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Envelope Dimension of Modules and the Simplified Radical Formula

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Abstract. We introduce and investigate the notion of envelope dimension of commutative rings and modules over them. In particular, we show that the envelope dimension of a ring, R, is equal to that of the R-module $R^{(\mathbb{N})}$. We also prove that the Krull dimension of a ring is no more than its envelope dimension and characterize Noetherian rings for which these two dimensions are equal. Moreover, we generalize and study the concept of simplified radical formula for modules, which we defined in an earlier paper.

1 Introduction

In this paper all rings are commutative and with identity, all modules are unitary, R denotes a ring, and M denotes an R-module. Also, by $\mathbb{N}$ we mean the set of positive integers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$. We indicate the relation of containment and strict containment by $\subseteq$ and $\subset$, respectively. Furthermore, $N \leq M$ (resp., $N < M$) means that N is a submodule (resp., proper submodule) of M.

Prime ideals of rings play an important role in commutative ring theory, hence many have tried to generalize this concept to modules. A proper submodule P of M is called *prime*, when from $rm \in P$ for some $r \in R$ and $m \in M$, we can conclude either $m \in P$ or $rM \subseteq P$ (see, for example, [2, 4, 11, 12, 14]). Let $(P:M)$ be the set of all $r \in R$ such that $rM \subseteq P$. If P is a prime submodule, then $\mathfrak{P} = (P:M)$ is a prime ideal of R and we say that P is $\mathfrak{P}$-prime.

Another generalization of prime ideals was proposed in [6]. There a proper submodule W of M is said to be *weakly prime* if from $rsm \in W$ for $r, s \in R$, and $m \in M$, we can conclude either $rm \in W$ or $sm \in W$. One can easily see that it is equivalent to asserting that $(W:m)$ is a prime ideal for every $m \in M \setminus W$.

If W is weakly prime, then we consider $\mathcal{C}(W)$ (or just $\mathcal{C}$) to be

$$\mathcal{C}(W) = \left\{ (W:m) \mid m \in M \setminus W \right\},$$

and we say that W is $\mathcal{C}$-weakly prime.

Recall that for an ideal $\mathfrak{I}$ of R, the intersection of all prime ideals of R containing $\mathfrak{I}$ is called the radical of $\mathfrak{I}$ and is denoted by $\sqrt{\mathfrak{I}}$. Similarly, if N is a submodule of M, the intersection of prime (resp., weakly prime) submodules of M containing N is called the *radical (resp., weak radical) of* N and we denote it by $\mathrm{rad}_M(N)$ (resp.,

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$\mathrm{wrad}_M(N)$) (or $\mathrm{rad}(N)$ (resp., $\mathrm{wrad}(N)$) if there is no subtlety). If M has no prime (resp., weakly prime) submodule containing N, then we say $\mathrm{rad}_M(N) = M$ (resp., $\mathrm{wrad}_M(N) = M$).

A well-known and very useful theorem in commutative ring theory is

$$\sqrt{\mathfrak{I}} = \{r \in R \mid r^k \in \mathfrak{I} \quad \text{for some } k \in \mathbb{N}\}.$$

To find a similar characterization for the radical of a submodule, the notion of envelope of a submodule was introduced in [11]. The *envelope* of a submodule N of M, $\mathrm{E}_M(N)$ (or $\mathrm{E}(N)$ if no ambiguity) is the set of all $x \in M$ for which, there exist $r \in R$, $m \in M$, and $k \in \mathbb{N}$ such that $x = rm$ and $r^k m \in N$. The envelope of a submodule is not necessarily itself a submodule (see [4, Proposition 2.1]), so we usually use the submodule generated by it, denoted by $R\mathrm{E}(N)$.

One can easily verify that for every submodule N of M, we have $N \subseteq R\mathrm{E}(N) \subseteq \mathrm{wrad}(N) \subseteq \mathrm{rad}(N)$. Now if $\mathrm{rad}(N) = R\mathrm{E}(N)$ (resp., $\mathrm{wrad}(N) = R\mathrm{E}(N)$), it is said that N *satisfies (resp., weakly satisfies) the radical formula* (r.f.) in M. A module M (weakly) satisfies the r.f. when every submodule of M (weakly) satisfies the r.f. in M. We also say that R (weakly) satisfies the r.f. if every R-module (weakly) satisfies the r.f. Clearly if a ring satisfies the r.f., then it weakly satisfies the r.f.

Many have studied when a ring or a module satisfies the r.f. (see [2, 9, 11, 12]). For example in [9], Noetherian rings that satisfy the r.f. are characterized, and in [12] it is proved that every finite dimensional *arithmetic ring* (that is, a ring in which for every three ideals I, J, and K, we have $I + J \cap K = (I + J) \cap (I + K)$) satisfies the r.f. Also, the weak radical of submodules of a module is investigated in [3, 5, 13].

In [4] we said that R satisfies the *simplified radical formula* (s.r.f.) if $\mathrm{rad}_M(0) = \mathrm{E}_M(0)$ (in particular, $\mathrm{E}_M(0)$ is a submodule) for every R-module M. There we proved (see [4, Theorem 2.15]) that a Noetherian ring satisfies the s.r.f. if and only if it is a *ZPI-ring* (a ring every non-zero ideal of which is a product of prime ideals).

In this paper, we will study a generalization of this notion. We will say that R *satisfies (resp., weakly satisfies) the simplified radical formula of degree k* for a $k \in \mathbb{N}^*$ if

$$\mathrm{rad}_M(N) = k\mathrm{E}_M(N) + N \quad \big(\text{resp., } \mathrm{wrad}_M(N) = k\mathrm{E}_M(N) + N\big)$$

for every submodule N of every R-module M. Therefore R satisfies the s.r.f. if and only if it satisfies the s.r.f. of degree 1 (see Lemma 2.3), and by [14, Theorem 2.1] R satisfies the s.r.f. of degree 0 if and only if it is absolutely flat.

Among some other results, in Section 4 we will show that for every valuation domain or Artinian ring R there exists a $k \in \mathbb{N}^*$ such that R satisfies the s.r.f. of degree k, and we find the smallest such k. Moreover, we will prove that if R weakly satisfies the s.r.f. of degree k, then it has Krull dimension $\leq k$. Furthermore, we will prove that every finite dimensional semi-local arithmetic ring satisfies the s.r.f. of degree k for some $k \in \mathbb{N}^*$.

2 Envelope Dimension and the Simplified Radical Formula

Let M be an R-module. If $M = R$, then $R\mathrm{E}(N) = \mathrm{E}(N)$ for every $N \leq M$, but for general M this is not true. Motivated by this, in this section we try to find rings, such

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as R, for which $R\mathrm{E}(N)$ is not "too far" from $\mathrm{E}(N)$, for every N of every R-module M, especially those rings that (weakly) satisfy the r.f.

Recall that in general $\mathrm{E}(N)$ is not a submodule of M, although $0 \in \mathrm{E}(N)$ and $\mathrm{E}(N)$ is closed under multiplication from R. So for each $k \in \mathbb{N}^*$, the sum of k copies of $\mathrm{E}(N)$ (denoted by $k\mathrm{E}(N)$) is a subset of $R\mathrm{E}(N)$ including $\mathrm{E}(N)$. Thus a reasonable question is for which k, do we have $R\mathrm{E}(N) = N + k\mathrm{E}(N)$ or $\mathrm{rad}(N) = N + k\mathrm{E}(N)$? This k is a kind of measure that shows how "far apart" $R\mathrm{E}(N)$ and $\mathrm{E}(N)$ are.

Definition 2.1 Let $k \in \mathbb{N}^*$ and let M be an R-module. If $\mathrm{rad}_M(N) = k\mathrm{E}_M(N) + N$ (resp., $\mathrm{wrad}_M(N) = k\mathrm{E}_M(N) + N$) for each $N \leq M$, then we will say that M satisfies (resp., weakly satisfies) the simplified radical formula *of degree k*. Also, we will say that R (weakly) satisfies the simplified radical formula of degree k when every R-module does so.

If $k = 1$ in the above definition, we drop "of degree 1" and simply say that R (weakly) satisfies the s.r.f. This concept was studied in [4].

Let $N \leq M$ and $k \in \mathbb{N}^*$. As $k\mathrm{E}_M(N) + N \subseteq R\mathrm{E}(N)$, if M (or R) (weakly) satisfies the s.r.f. of degree k, then obviously M (or R) (weakly) satisfies the r.f.

Note that if M satisfies the s.r.f. of degree k, then evidently M satisfies the s.r.f. of degree k' for each $k' > k$, hence we need to consider the smallest such k.

Definition 2.2 If $k \in \mathbb{N}^*$ is the smallest integer such that $R\mathrm{E}_M(N) = k\mathrm{E}_M(N) + N$, for every $N \leq M$, then we will say that k is the *envelope dimension* of M and write $\mathrm{edim}\, M = k$; otherwise we write $\mathrm{edim}\, M = \infty$. Also, we define

$$\mathrm{edim}\, R = \sup\{\mathrm{edim}\, M \mid M \text{ is an } R\text{-module}\}.$$

One may ask why we have chosen $k\mathrm{E}(N) + N$ and not $k\mathrm{E}(N)$. One reason is that, in the case $k = 1$, if $R\mathrm{E}(N) = \mathrm{E}(N)$ for every $N \leq M$, then by [4, Proposition 2.1] R is an absolutely flat ring, which implies that $R\mathrm{E}(N) = N$. Another reason is that in general $k\mathrm{E}(N) + N$ is a subset of $(k+1)\mathrm{E}(N)$. Also, elements of $k\mathrm{E}(N) + N$ have a simpler form in comparison with those of $(k+1)\mathrm{E}(N)$. Furthermore, if we used $k\mathrm{E}(N)$ instead of $k\mathrm{E}(N) + N$, part (vii) of the following useful lemma could not be asserted.

Lemma 2.3 *Let $k \in \mathbb{N}^*$ and let M be an R-module, N and K submodules of M with $K \subseteq N$. Also suppose that S is a multiplicatively closed subset of R and L is a subset of M containing K.*

(i) $E_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{E_M(N)}{K}$; $\mathrm{rad}_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{\mathrm{rad}_M(N)}{K}$; $\mathrm{wrad}_{\frac{M}{K}}\left(\frac{N}{K}\right) = \frac{\mathrm{wrad}_M(N)}{K}$.

(ii) $\frac{N}{K} = \frac{L}{K}$ *if and only if* $N = L + K$.

(iii) $E(N_S) = (E(N))_S$; $(\mathrm{rad}(N))_S \subseteq \mathrm{rad}(N_S)$; $(\mathrm{wrad}(N))_S \subseteq \mathrm{wrad}(N_S)$.

(iv) *If* $M = \bigoplus_{i\in I} M_i$ *and* $N = \bigoplus_{i\in I} N_i$, *where* $N_i \leq M_i \leq M$ *for each* $i \in I$, *then* $RE_M(N) = \bigoplus_{i\in I} RE_{M_i}(N_i)$.

(v) *If* $M_{\mathfrak{M}}$ *(weakly) satisfies the r.f. for all maximal ideals* $\mathfrak{M}$ *of* R, *then* M *(weakly) satisfies the r.f.*

(vi) *The ring* R *satisfies (resp., weakly satisfies) the r.f. if and only if* $\mathrm{rad}_{M'}(0) = RE_{M'}(0)$ *(resp.,* $\mathrm{wrad}_{M'}(0) = RE_{M'}(0)$*) for every* R*-module* M'.

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(vii) *A ring R satisfies the s.r.f. of degree k, (resp. weakly satisfies the s.r.f. of degree k),* $\operatorname{edim} R \leq k$, *if and only if* $\operatorname{rad}_M(0) = kE_M(0)$ *(resp.* $\operatorname{wrad}_M(0) = kE_M(0)$, $RE_M(0) = kE_M(0)$*), for every R-module M.*

Proof Parts (i), (ii), and (iv) are easy. For part (iii), see, for example, [5, Proposition 2.1] and the proof of [14, Proposition 1.6]. Part (v) follows from (iii), (vi) follows from (i), and (vii) can be deduced from (i) and (ii). ■

Note that in part (ii) of the lemma, although $K \subseteq L$, since L is not necessarily a submodule, the set $L + K$ need not be equal to L.

Notation 2.4 Let $N \leq M$ and $x \in \mathrm{RE}(N)$. If k_x is the smallest integer such that

$$(*) \qquad x = n + \sum_{i=1}^{k_x} x_i,$$

where $n \in N$ and $x_i \in \mathrm{E}(N)$ for each $1 \leq i \leq k_x$, then we say $(*)$ is a *reduced summation* in $\mathrm{E}(N)$ for x.

Suppose $x = n + \sum_{i=1}^{k_x} r_i m_i$ is a reduced summation in $\mathrm{E}(N)$ for x, where $r_i \in R$ and $m_i \in M$ with $r_i \in \sqrt{(N:m_i)}$. Then all r_is are non-zero and no r_i is a unit element of R. Also, note that we can find a $t \in \mathbb{N}$ such that $r_i^t m_i \in N$ for all $1 \leq i \leq k_x$, and clearly $\operatorname{edim} M = \sup\{k_x \mid x \in \mathrm{RE}(N)\}$.

Obviously R (weakly) satisfies the s.r.f. of degree k if and only if R (weakly) satisfies the r.f. and $\operatorname{edim} R \leq k$. So first we focus on rings with the envelope dimension $\leq k$.

Theorem 2.5 *Suppose that* $(R, \mathfrak{M})$ *is a local ring and* $n \in \mathbb{N}$*. Then the following are equivalent:*

(i) $\operatorname{edim} M \leq n$, *for the R-module* $M = R^{n+1}$.

(ii) *For every* $1 < \alpha \in \mathbb{N}$ *and* $a_i \in \mathfrak{M}$, $1 \leq i \leq n+1$, *there exist* $\beta \in \mathbb{N}$, $r_j \in \mathfrak{M}$ *and* $c_{ij} \in R$, $1 \leq j \leq n$ *such that* $a_i = \sum_{j=1}^{n} r_j c_{ij}$ *and* $r_j^{\beta} c_{ij} \in Ra_i^{\alpha}$, *for all* $1 \leq i \leq n+1$, $1 \leq j \leq n$.

(iii) $\operatorname{edim} R \leq n$.

Proof (i) ⇒ (ii): Let $1 < \alpha \in \mathbb{N}$ and $a_1, a_2, \dots, a_{n+1} \in \mathfrak{M}$ and set $N = Ra_1^{\alpha} \oplus Ra_2^{\alpha} \oplus \cdots \oplus Ra_{n+1}^{\alpha}$. Then, according to Lemma 2.3(iv),

$$\mathrm{RE}_M(N) = \bigoplus_{i=1}^{n+1} \mathrm{RE}_R(Ra_i^{\alpha}) = \bigoplus_{i=1}^{n+1} \sqrt{Ra_i^{\alpha}} = \bigoplus_{i=1}^{n+1} \sqrt{Ra_i}.$$

Therefore $(a_i) \in RE_M(N)$ (where (a_i) denotes $(a_1, \dots, a_{n+1})$).

Let $(a_i) = (b_i a_i^{\alpha}) + \sum_{j=1}^{k} r_j (c'_{ij})$ be a reduced summation in $\mathrm{E}(N)$ for (a_i), where for each $1 \leq i \leq n+1$ and $1 \leq j \leq k$ we have $b_i, r_j, c'_{ij} \in R$ and there is a $\beta \in \mathbb{N}$ such that $r_j^{\beta}(c'_{ij}) \in N$. But since the envelope dimension of $M \leq n$ and this summation is reduced, the r_j s are in $\mathfrak{M}$ and $k \leq n$. One can assume that $k = n$, because if necessary we can consider $r_j = c_{ij} = 0$ for all $k < i \leq n$ and $1 \leq j \leq n$.

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Then $a_i = b_i a_i^{\alpha} + \sum_{j=1}^{n} r_j c'_{ij}$, for each $1 \le i \le n+1$, and hence

$$a_i(1 - b_i a_i^{\alpha-1}) = \sum_{j=1}^{n} r_j c'_{ij}.$$

Note that $\alpha - 1 > 0$, so $u_i = 1 - b_i a_i^{\alpha-1}$ is a unit in R. Now if $c_{ij} = u_i^{-1} c'_{ij}$, then β, the r_j s and the c_{ij} s satisfy the claimed conditions of (ii).

(ii) $\Rightarrow$ (iii): Let N be a submodule of an arbitrary R-module M and $x \in R\mathrm{E}(N)$. Assume that $x = a + \sum_{i=1}^{t} a_i m_i$ is a reduced summation in $\mathrm{E}(N)$ for x, where $a_i^{\alpha} m_i \in N$ for each $1 \le i \le t$, for some $1 < \alpha \in \mathbb{N}$. We must show that $t \le n$. Suppose not and apply (ii) to $a_1, \dots, a_{n+1}$ and α to get a natural number β and r_j s in $\mathfrak{M}$ and c_{ij} s in R satisfying the conditions of (ii).

Now

$$\sum_{i=1}^{n+1} a_i m_i = \sum_{i=1}^{n+1} \sum_{j=1}^{n} r_j c_{ij} m_i = \sum_{j=1}^{n} r_j \Big(\sum_{i=1}^{n+1} c_{ij} m_i \Big).$$

Note that $r_j^{\beta} c_{ij} \in R a_i^{\alpha}$, so for $m'_j = \sum_{i=1}^{n+1} c_{ij} m_i$, we have $r_j^{\beta} m'_j \in \sum_{i=1}^{n+1} R a_i^{\alpha} m_i \subseteq N$. Consequently, by replacing $\sum_{i=1}^{n+1} a_i m_i$ with $\sum_{j=1}^{k} r_j m'_j$, we get a summation for x as elements of $\mathrm{E}(N)$, in which the number of summands is less than t. But this is contrary to $x = a + \sum_{i=1}^{t} a_i m_i$ being a reduced summation in $\mathrm{E}(N)$ for x, and the result follows.

(iii) $\Rightarrow$ (i): This is trivial. ■

Throughout this paper, the Krull dimension of a ring R is denoted by $\dim R$.

Lemma 2.6 *Let M be an R-module, and I an ideal of R. Then*

(i) $\operatorname{edim} M_{\mathfrak{P}} \le \operatorname{edim} M$;
(ii) $\operatorname{edim} R_{\mathfrak{P}} \le \operatorname{edim} R$;
(iii) $\operatorname{edim} \frac{R}{I} \le \operatorname{edim} R$;
(iv) $\dim R = \sup\{\dim R/\mathfrak{P} \mid \mathfrak{P}$ *is a prime ideal of* $R\}$, *and if* $\dim R$ *is finite, then there exists a prime ideal* $\mathfrak{P}$ *of* R *such that* $\dim R = \dim R/\mathfrak{P}$.

Proof (i) Suppose $\operatorname{edim} M = k \in \mathbb{N}^*$. Evidently, for each $N \le M$, $\mathrm{E}(N)$ is closed with respect to multiplication from R, thus $(k\mathrm{E}(N))_{\mathfrak{P}} = k(\mathrm{E}(N))_{\mathfrak{P}}$, and hence by Lemma 2.3(iii), $\operatorname{edim} M_{\mathfrak{P}} \le k$.

Parts (ii), (iii), and (iv) are easy. ■

Corollary 2.7 *If $n \in \mathbb{N}$ and $\operatorname{edim} R \le n$, then for every prime ideal $\mathfrak{P}$ of R, every finitely generated proper ideal of $R_{\mathfrak{P}}$ is contained in a proper ideal that can be generated by n elements. If $(R, \mathfrak{M})$ is a zero dimensional local ring or a one dimensional local domain, the converse also holds.*

Proof By Lemma 2.6(ii), $\operatorname{edim} R_{\mathfrak{P}} \le n$. Now let I be a proper ideal of $R_{\mathfrak{P}}$ generated by $a_1, a_2, \dots, a_m$. As $\operatorname{edim} M \le n$, by Theorem 2.5(ii), there exist $r_1, r_2, \dots, r_n \in \mathfrak{P}_{\mathfrak{P}}$ such that each $a_i \in \sum_{i=1}^{n} R r_i$, that is $I \subseteq \sum_{i=1}^{n} R r_i$.

Now assume that R is zero dimensional or a one dimensional integral domain. Let $a_1, a_2, \dots, a_{n+1} \in \mathfrak{M}$ and $\alpha \in \mathbb{N}$ be given. If one of the a_i s, say a_{n+1}, is zero, then for

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all $1 \le i \le n+1$ and $1 \le j \le n$, choose $\beta = \alpha$, $r_j = a_j$, and $c_{ij} = \delta_{ij}$, where δ denotes the Kronecker delta. Then these elements satisfy the conditions of Theorem 2.5(ii), which completes the proof.

If all a_i s are non-zero, then by our assumption there are $r_1, \dots, r_n$ such that each $a_i \in \sum_{i=1}^n Rr_i \subseteq \mathfrak{M}$. Now because $r_j \in \mathfrak{M} = \sqrt{Ra_i}$ for all i s and j s, hence $r_j^\beta \in Ra_i^\alpha$ for some $\beta \in \mathbb{N}$. Therefore again the result follows from Theorem 2.5(ii). ∎

The first part of the following result generalizes [4, Theorem 2.6].

Corollary 2.8 *Let $(R, \mathfrak{M})$ be a local ring and $k \in \mathbb{N}^*$.*

(i) *If R is zero dimensional, then R satisfies the s.r.f. of degree k if and only if every finitely generated proper ideal of R is contained in a proper ideal that can be generated by k elements.*

(ii) *If R is a one dimensional integral domain, then R weakly satisfies the s.r.f. of degree k if and only if every finitely generated proper ideal of R is contained in a proper ideal that can be generated by k elements.*

Proof If $k \in \mathbb{N}$, then the proof follows from Corollary 2.7, [13, Corollary 3.3], and [14, Theorem 2.8], which state that zero dimensional rings (resp., one dimensional domains) satisfy (resp., weakly satisfy) the r.f. For the case $k = 0$ use [4, Proposition 2.1], which shows that R satisfies the s.r.f. of degree 0 if and only if R is absolutely flat. ∎

Throughout this paper, the set of maximal ideals of R will be denoted by $\operatorname{Max}(R)$.

Theorem 2.9 *For every ring R,* $\dim R \le \operatorname{edim} R$.

Proof Let $\operatorname{Min}(R)$ be the set of minimal prime ideals of R. We will prove that $\dim R_{\mathfrak{M}}/\mathfrak{P} \le \operatorname{edim} R_{\mathfrak{M}}/\mathfrak{P}$, for every $\mathfrak{M} \in \operatorname{Max}(R)$ and $\mathfrak{P} \in \operatorname{Min}(R_{\mathfrak{M}})$. So by Lemma 2.6(iv), (iii), and (ii),

$$\dim R_{\mathfrak{M}} = \sup\Big\{ \dim \frac{R_{\mathfrak{M}}}{\mathfrak{P}} \;\Big|\; \mathfrak{P} \in \operatorname{Min}(R_{\mathfrak{M}}) \Big\} \le \sup\Big\{ \operatorname{edim} \frac{R_{\mathfrak{M}}}{\mathfrak{P}} \;\Big|\; \mathfrak{P} \in \operatorname{Min}(R_{\mathfrak{M}}) \Big\}$$
$$\le \operatorname{edim} R_{\mathfrak{M}} \le \operatorname{edim} R.$$

Thus $\dim R = \sup\{\dim R_{\mathfrak{M}} \mid \mathfrak{M} \in \operatorname{Max}(R)\} \le \operatorname{edim} R$.

Note that $\frac{R_{\mathfrak{M}}}{\mathfrak{P}_{\mathfrak{M}}}$ is a local integral domain, hence we can assume that R is a local integral domain. If $\operatorname{edim} R = \infty$, there is nothing to prove. Now suppose that $\operatorname{edim} R = n \in \mathbb{N}^*$. On the contrary, assume that there is a chain of prime ideals of R with length $n+1$, such as $0 = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_{n+1}$. Let $a_i \in \mathfrak{P}_i \setminus \mathfrak{P}_{i-1}$ $(1 \le i \le n+1)$. Assume that the r_i s in $\mathfrak{M}$, the c_{ij} s in R and $\beta, k \in \mathbb{N}$ are as in Theorem 2.5(ii) for the above a_i s and $\alpha = 2$.

If $r = (r_1 r_2 \cdots r_k)^\beta$, then $ra_1 \in \sum_{j=1}^k Rr_j^\beta c_{1j} \in Ra_1^2$. But since R is an integral domain and $a_1 \neq 0$, we have $(Ra_1^2 : Ra_1) = Ra_1$ and thus $r \in Ra_1 \subseteq \mathfrak{P}_1$. Hence one of the r_i s, say r_1, must be in $\mathfrak{P}_1$.

Now consider $\bar{R} = R/\mathfrak{P}_1$. If for each $x \in R$ we denote its image in $\bar{R}$ by $\bar{x}$, then for $2 \le i \le n+1$ and $2 \le j \le k$ we have $\bar{a}_i \neq \bar{0}$, $\bar{a}_i = \sum_{j=2}^k \bar{r}_j \bar{c}_{ij}$ and $\bar{r}_i^\beta \bar{c}_{ij} \in \bar{R}\bar{a}_i^2$.

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Therefore, by an argument similar to that in the last paragraph, we conclude that some $\bar{r}_i \in \bar{\mathfrak{P}}_2$, say $\bar{r}_2$. Whence $r_2 \in \mathfrak{P}_2$ and continuing this way we see that after a possible reordering of r_i s, we can assume that $r_i \in \mathfrak{P}_i$, for each $1 \leq i \leq k$.

But this implies that $a_{n+1} \in \sum_{i=1}^{k} Rr_i \subseteq \mathfrak{P}_n$, in contrast to how we chose a_{n+1}. Therefore, $n \leq k$, as asserted. ∎

Now we are in a position to determine some types of rings that weakly satisfy the s.r.f. The following corollaries characterize local integral domains and Noetherian rings that weakly satisfy the s.r.f. In [4], we called a ring R *weakly Bezout* if each finitely generated proper ideal of R is contained in a proper principal ideal.

Corollary 2.10 *If R is an integral domain, then the following are equivalent:*

(i) *R weakly satisfies the s.r.f.;*
(ii) $\operatorname{edim} R \leq 1$.

If moreover R is local, then the above statements are also equivalent to R being weakly Bezout and dim $R \leq 1$.

Proof (i) ⇒ (ii): This is trivial. (ii) ⇒ (i): This follows from Theorem 2.9 and [13, Corollary 3.3], which states that one dimensional domains weakly satisfy the r.f.. The assertion on local domains is clear by Theorem 2.9 and Corollary 2.8(ii). ∎

Lemma 2.11 *A ring R is a ZPI-ring if and only if it is Noetherian and the localization of R at every maximal ideal is a ZPI-ring.*

Proof This follows from [8, p. 224, Exercise 10] and [8, Theorem 9.27]. ∎

Corollary 2.12 *If R is Noetherian ring then the following are equivalent:*

(i) *R satisfies the s.r.f.;*
(ii) *R weakly satisfies the s.r.f.;*
(iii) $\operatorname{edim} R \leq 1$*;*
(iv) *R is a ZPI-ring.*

Proof (i) ⇒ (ii) ⇒ (iii) is trivial, and for (iv) ⇒(i), see [4, Theorem 2.15].

(iii) ⇒ (iv) By Lemma 2.6(ii), $\operatorname{edim} R_{\mathfrak{M}} \leq 1$ for each maximal ideal $\mathfrak{M}$ of R, and due to Lemma 2.11, we can assume that R is local with the maximal ideal $\mathfrak{M}$. Since $\mathfrak{M}$ is finitely generated and R is weakly Bezout by Corollary 2.7, we deduce that $\mathfrak{M}$ is principal, say $\mathfrak{M} = Rx$. Now, by the Krull intersection theorem, $\bigcap_{n=1}^{\infty} Rx^n = 0$. If I is an ideal of R such that $\mathfrak{I} \subseteq Rx^n$ but $\mathfrak{I} \nsubseteq Rx^{n+1}$, then it is easy to see that $\mathfrak{I} = Rx^n$. Therefore every non-zero proper ideal of R is of the form Rx^n for some $n \in \mathbb{N}$ and R is either a discrete valuation domain or an SPIR. In particular, R is a ZPI-ring, as required. ∎

An immediate consequence of Theorem 2.5 is that if R is a local ring, then $\operatorname{edim} R = \operatorname{edim} M$, where M is the R-module $\bigoplus_{i \in \mathbb{N}} R$. The following proposition implies that this equality holds, even if R is not local.

Proposition 2.13 *Let I be a directed set and let $(\{M_i\}_{i \in I}, \{\phi_{ij}\}_{i \leq j \in I})$ be a directed system of modules and homomorphisms. Also suppose that for each $i \in I$, N_i is a submodule of M_i such that $\phi_{ij}(N_i) \subseteq N_j$ for all $i \leq j \in I$. Set $M = \varinjlim M_i$ and $N = \varinjlim N_i$. Then*

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(i) $E_M(N) = \varinjlim E_{M_i}(N_i)$;
(ii) $\varinjlim \operatorname{rad}_{M_i}(N_i) \subseteq \operatorname{rad}_M(N)$ *and* $\varinjlim \operatorname{wrad}_{M_i}(N_i) \subseteq \operatorname{wrad}_M(N)$;
(iii) *if* N *satisfies (resp., weakly satisfies) the r.f. in* M, *then*

$$\varinjlim \operatorname{rad}_{M_i}(N_i) = \operatorname{rad}_M(N) \quad (\textit{resp.}, \varinjlim \operatorname{wrad}_{M_i}(N_i) = \operatorname{wrad}_M(N)).$$

Proof (i) First note that if $\phi\colon A \to B$ and $C \leq A$, then $\phi(E_A(C)) \subseteq E_B(\phi(C))$. Therefore for all $i \leq j \in I$, we have

$$\phi_{ij}\big(E_{M_i}(N_i)\big) \subseteq E_{M_j}\big(\phi_{ij}(N_i)\big) \subseteq E_{M_j}(N_j).$$

Thus $\varinjlim E_{M_i}(N_i)$ exists. Also, if $\phi_i\colon M_i \to M$ is the canonical mapping, then

$$\phi_i\left(E_{M_i}(N_i)\right) \subseteq E_M(N),$$

and since every $x \in \varinjlim E_{M_i}(N_i)$ is in $\phi_i\left(E_{M_i}(N_i)\right)$ for some $i \in I$, we deduce that

$$\varinjlim E_{M_i}(N_i) \subseteq E_M(N).$$

Now suppose that $x \in \mathrm{E}_M(N)$. Then there exist $r \in R$, $m \in M$, $t \in \mathbb{N}$ and $n \in N$ such that $x = rm$ and $r^t m = n$. So for a large enough i, there are $x_i, m_i \in M_i$, and $n_i \in N_i$ such that $\phi_i(x_i) = x$, $\phi_i(m_i) = m$, $\phi_i(n_i) = n$, $x_i = rm_i$, and $r^t m_i = n_i$. But this means that $x_i \in \mathrm{E}_{M_i}(N_i)$, whence $x \in \varinjlim \mathrm{E}_{M_i}(N_i)$.

(ii) This is similar to the first paragraph of proof of (i).

(iii) By (i) and (ii), if N satisfies the r.f. in M, then

$$\operatorname{rad}_M(N) = R\mathrm{E}_M(N) = \varinjlim \left(R\mathrm{E}_{M_i}(N_i)\right) \subseteq \varinjlim \left(\operatorname{rad}_{M_i}(N_i)\right) \subseteq \operatorname{rad}_M(N),$$

whence all the inequalities must be equality. The claim on weak radicals can be established similarly. ∎

Corollary 2.14 *We have* $\operatorname{edim} R = \operatorname{edim} M$, *where* M *is the* R*-module* $\bigoplus_{i\in\mathbb{N}} R$.

Proof Let $\operatorname{edim} R = d$ and $\operatorname{edim} M = d'$. Trivially $d' \leq d$. If $d' = \infty$, then the claim is obvious. Now assume that $d' < \infty$. According to Lemma 2.3(i) and (ii), every quotient module of M, particularly every finitely generated R module, has the envelope dimension $\leq d'$. Let M' be an arbitrary R-module. Then $M' = \varinjlim M'_f$, where M'_f s are all the finitely generated submodules of M'. therefore by 2.13 we have

$$R\mathrm{E}_{M'}(0) = \varinjlim R\mathrm{E}_{M'_f}(0) = \varinjlim d'\mathrm{E}_{M'_f}(0) = d'\varinjlim E_{M'_f}(0) = d'\mathrm{E}_{M'}(0),$$

hence $d \leq d'$. ∎

Recall that a *chained ring* is a ring in which every pair of ideals is comparable. In particular, every valuation domain is a chained ring. Since for each $k \in \mathbb{N}^*$ there is a valuation domain R with $\dim R = k$, the following result supplies us with examples of rings that satisfy the s.r.f. of degree k but not of $k - 1$.

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Theorem 2.15 *Let R be a chained ring and $\dim R = k \in \mathbb{N}^*$. Then R satisfies the s.r.f. of degree $k+1$. If R is an integral domain, then R satisfies the s.r.f. of degree k and k is the least such integer.*

Proof According to [12, Theorem 2.8], R satisfies the r.f. hence we just need to show that $R\mathrm{E}(0) = k\mathrm{E}(0)$ for every R-module M. Let $x \in R\mathrm{E}(0)$ and suppose that $x = \sum_{i=1}^{n} r_i m_i$ is a reduced summation in $\mathrm{E}(0)$ for x, where $r_i^{\alpha} m_i = 0$. We must prove that $n \leq k+1$. Otherwise, by the pigeon hole principle there are $1 \leq i \neq j \leq n$, such that $\operatorname{ht} Rr_i = \operatorname{ht} Rr_j$, where $\operatorname{ht} I$ is the height of the ideal I.

We can assume that $Rr_i \subseteq Rr_j$, say $r_i = ar_j$. Let $\mathfrak{P} = \bigcap_{t=1}^{\infty} Rr_j^t$. Then $\mathfrak{P} = 0$ or $\mathfrak{P}$ is a prime ideal of R by [2, Lemma 2.3]. We show that $r_i \notin \mathfrak{P}$.

On the contrary, suppose that $r_i \in \mathfrak{P}$. If $\mathfrak{P} = 0$, then obviously $r_i = 0$, which is impossible. Hence $\mathfrak{P}$ is a prime ideal and $Rr_i \subseteq \mathfrak{P}$. Note that $r_j \notin \mathfrak{P}$, otherwise $r_j = rr_j^2$ for some $r \in R$, then $r_j(1 - rr_j) = 0$ and $1 - rr_j$ is a unit, thus $r_j = 0$, which is impossible. Hence $Rr_i \subseteq \mathfrak{P} \subseteq Rr_j$, and consequently $\operatorname{ht} Rr_i < \operatorname{ht} Rr_j$, which is a contradiction.

Then $r_i \notin \mathfrak{P}$, and so $r_i \notin Rr_j^t$ for some $t \in \mathbb{N}$, and whence $Rr_j^t \subseteq Rr_i$. Consequently,

$$r_i m_i + r_j m_j = r_j(am_i + m_j) \quad \text{and} \quad r_j^{t\alpha}(am_i + m_j) = r_j^{t\alpha} am_i \in Rr_i^{\alpha} am_i = 0.$$

Thus we can replace $r_i m_i + r_j m_j$ with $r_j(am_i + m_j)$ and get that $\sum_{i=1}^{n} r_i m_i$ is not a reduced summation in $\mathrm{E}(0)$ for x, which is a contradiction.

Now if R is an integral domain, then since the above r_i s are non-zero, $1 \leq \operatorname{ht} Rr_i$ for each i. Hence if $n > k$, then, again by the pigeon hole principle, there exist $i \neq j$ with $\operatorname{ht} Rr_i = \operatorname{ht} Rr_j$, which leads to a similar contradiction. So $n \leq k$ when R is a valuation domain, and therefore R satisfies the s.r.f. of degree k. Also, $k \leq \operatorname{edim} R$ by Theorem 2.9, so k is the least integer that R satisfies the s.r.f. of degree k. ■

It follows from Lemma 2.3(iii), that if $R_{\mathfrak{M}}$ (weakly) satisfies the r.f. for each maximal ideal $\mathfrak{M}$ of R, then R (weakly) satisfies the r.f. But we neither could prove nor reject the similar assertion for the simplified radical formula (see [4, p. 12, Question]). Nevertheless we have the following result.

Proposition 2.16 *Let R be a ring and let N be a submodule of M such that the only maximal ideals of R containing $(N:M)$ are $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$. If for each $1 \leq i \leq n$ there exists $k_i \in \mathbb{N}$ such that $RE_{M_{\mathfrak{M}_i}}(N_{\mathfrak{M}_i}) = k_i E_{M_{\mathfrak{M}_i}}(N_{\mathfrak{M}_i}) + N_{\mathfrak{M}_i}$, then $RE_M(N) = kE_M(N) + N$ where $k = \sum_{i=1}^{n} k_i$. Hence if R is semi-local, then* $\operatorname{edim} R \leq \sum_{\mathfrak{M} \in Max(R)} \operatorname{edim} R_{\mathfrak{M}}$.

Proof Let $x \in R\mathrm{E}(N)$. Then

$$\frac{x}{1} \in R\mathrm{E}\left(N_{\mathfrak{M}_i}\right) = k_i \mathrm{E}\left(N_{\mathfrak{M}_i}\right) + N_{\mathfrak{M}_i} = \left(k_i \mathrm{E}(N) + N\right)_{\mathfrak{M}_i}.$$

Therefore there is an $s_i \in R \setminus \mathfrak{M}_i$, such that $s_i x \in k_i \mathrm{E}(N) + N$ (note that although $k_i \mathrm{E}(N) + N$ is not necessarily a submodule, it is closed under multiplication from R).

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If $\mathfrak{I} = \langle s_1, s_2, \dots, s_n \rangle + (N{:}M)$, then $\mathfrak{I}$ is not contained in any maximal ideal of R. Hence for some r_i in R and $r \in (N{:}M)$, we have $1 = r + \sum_{i=1}^{n} r_i s_i$. Now

$$x = rx + \sum_{i=1}^{n} r_i s_i x \in N + \sum_{i=1}^{n} k_i \mathrm{E}(N) = k\mathrm{E}(N) + N,$$

as required. ■

Corollary 2.17 *If R is a finite dimensional semi-local arithmetic ring (resp., Prüfer domain) with maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_n$, then R satisfies the s.r.f. of degree $n + \sum_{i=1}^{n} \operatorname{ht}(\mathfrak{M}_i)$ (resp., degree $\sum_{i=1}^{n} \operatorname{ht}(\mathfrak{M}_i)$).*

Proof The proof is an immediate consequence of Proposition 2.16, Theorem 2.15, and the well-known fact that each localization of an arithmetic ring at a maximal ideal is a chained ring. ■

By Corollary 2.8 a local Artinian ring with maximal ideal $\mathfrak{M}$ satisfies the s.r.f. of degree k and not of $k-1$, where k is the minimum number of generators of $\mathfrak{M}$. Thus one can apply Proposition 2.16 to Artinian rings to show that they satisfy the s.r.f. of degree k for some $k \in \mathbb{N}^*$; but more can be proved.

The proof of the following lemma is easy and it is similar to that of [4, Lemma 2.7].

Lemma 2.18 *Let $R = R_1 \times R_2$, where R_1 and R_2 are rings. Then for the ring R, $\operatorname{edim} R = \max\{\operatorname{edim} R_1, \operatorname{edim} R_2\}$.*

Theorem 2.19 *Let R be an Artinian ring. Then*

$$\operatorname{edim} R = \max\Big\{ \dim_{\frac{R}{\mathfrak{M}}} \frac{\mathfrak{M}}{\mathfrak{M}^2} \;\Big|\; \mathfrak{M} \in \max(R) \Big\}$$

and R satisfies the s.r.f. of degree $k = \operatorname{edim} R$.

Proof Let $\mathfrak{M}_1, \dots, \mathfrak{M}_n$ be all of the maximal ideals of R. According to [1, Theorem 8.7], $R \cong \prod_{i=1}^{n} R_i$, where the Artinian local ring R_i is $R/\mathfrak{M}_i^{k_i}$ for some $k_i \in \mathbb{N}$. Hence by the previous lemma and the fact that every zero dimensional ring satisfies the r.f., we just need to show that $\operatorname{edim} R_i = \dim_{R/\mathfrak{M}_i} \mathfrak{M}_i/\mathfrak{M}_i^2$.

If we set $\overline{\mathfrak{M}}_i = \mathfrak{M}_i/\mathfrak{M}_i^{k_i}$ to be the maximal ideal of R_i, then $R/\mathfrak{M}_i \cong R_i/\overline{\mathfrak{M}}_i$ and $\mathfrak{M}_i/\mathfrak{M}_i^2 \cong \overline{\mathfrak{M}}_i/\overline{\mathfrak{M}}_i^2$. If $\dim_{R_i/\overline{\mathfrak{M}}_i} \overline{\mathfrak{M}}_i/\overline{\mathfrak{M}}_i^2 = 0$, then $\overline{\mathfrak{M}}_i = \overline{\mathfrak{M}}_i^2$, and by Nakayama's Lemma, $\overline{\mathfrak{M}}_i = 0$, whence R_i is a field and $\operatorname{edim} R_i = 0$. Now assume that

$$\dim_{R_i/\overline{\mathfrak{M}}_i} \frac{\overline{\mathfrak{M}}_i}{\overline{\mathfrak{M}}_i^2} \neq 0.$$

In this case, by [1, Proposition 2.8], $\dim_{R_i/\overline{\mathfrak{M}}_i} \overline{\mathfrak{M}}_i/\overline{\mathfrak{M}}_i^2$ is equal to the number of generators of $\overline{\mathfrak{M}}_i$, which according to Corollary 2.7 is $\operatorname{edim} R_i$, as required. ■

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The following theorem shows that, at least for Noetherian rings, the condition that $\dim R = \operatorname{edim} R$ is a strong condition. Recall that in commutative algebra, a Noetherian local ring of dimension n is called regular if its maximal ideal can be generated by n elements.

Theorem 2.20 *Let R be a Noetherian ring. Then $\dim R = \operatorname{edim} R < \infty$ if and only if either R is a finite direct product of fields or is a ZPI-ring with $\dim R = 1$.*

Proof If R is a one dimensional ZPI-ring, then by Corollary 2.12 and Theorem 2.9, $1 = \dim R \leq \operatorname{edim} R \leq 1$. Also, if R is a finite direct product of fields, then the result follows by Lemma 2.18.

Conversely suppose that $\dim R = \operatorname{edim} R < \infty$. Then for some maximal ideal $\mathfrak{M}$ of R, $\dim R = \operatorname{ht} \mathfrak{M}$. For any such $\mathfrak{M}$ we have $\dim R_{\mathfrak{M}} \leq \operatorname{edim} R_{\mathfrak{M}} \leq \operatorname{edim} R = \dim R = \dim R_{\mathfrak{M}}$, and hence $\dim R_{\mathfrak{M}} = \operatorname{edim} R_{\mathfrak{M}}$. Since $R_{\mathfrak{M}}$ is Noetherian, $\mathfrak{M}_{\mathfrak{M}}$ is finitely generated, say $\mathfrak{M}_{\mathfrak{M}} = \langle a_1, \dots, a_n \rangle$. If $n > \dim R_{\mathfrak{M}} = \operatorname{edim} R_{\mathfrak{M}}$, then according to Theorem 2.5, $\mathfrak{M}_{\mathfrak{M}}$ can be generated by $n-1$ elements. Therefore we can assume that $n = \dim R_{\mathfrak{M}}$ and $R_{\mathfrak{M}}$ is regular.

Suppose that $n > 1$. Then $R_{\mathfrak{M}}$ has infinitely many height one prime ideals. (Otherwise, if $\mathfrak{P}_1, \dots, \mathfrak{P}_t$ are the height one primes of $R_{\mathfrak{M}}$, then by the Principal Ideal Theorem, every $x \in \mathfrak{M}_{\mathfrak{M}}$ is in some $\mathfrak{P}_i$ and hence $\mathfrak{M}_{\mathfrak{M}} \subseteq \bigcup_{i=1}^{t} \mathfrak{P}_i$. Consequently, $\mathfrak{M}_{\mathfrak{M}} \subseteq \mathfrak{P}_i$ for some i and $\dim R \leq 1$, which is a contradiction.) Also, according to the Auslander–Buchsbaum theorem ([10, Theorem 20.3]) every regular local ring is a unique factorization domain (UFD). So every height one prime of $R_{\mathfrak{M}}$ is principal and $R_{\mathfrak{M}}$ has infinitely many prime elements. In particular, $R_{\mathfrak{M}}$ has a prime element a_{n+1} that is not associated with a_i for each $1 \leq i \leq n$. Note that since $R_{\mathfrak{M}}$ is regular local and $\{a_1, \dots, a_n\}$ is a minimal generating set for its maximal ideal, each a_i is prime.

Now apply Theorem 2.5(ii) to a_i s and $\alpha = 2$ to get $r_1, \dots, r_n$, c_{ij} s and β as in Theorem 2.5(ii). In particular we have

$$(**) \qquad r_j^{\beta} c_{ij} \in \langle a_i^2 \rangle$$

for all i, j s. Since $a_1, \dots, a_n \in \langle r_1, \dots, r_n \rangle \subseteq \mathfrak{M}_{\mathfrak{M}}$, $r_1, \dots, r_n$ form a minimal generating set for $\mathfrak{M}_{\mathfrak{M}}$, and therfore are prime. Assume that there is an i such that for all j, $r_j \notin \langle a_i \rangle$. Then since $R_{\mathfrak{M}}$ is a UFD and r_j s, and a_i are prime, from $(**)$ we deduce that for all j we have $c_{ij} \in \langle a_i^2 \rangle$. Thus $a_i \in \langle c_{i1}, \dots, c_{in} \rangle \subseteq \langle a_i^2 \rangle$. That is, a_i is a unit, which is a contradiction.

Therefore, we can assume that for each i there is a j_i such that $r_{j_i} \in \langle a_i \rangle$. Because the a_i s and r_i s are primes we deduce that $\langle a_i \rangle = \langle r_{j_i} \rangle$. Since $1 \leq i \leq n+1$ and $1 \leq j_i \leq n$, there must exist $i_1 \neq i_2$ such that $\langle a_{i_1} \rangle = \langle a_{i_2} \rangle$, which is contrary with the choice of the a_i s. From this contradiction we deduce that $\dim R_{\mathfrak{M}} = \dim_R \leq 1$.

If $\dim R = \operatorname{edim} R = 0$, then it follows Theorem 2.19 that R is a finite direct product of fields. Thus we suppose that $\dim R = 1$. By the above argument, for each height one maximal ideal $\mathfrak{M}$ of R, $R_{\mathfrak{M}}$ is regular local and hence a discrete valuation domain. Also, if $\mathfrak{M}$ is a height zero maximal ideal of R, then $\operatorname{edim} R_{\mathfrak{M}} \leq \operatorname{edim} R = 1$, so $\mathfrak{M}_{\mathfrak{M}}$ must be principal; that is, $R_{\mathfrak{M}}$ is an SPIR. Consequently, localization of R at each maximal ideal is a ZPI-ring and due to Lemma 2.11, R itself is a ZPI-ring with $\dim R = 1$, as required. ■

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Finally, we present an example of a ring that satisfies the r.f., but does not satisfy the s.r.f. of degree k for any $k \in \mathbb{N}^*$.

Example 2.21 Let $S_n = \mathbb{Z}_2[x_1, x_2, \dots, x_n]$ and let $\mathfrak{I}_n$ be the ideal of S_n generated by $\{x_i^2, x_i x_j \mid 1 \le i \ne j \le n\}$. Set $R_n = \frac{S_n}{\mathfrak{I}_n}$ and $R = \prod_{n=1}^{\infty} R_n$. Then R satisfies the r.f., however $\operatorname{edim} R = \infty$.

Proof First note that R_n is zero dimensional local with maximal ideal $\mathfrak{M}_n = \langle x_1, \dots, x_n \rangle / \mathfrak{I}_n$, which can clearly be generated by n elements but not by less. Therefore, according to Lemma 2.18, $\operatorname{edim} R \ge \operatorname{edim} R_n = n$, by Corollary 2.7. Thus $\operatorname{edim} R = \infty$.

On the other hand, note that if $r \in R_n$, then either $r = 1$ or $r^2 = 0$, hence always $r^4 = r^2$. Consequently, the same equation holds for every $r \in R$, which implies that $\dim R = 0$ by [7, Theorem 3.1], so R satisfies the r.f. by [14, Theorem 2.8]. ■

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