NOTE ON THE REGION OF OVERCONVERGENCE OF DIRICHLET SERIES WITH OSTROWSKI GAPS

by J. P. EARL and J. R. SHACKELL

(Received 4 March, 1965)

1. The main object of this note is to show that a proof given by A. J. Macintyre [2] of a result on the overconvergence of partial sums of power series works more easily in the context of Dirichlet series. Applying this observation to the particular Dirichlet series $\sum a_n e^{-ns}$, we can remove certain restrictions which Macintyre finds necessary in the direct treatment of power series.

We consider a Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \qquad (s = \sigma + it),$$

where $\lambda_n = \mu_n + i\nu_n$ (μ_n and ν_n real), with μ_n increasing and tending to infinity and $\nu_n = o(\mu_n)$. We assume that the series has a finite abscissa of absolute convergence, which we may take to be $\sigma = 0$. Our main result is then

THEOREM 1. *Suppose* (i) *that* $f(s) = \sum a_n e^{-\lambda_n s}$ *has abscissa of absolute convergence* $\sigma = 0$, *and is continuable in some neighbourhood of the origin throughout the angle* $\phi_1 < \arg s < \phi_2$; i.e. *in the region* $0 < |s| < \delta$, $\phi_1 < \arg s < \phi_2$ *for some* $\delta > 0$, *with* $-\frac{3}{2}\pi < \phi_1 \leqq -\frac{1}{2}\pi$, $\frac{1}{2}\pi \leqq \phi_2 < \frac{3}{2}\pi$; (ii) $\lambda_n = \mu_n + i\nu_n$, *where* μ_n *increases and tends to infinity and* $\nu_n = o(\mu_n)$; (iii) *there exists an increasing sequence of integers* $\{n_k\}$, *where* $n_k \to \infty$ *as* $k \to \infty$, *such that*

$$\frac{\mu_{n_k+1}}{\mu_{n_k}} \geqq 1 + h,$$

where $h > 0$. *Then, if* ϕ_1', ϕ_2' *are angles satisfying*

$$\phi_1 < \phi_1' < \phi_2' < \phi_2,$$

there exists a neighbourhood of the origin in which

$$\sum_{p=1}^{n_k} a_p e^{-\lambda_p s} \to f(s), \quad \text{as} \quad k \to \infty$$

throughout the angle $\phi_1' \leqq \arg s \leqq \phi_r'$.

If $1/\lambda_n = o(1/\log n)$, then the abscissa of absolute convergence coincides with that of convergence.

Cases of particular interest occur when there is an easily approachable, or a virtually isolated, singularity at the origin.

M

2. Proof of Theorem 1. Write

$$S_n(s) = \sum_{p=1}^{n} a_p e^{-\lambda_p s},$$

and

$$R_n(s) = f(s) - S_n(s).$$

We now obtain some estimates for $|R_n(s)|$.

LEMMA 1. *If $\bar{D}$ is any compact subset of the domain of continuability of $f(s) = \sum a_p e^{-\lambda_p s}$ and $\gamma > 0$, then for every $\varepsilon > 0$ there exists $n_0(\varepsilon, \bar{D})$, such that, if $n \geqq n_0$,*

(i) $\{\log |R_n(s)|\}/\mu_n \leqq -\sigma + \varepsilon$, *for* $\sigma \leqq 0$, s in $\bar{D}$,

(ii) $\{\log |R_n(s)|\}/\mu_n \leqq \varepsilon$, *for* $\sigma > 0$, s in $\bar{D}$,

(iii) $\{\log |R_n(s)|\}/\mu_{n+1} \leqq -\sigma + \varepsilon$, *for* $\sigma \geqq \gamma > 0$.

Proof. Case (i): $\sigma \leqq 0$. Suppose that $|t| \leqq T$ in $\bar{D}$ and define

$$\omega_n = \sup_{1 \leqq p \leqq n} |\nu_p|.$$

Then $\omega_n = o(\mu_n)$,

$$|S_n(s)| \leqq e^{\omega_n |t| - \mu_n \sigma} \sum_{p=1}^{n} |a_p|,$$

and since $\sum a_p e^{-\lambda_p s}$ is absolutely convergent for every $\sigma > 0$ we have, with $\varepsilon_1 = \frac{1}{4}\varepsilon$,

$$e^{-\mu_n \varepsilon_1} \sum_{p=1}^{n} |a_p| \leqq \sum_{p=1}^{n} |a_p| e^{-\mu_p \varepsilon_1} \leqq K(\varepsilon).$$

Therefore, for $n \geqq n_1(\varepsilon)$,

$$\frac{1}{\mu_n} \log \left(\sum_{p=1}^{n} |a_p| \right) \leqq 2\varepsilon_1,$$

and

$$\frac{1}{\mu_n} \log |S_n(s)| \leqq -\sigma + 2\varepsilon_1 + \frac{\omega_n T}{\mu_n}$$

$$\leqq -\sigma + 3\varepsilon_1,$$

for $n \geqq n_2(\varepsilon)$. Now write

$$M = \sup_{\bar{D}} \{|f(s)|\}.$$

Then

$$\{\log |R_n(s)|\}/\mu_n \leqq \{\log (M + |S_n(s)|)\}/\mu_n \leqq -\sigma + 4\varepsilon_1 = -\sigma + \varepsilon,$$

for $n \geqq n_3(\varepsilon, \bar{D})$.

Case (ii): $0 < \sigma < \gamma$. Let $\varepsilon_1 = \frac{1}{3}\varepsilon$. We have

$$e^{-\mu_n \varepsilon_1} \left| \sum_{p=1}^{n} a_p e^{-\lambda_p s} \right| \leqq \sum_{p=1}^{n} \left| a_p e^{-\mu_p (s+\varepsilon_1) - i\nu_p s} \right|$$
$$\leqq e^{\omega_n T} \sum_{p=1}^{n} |a_p| e^{-\mu_p \varepsilon_1}$$
$$\leqq e^{\omega_n T} K(\varepsilon).$$

Therefore

$$\left| \sum_{p=1}^{n} a_p e^{-\lambda_p s} \right| \leqq K(\varepsilon) \exp\{\omega_n T + \mu_n \varepsilon_1\}$$

and, for $n \geqq n_4(\varepsilon, \bar{D})$,

$$\{\log |R_n(s)|\}/\mu_n \leqq 3\varepsilon_1 = \varepsilon:$$

Case (iii): $\sigma \geqq \gamma > 0$. Choose $\varepsilon < \frac{1}{2}\gamma$, $\varepsilon_2 = \frac{1}{2}\varepsilon$. Since

$$R_n(s) = \sum_{p=n+1}^{\infty} a_p e^{-\lambda_p s},$$

$$|R_n(s)| \leqq e^{-\mu_{n+1}\sigma} \sum_{p=n+1}^{\infty} |a_p e^{-i\nu_p s} e^{(\mu_{n+1} - \mu_p)s}|$$
$$\leqq e^{-\mu_{n+1}\sigma} \sum_{p=n+1}^{\infty} |a_p e^{i\nu_p s}| e^{(\mu_{n+1} - \mu_p)\varepsilon_2}$$
$$\leqq e^{-\mu_{n+1}(\sigma - \varepsilon_2)} \sum_{p=n+1}^{\infty} |a_p| e^{\nu_p t - \mu_p \varepsilon_2}$$
$$\leqq K(\varepsilon) e^{-\mu_{n+1}(\sigma - \varepsilon_2)}.$$

Therefore, for $n \geqq n_5(\varepsilon)$,

$$\{\log |R_n(s)|\}/\mu_{n+1} \leqq -\sigma + 2\varepsilon_2 = -\sigma + \varepsilon.$$

This completes the proof of the lemma.

We show that, if $\phi_2' < \phi_2$, then $S_{n_k}(s) \to f(s)$ in some neighbourhood of the origin, throughout the angle $-\frac{1}{2}\pi < \arg s \leqq \phi_2'$. A similar argument shows that, if $\phi_1' > \phi_1$, $S_{n_k}(s) \to f(s)$ in some neighbourhood of the origin throughout the angle $\phi_1' \leqq \arg s < \frac{1}{2}\pi$. If $\phi_2 \leqq \frac{1}{2}\pi$ we have nothing to prove, and hence we may assume that $\phi_2 > \frac{1}{2}\pi$.

There exists a sequence $\{n_k\}$ such that $\mu_{n_k+1}/\mu_{n_k} \geqq 1+h$, where h is a positive constant. From Lemma 1 we have, for every $\gamma > 0$,

(i) $\{\log |R_{n_k}(s)|\}/\mu_{n_k} \leqq -\sigma + \varepsilon_{n_k}$, for $\sigma \leqq 0$, s in $\bar{D}$,

(ii) $\{\log |R_{n_k}(s)|\}/\mu_{n_k} \leqq \varepsilon_{n_k}$, for $\sigma > 0$, s in $\bar{D}$,

(iii) $\{\log |R_{n_k}(s)|\}/\mu_{n_k} \leqq -(1+h)\sigma + \varepsilon_{n_k}$, for $\sigma \geqq \gamma > 0$,

where $\varepsilon_{n_k} \to 0$ as $k \to \infty$. Now $f(s)$ is regular in some neighbourhood of the origin throughout an angle $-\frac{1}{2}\pi < \arg s < \phi_2$, where $\frac{1}{2}\pi < \phi_2 < \frac{3}{2}\pi$. Then we may choose $\delta > 0$ such that $f(s)$

is regular in $0 < |s| < 2\delta$, $-\frac{1}{2}\pi < \arg s < \phi_2$. Hence, for every η in $0 < \eta < \delta$, $f(s)$ is regular in the region $\bar{D}_\eta$, where D_η is the half disc

$$|s-i\eta| < \delta, \quad \phi_2-\pi < \arg(s-i\eta) < \phi_2.$$

Let Γ_η be the boundary of D_η. Now $\{\log|R_{n_k}(s)|\}/\mu_{n_k}$ is a subharmonic function; hence if $u_{n_k}(s)$ is function harmonic in D_η and taking on Γ_η boundary values $-\sigma+\varepsilon_{n_k}$ for $\sigma \leqq 0$, ε_{n_k} for $0 < \sigma < \gamma$, and $-(1+h)\sigma+\varepsilon_{n_k}$ for $\sigma \geqq \gamma$, then $\{\log|R_{n_k}(s)|\}/\mu_{n_k} \leqq u_{n_k}(s)$ in D_η. As $k \to \infty$, $\varepsilon_{n_k} \to 0$ and $u_{n_k}(s) \to u(s)$, where $u(s)$ is harmonic in D_η and takes on Γ_η the boundary values $-\sigma$ for $\sigma \leqq 0$, 0 for $0 < \sigma < \gamma$, and $-(1+h)\sigma$ for $\sigma \geqq \gamma$. Also in D_η

$$\limsup_{k\to\infty} \frac{1}{\mu_{n_k}} \log|R_{n_k}(s)| \leqq u(s).$$

If $u(s) < 0$, then $R_{n_k}(s) \to 0$ and so $S_{n_k}(s) \to f(s)$.

We can take γ as small as we please, and therefore $u(s)$ may be taken to differ by as little as we please from $v(s)$, where $v(s)$ is harmonic in D_η and takes on Γ_η the boundary values $-\sigma$ for $\sigma \leqq 0$ and $-(1+h)\sigma$ for $\sigma > 0$.

We now take a new variable $z = re^{i\theta} = (s-i\eta)e^{i(\phi_2-\pi)}$. Consider $\operatorname{Im}\{z\log z\}$, which is harmonic for $\operatorname{Im} z > 0$ and takes on $y = \operatorname{Im} z = 0$ the boundary values πx for $x = \operatorname{Re} z < 0$ and 0 for $x \geqq 0$.

Let $v^*(z) = v(s)$. Consider

$$g(z) = v^*(z)-(h/\pi)\cos(\phi_2-\pi)\operatorname{Im}\{z\log z\}+(1+h)x\cos(\phi_2-\pi),$$

which is harmonic for s in D_η and zero on $y = 0$. Hence for s in D_η,

$$g(re^{i\theta}) = \frac{1}{2\pi}\int_0^\pi g(\delta e^{i\theta}) \frac{4(\delta^2-r^2)\,\delta r\sin\theta\sin\phi}{[\delta^2-2\delta r\cos(\theta-\phi)+r^2][\delta^2-2\delta r\cos(\theta+\phi)+r^2]}\,d\phi,$$

and therefore

$$|g(re^{i\theta})| \leqq r\sin\theta \,.\, \frac{2(\delta+r)}{(\delta-r)^3} \sup_{0\leqq\phi\leqq\pi} |g(\delta e^{i\phi})|$$

$$\leqq K(r_0)r\sin\theta$$

for $r \leqq r_0 < \delta$. Therefore

$$\begin{aligned} v^*(re^{i\theta}) &= H(r,\theta)r\sin\theta+(h/\pi)\cos(\phi_2-\pi)\operatorname{Im}\{re^{i\theta}\log(re^{i\theta})\}-(1+h)r\cos\theta\cos(\phi_2-\pi), \\ &= r\{H(r,\theta)\sin\theta+(h/\pi)\cos(\phi_2-\pi)\sin\theta\log r+(\theta h/\pi)\cos\theta\cos(\phi_2-\pi) \\ &\qquad -(1+h)\cos\theta\cos(\phi_2-\pi)\}, \end{aligned}$$

where $H(r, \theta)$ is bounded.

Now $\log r \to -\infty$ as $r \to 0+$. Hence for every θ in $0 < \theta < \pi$ there exists $r_0(\theta)$ such that, if $r \leqq r_0(\theta)$, $v^*(re^{i\theta}) < 0$. Thus v is negative in a region R_η, fixed with respect to D_η, the boundary of which touches Γ_η at $z = 0$, and $R_{n_k}(s) \to 0$ as $k \to \infty$ at any point of R_η.

Let $\eta \to 0$; then $R_\eta \to R$, where R is a region whose boundary touches the line $\arg s = \phi_2$ at $s = 0$. This is sufficient to establish Theorem 1.

3. As an almost immediate consequence of Theorem 1 we have

THEOREM 2. (Bourion [**1**]) *If* $\sum a_n z^n = f(z)$ *is a power series with radius of convergence unity such that*

$$a_n = 0 \quad for \quad n_k < n \leqq N_k,$$

where $\{n_k\}$ *and* $\{N_k\}$ *are two sequences of integers such that*

$$\frac{N_k}{n_k} \geqq 1 + h > 1$$

and if $f(z)$ *is regular near* $z = 1$ *for* $-\alpha_1 < \arg(1-z) < \alpha_2$, *then the sequence* $\{S_{n_k}(z)\}$ *of partial sums converges to* $f(z)$ *in some neighbourhood of* $z = 1$ *in the angle* $-\beta_1 \leqq \arg(1-z) \leqq \beta_2$, *provided that* $-\alpha_1 < -\beta_1 < \frac{1}{2}\pi$ *and* $\frac{1}{2}\pi < \beta_2 < \alpha_2$.

As mentioned in the introduction, this follows from consideration of the Dirichlet series $\sum a_n e^{-ns}$ and the conformal map $z = e^{-s}$.

Macintyre's work concerned only the case $\alpha_1 > \pi$, $\alpha_2 > \pi$, and he showed that there exist angles γ_1 and γ_2, each depending on the value of h, satisfying

$$\alpha_1 > \gamma_1 > \pi, \quad \alpha_2 > \gamma_2 > \pi,$$

such that $S_{n_k}(z) \to f(z)$ in some neighbourhood of $z = 1$ throughout the angle

$$-\gamma_1 < \arg(1-z) < \gamma_2.$$

4. We note that Theorems A and B of [**2**] can be established under slightly weaker conditions by using Bourion's result. Instead of requiring $f(z)$ to be continuable across the real axis $z > 1$, from the upper half-plane into a definite angle $0 > \arg(z-1) > -(\alpha_1 - \pi)$ of the lower half-plane, and similarly from the lower half-plane into an equal angle of the upper half-plane, all that is needed is that the regions of continuability overlap in a neighbourhood of $z = 1$ throughout some definite angle outside $|z| \leqq 1$.

Finally we wish to express our gratitude to M. E. Noble for his help and guidance, to the referee and editor of these *Proceedings* for their help in the presentation of this note, and to D.S.I.R. for financial support during the period involved.

REFERENCES

1. G. Bourion, *L'ultraconvergence dans les séries de Taylor*, Actualités Scientifiques et Industrielles no. 472 (Paris 1937).

2. A. J. Macintyre, An overconvergence theorem of G. Bourion and its application to coefficients of certain power series, *Ann. Acad. Sci. Fenn. Ser. A.I.* **250/23** (1958).

UNIVERSITY OF KENT AT CANTERBURY