q}_n = Q((\widetilde{A}_{n_{ij}})_{1\le i,j,\le k}, (\widetilde{A}_{n_{ij}}^*)_{1\le i,j,\le k})$, then $P_{1,n}\widetilde{Q}_n P_{1,n} = \widetilde{Q}_n$, and the $1,1$ block of this random matrix is $Q_n = Q((A_{n_{ij}})_{1\le i,j,\le k}, (A_{n_{ij}}^*)_{1\le i,j,\le k})$. By Theorem 4.3,

$$\mathrm{tr}(Q_n^m) = \int_{\mathbb{R}} x^m d\mu_Q(x)$$

with μ_Q compactly supported. Since the law of $Q((X_{\epsilon_{ij}})_{1\le i,j,\le k}, (X_{\epsilon_{ij}^{\mathrm{op}}})_{1\le i,j,\le k}))$ has no atoms in $p_1\mathcal{S}(\Gamma,\mu)p_1$, it follows that μ_Q has no atoms. Moreover, the absolute continuity of μ_Q and the rest of Theorem 4.2(ii) follows from Corollary 3.17. ∎

4.2 An Application to Planar Algebras

Let $\mathcal{P}_\bullet$ be a (sub)factor planar algebra. For the definition and basic properties of (sub)factor planar algebras, see [Pet10, BHP12]. We briefly recall the construction of [GJS10]. Let $\mathrm{Gr}_0(\mathcal{P}_\bullet) = \bigoplus_{n\ge 0} \mathcal{P}_n$. We endow $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ with the following multiplication

$$x \wedge y = \boxed{x}^{n} \quad \boxed{y}^{m}$$

for $x \in \mathcal{P}_n$, and $y \in \mathcal{P}_m$. $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ is endowed with the Voiculescu trace

$$\mathrm{tr}(x) = \boxed{x}_n^{\mathcal{TL}},$$

where $\mathcal{TL}$ represents the sum of all Temperley–Lieb diagrams with all strings having endpoints at the bottom of the box. We have the following facts about $\mathrm{Gr}_0(\mathcal{P})$.

Theorem 4.4 (i) tr *is positive definite on $\mathrm{Gr}_0(\mathcal{P}_\bullet)$, and the action of $x \in \mathrm{Gr}_0(\mathcal{P}_\bullet)$ on $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ by left multiplication extends to an action on $L^2(\mathcal{P}_\bullet, \mathrm{tr})$ by bounded operators* [GJS10].

(ii) *Let $\mathcal{M}$ be the von-Neumann algebra generated by the action of $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ on $L^2(\mathrm{Gr}_0(\mathcal{P}_\bullet), \mathrm{tr})$. Then $\mathcal{M}$ is a II_1 factor [GJS10]. If $\mathcal{P}_\bullet$ is finite depth, then $\mathcal{M} \cong L(\mathbb{F}_{1+2(\delta-1)I})$ where I is the global index of $\mathcal{P}_\bullet$ and δ is the loop parameter [GJS11]. If $\mathcal{P}_\bullet$ is infinite-depth, then $\mathcal{M} \cong L(\mathbb{F}_\infty)$ [Har13].*

(iii) *Let $\mathcal{B}$ be the C^*-algebra generated by the action of $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ on $L^2(\mathrm{Gr}_0(\mathcal{P}_\bullet), \mathrm{tr})$. If Γ is the principal graph of $\mathcal{P}$, then $K_0(\mathcal{B}) \cong \mathbb{Z}\{[\alpha] \mid \alpha \in V(\Gamma)\}$, moreover $\mathcal{B}$ is isomorphic to $p_\star \mathcal{S}(\Gamma, \mu) p_\star$ with $\star$ the unique depth-zero vertex of Γ and μ the induced weighting from $\mathcal{P}_\bullet$ [HP14b]. If $\mathcal{A}$ is the $*$-algebra generated by*

$$\left\{ p_\alpha, X_\epsilon \mid \alpha \in V(\Gamma), \text{ and } \epsilon \in E(\vec{\Gamma}) \right\},$$

then this isomorphism carries $\mathrm{Gr}_0(\mathcal{P}_\bullet)$ onto $p_\star \mathcal{A} p_\star$.

Our work on the laws of elements in $\mathcal{A}$ immediately implies the following corollary.

Corollary 4.5 *If $x = x^* \in \mathrm{Gr}_0(\mathcal{P}_\bullet)$ is not a scalar, then the Cauchy transform of x with respect to tr is algebraic, the law of x with respect to tr has no atoms, and the spectrum of x is a finite union of closed intervals, each of which having measure in $(0, 1] \cap \mathbb{Z}\{\mu(\alpha) \mid \alpha \in V(\Gamma)\}$*

Proof Since $\star$ is of minimal weight in $V(\Gamma)$, this is immediate from Theorem 4.4(iii) as well as Corollary 3.17. ∎

References

[Avi82]   D. Avitzour, *Free products of C*-algebras*. Trans. Amer. Math. Soc. **271**(1982), no. 2, 423–435.   http://dx.doi.org/10.2307/1998890

[BG05]   F. Benaych-Georges, *Rectangular random matrices, related free entropy and free fisher's information*. arxiv:math/0512081

[BHP12]   A. Brothier, M. Hartglass, and D. Penneys, *Rigid C*-tensor categories of bimodules over interpolated free group factors*. J. Math. Phys. **53**(2012), no. 12, 123525. http://dx.doi.org/10.1063/1.4769178

[Bis97]   D. Bisch, *Bimodules, higher relative commutants and the fusion algebra associated to a subfactor*. In: Operator algebras and their applications (Waterloo, ON, 1994/1995), Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997, pp. 13–63.

[CDS14]   S. Curran, Y. Dabrowski, and D. Shlyakhtenko, *Free analysis and planar algebras*. arxiv:1411.0268

[CK80]   J. Cuntz and W. Krieger, *A class of C*-algebras and topological Markov chains*. Invent. Math. **56**(1980), no. 3, 251–268.   http://dx.doi.org/10.1007/BF01390048

[CS15]   I. Charlesworth and D. Shlyakhtenko, *Regularity of polynomials in free variables*. arxiv:1408.0580v2

[Dyk93]   K. Dykema, *Free products of hyperfinite von Neumann algebras and free dimension*. Duke Math. J. **69**(1993), no. 1, 97–119.   http://dx.doi.org/10.1090/S0002-9947-99-02180-7

[Dyk99]   K. J. Dykema, *Simplicity and the stable rank of some free product C*-algebras*. Trans. Amer. Math. Soc. **351**(1999), no. 1, 1–40.   http://dx.doi.org/10.1090/S0002-9947-99-02180-7

[DHR97]   K. Dykema, U. Haagerup, and M. Rørdam, *The stable rank of some free product C*-algebras*. Duke Math. J. **90**(1997), no. 1, 95–121.   http://dx.doi.org/10.1215/S0012-7094-97-09004-9

[DR13]   K. J. Dykema and D. Redelmeier, *The amalgamated free product of hyperfinite von Neumann algebras over finite dimensional subalgebras*. Houston J. Math. **39**(2013), no. 4, 1313–1331.

[DR98]   K. J. Dykema and M. Rørdam, *Projections in free product C^*-algebras*. Geom. Funct. Anal.
           8(1998), no. 1, 1–16.   http://dx.doi.org/10.1007/s000390050046

[Ger]   E. Germain, *KK-theory of C^*-algebras related to Pimsner algebras*.
           http://www.math.jussieu.fr/~germain

[GJS10]   A. Guionnet, V. F. R. Jones, and D. Shlyakhtenko, *Random matrices, free probability, planar
           algebras and subfactors*. In: Quanta of maths, Clay Math. Proc., 11, Amer. Math. Soc.,
           Providence, RI, 2010, pp. 201–239.

[GJS11]   _____, *A semi-finite algebra associated to a subfactor planar algebra*. J. Funct. Anal.
           261(2011), no. 5, 1345–1360.   http://dx.doi.org/10.1016/j.jfa.2011.05.004

[Har13]   M. Hartglass, *Free product von Neumann algebras associated to graphs, and Guionnet, Jones,
           Shlyakhtenko subfactors in infinite depth*. J. Funct. Anal. **265**(2013), no. 12, 3305–3324.
           http://dx.doi.org/10.1016/j.jfa.2013.09.011

[HP14a]   M. Hartglass and D. Penneys, *C^*-algebras from planar algebras I: canonical C^*-algebras
           associated to a planar algebra*. Trans. Amer. Math. Soc., to appear. arxiv:1401.2485

[HP14b]   _____, *C^*-algebras from planar algebras II: The Guionnet-Jones-Shlyakhtenko C^*-algebras*.
           J. Funct. Anal. **267**(2014), no. 10, 3859–3893.   http://dx.doi.org/10.1016/j.jfa.2014.08.024

[Iva11]   N. A. Ivanov, *On the structure of some reduced amalgamated free product C^*-algebras*.
           Internat. J. Math. **22**(2011), no. 2, 281–306.   http://dx.doi.org/10.1142/S0129167X11006799

[Jon83]   V. F. R. Jones, *Index for subfactors*. Invent. Math. **72**(1983), no. 1, 1–25,
           http://dx.doi.org/10.1007/BF01389127

[MSW14]   T. Mai, R. Speicher, and M. Weber, *Absence of algebraic relations and of zero divisors under
           the assumption of finite non-microstates free fisher information*. arxiv:1407.5715

[Pet10]   E. Peters, *A planar algebra construction of the Haagerup subfactor*. Internat. J. Math.
           21(2010), no. 8, 987–1045.   http://dx.doi.org/10.1142/S0129167X10006380

[Pim97]   M. V. Pimsner, *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed
           products by $\mathbf{Z}$*. In: Free probability theory (Waterloo, ON, 1995), Fields Inst. Commun., 12,
           American Mathematical Society, Providence, RI, 1997, pp. 189–212.

[Pop95]   S. Popa, *An axiomatization of the lattice of higher relative commutants of a subfactor*. Invent.
           Math. **120**(1995), no. 3, 427–445.   http://dx.doi.org/10.1007/BF01241137

[Rie83]   M. A. Rieffel, *Dimension and stable rank in the K-theory of C^*-algebras*. Proc. London
           Math. Soc. (3) **46**(1983), no. 2, 301–333.   http://dx.doi.org/10.1112/plms/s3-46.2.301

[Sau03]   R. Sauer, *Power series over the group ring of a free group and applications to Novikov-Shubin
           invariants*. In: High-dimensional manifold topology, World Sci. Publ., River Edge, NJ,
           2003, pp. 449–468.   http://dx.doi.org/10.1142/9789812704443_0020

[Sch62]   M. P. Schützenberger, *On a theorem of R. Jungen*. Proc. Amer. Math. Soc. **13**(1962), 885–890.
           http://dx.doi.org/10.2307/2034080

[Shl99]   D. Shlyakhtenko, *A-valued semicircular systems*. J. Funct. Anal. **166**(1999), no. 1, 1–47.
           http://dx.doi.org/10.1006/jfan.1999.3424

[SS15]   D. Shlyakhtenko and P. Skoufranis, *Freely independent random variables with non-atomic
           distributions*. Trans. Amer. Math. Soc. **367**(2015), no. 9, 6267–6291.
           http://dx.doi.org/10.1090/S0002-9947-2015-06434-4

[VDN92]   D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*. CRM Monograph
           Series, 1, American Mathematical Society, Providence, RI, 1992.

[Voi93]   _____, *The analogues of entropy and of Fisher's information measure in free probability
           theory. I*. Comm. Math. Phys. **155**(1993), no. 1, 71–92.   http://dx.doi.org/10.1007/BF02100050

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