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Connected sums of codimension two locally flat submanifolds

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Let X and Y be oriented topological manifolds of dimension $n+2$, and let $K \subset X$ and $J \subset Y$ be connected, locally-flat, oriented, n–dimensional submanifolds. We show that up to orientation preserving homeomorphism there is a well-defined connected sum $(X, K)\#(Y, J)$. For $n = 1$, the proof is classical, relying on results of Rado and Moise. For dimensions $n = 3$ and $n \geqslant 6$, results of Edwards-Kirby, Kirby, and Kirby-Siebenmann concerning higher dimensional topological manifolds are required. For $n = 2, 4$, and 5, Freedman and Quinn's work on topological four-manifolds is required along with the higher dimensional theory.

Keywords: Connected sum; Locally flat

1. Introduction

In the smooth category, the fact that the connected sum of manifolds is well-defined depends on a result first proved by Cerf [7] and Palais [18]. In the topological category the result is much deeper. For $n \geqslant 6$, the theorem that the connected sum of topological n–manifolds is well-defined is a consequence of the Annulus Conjecture, proved by Kirby [14]. In dimensions $n = 4$ and $n = 5$ the proof relies on Freedman and Quinn's work concerning topological 4–manifolds [10, 11].

For smooth manifolds, proving that the connected sum of submanifolds is well-defined is also a consequence of the result proved by Palais and by Cerf; a summary is presented in an appendix to this paper. Proving that connected sums of locally flat n–dimensional submanifolds of topological manifolds of dimension $n + 2$ is well-defined is more challenging: the proof presented here relies on the existence and uniqueness of normal bundles in codimension two, results that call on the s–cobordism theorem with fundamental group $\mathbb{Z}$, proved by Kirby-Siebenmann [16, § III, 3.4] for higher dimensions and appearing in [11, theorem 7.1A] for cobordisms of dimension five.

Cappell and Shaneson [6, proposition, Page 34] briefly sketched a proof of the key result that is needed in showing that connected sums are well-defined for $n \geqslant 3$ in the special case of knotted spheres in S^n. The proof here roughly follows their approach, but it addresses some issues that were not considered in [6]. In particular, the cases of $n = 2, n = 4$ and $n = 5$ require the use of results of Freedman-Quinn. In addition, at the time [6] was written, details of the necessary topological manifold theory had not been published.

Conventions. Manifolds of dimension n are second countable Hausdorff spaces with the property that every point has a neighbourhood homeomorphic to $\mathbb{R}^n$. A k–submanifold N of an n–manifold M is a pair (M, N) of manifolds, $N \subset M$, that is locally modelled on $(\mathbb{R}^k \times \mathbb{R}^{n-k}, \mathbb{R}^k \times \{0\})$. We will refer to these as *locally-flat* submanifolds to emphasize the structure. We will work exclusively with manifolds that are oriented. As will be made precise in § 3, we will show that if F_1 and F_2 are connected, n–dimensional submanifolds of $(n+2)$–manifolds W_1 and W_2, then $(W_1, F_1)\#(W_2, F_2)$ is a well-defined homeomorphism class of topological pairs.

The results here extend to the case of interior connected sums of manifold pairs with boundary. We do not consider the case of boundary connected sums of manifold pairs.

References. The key background material for our work is in the books by Kirby-Siebenmann [16] and by Freedman-Quinn [11]. An early version of the current paper that included an appendix with an overview of that material is available at [17]. The book [1] provides a detailed exposition of the foundations of topological four-manifold theory.

2. Connected sums of manifolds

Initially we want to work with manifolds, not with homeomorphism classes of manifolds. Constructions of spaces, such as connected sums, are built from quotient spaces and these are typically not well-defined spaces, but rather are defined up to homeomorphism. To remedy this, we will assume that one has chosen a convention so that the disjoint union of spaces is a well-defined space. With this, adjunction spaces are well-defined as spaces.

Let $n > 0$ and let W_1 and W_2 be oriented connected $(n+2)$–manifolds. Choose embeddings $\phi\colon \mathbb{R}^{n+2} \to W_1$ and $\psi\colon \mathbb{R}^{n+2} \to W_2$, where ϕ is orientation preserving and ψ is orientation reversing. The connected sum of manifolds is defined as

$$W_1 \#_{\phi,\psi} W_2 := \big(\big(W_1 \backslash \mathrm{Int}(\phi(B^{n+2}))\big) \sqcup \big(W_2 \backslash \mathrm{int}(\psi(B^{n+2}))\big)\big) \sim .$$

The equivalence relation identifies $\phi(\theta)$ with $\psi(\theta)$ for $\theta \in S^{n+1}$. There is a canonical embedding $\phi'\colon S^{n+1} \to W_1 \#_{\phi,\psi} W_2$, yielding a pair that we denote $(W_1 \#_{\phi,\psi} W_2, \phi'(S^{n+1}))$. We leave the following result to the reader.

THEOREM 2.1. *The space $W_1 \#_{\phi,\psi} W_2$ is an oriented topological manifold. The natural inclusions of $W_1 \backslash Int(\phi(B^{n+2}))$ and $W_2 \backslash Int(\psi(B^{n+2}))$ into $W_1 \#_{\phi,\psi} W_2$ are orientation preserving embeddings. The pair $(W_1 \#_{\phi,\psi} W_2, \phi'(S^{n+1}))$ is locally flat.*

The following result implies that the connected sum of connected oriented topological manifolds W_1 and W_2 is a well-defined homeomorphism class, usually denoted $W_1 \# W_2$. Notice that in the following theorem, we are working with manifolds, not homeomorphism classes.

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THEOREM 2.2. *Given embeddings ϕ_1, ϕ_2, ψ_1 and ψ_2 as above, there exists an orientation preserving homeomorphism* $(W_1 \#_{\phi_1,\psi_1} W_2, \phi_1'(S^{n+1})) \to (W_1 \#_{\phi_2,\psi_2} W_2, \phi_2'(S^{n+1}))$.

Proof. We focus on ϕ_1 and ϕ_2. Here is a summary of the key steps.

(1) There exists a homeomorphism $F\colon W_1 \to W_1$ for which $F(\phi_2(0)) = \phi_1(0)$. Using this, we can assume that $\phi_2(0) = \phi_1(0)$.

(2) By composing with another homeomorphism of W_1, we can also arrange that $\phi_2(B^{n+2}) \subset \mathrm{Int}(\phi_1(B^{n+2}))$.

(3) The Annulus Conjecture permits us to arrange that $\phi_2(B^{n+2}) = \phi_1(B^{n+2})$. (The Annulus Conjecture was proved by Kirby [**14**] for $n + 2 \geqslant 5$; in dimension four it was proved by Quinn [**19**, § 2.2]. See Edwards [**9**] for a survey.)

(4) We need to arrange that ϕ_1 and ϕ_2 agree on S^{n+1}. Consider $F\colon S^{n+1} \to S^{n+1}$ defined by $\phi_2^{-1} \circ \phi_1$. The truth of the Stable Homeomorphism Conjecture implies that F is a composition of maps, each of which is the identity on some non-empty open set. (For $n \geqslant 5$, the Stable Homeomorphism Conjecture was proved by Kirby [**14**]. Quinn's proof of the Annulus Conjecture for $n = 4$ yields a proof of the Stable Homeomorphism Conjecture in that dimension. The relationships between the Annulus Conjecture and the Stable Homeomorphism Conjecture was first identified by Brown and Gluck in a series of papers [**3**–**5**]. In particular, see [**3**, § 9].)
Assuming that F is the identity on a closed ball B in S^{n+1}, we can use the Alexander Trick [**2**] applied to the closure of the complement of B to find an isotopy from F to the identity map.

(5) A tubular neighbourhood of $\phi_1(S^{n+1})$ can be used to extend the isotopy constructed in the previous step to build an isotopy of W_1 that carries ϕ_2 to a new embedding that agrees with ϕ_1 on S^{n+1}, as needed to complete the proof. □

3. Connected sums of pairs

Let $n > 0$ and suppose that F_1 and F_2 are n–dimensional oriented, locally flat, connected submanifolds of $(n+2)$–dimensional oriented manifolds W_1 and W_2. Local flatness ensures that we can choose an orientation preserving embedding $\phi\colon \mathbb{R}^{n+2} \to W_1$ such that $\phi^{-1}(F_1) = \mathbb{R}^n \times \{0\}$. We view such an embedding as a map of pairs: $\phi\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (W_1, F_1)$. Similarly, choose an orientation *reversing* embedding $\psi\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (W_2, F_2)$. We have the unit balls $B^{n+2} \subset \mathbb{R}^{n+2}$ and $B^n \subset \mathbb{R}^n \subset \mathbb{R}^{n+2}$.

The connected sum of pairs $(W_1, F_1) \#_{\phi,\psi} (W_2, F_2)$ is defined as follows.

$$(W_1, F_1) \#_{\phi,\psi} (W_2, F_2) = ((W_1 \backslash \mathrm{Int}(\phi(B^{n+2})), F_1 \backslash \mathrm{Int}(\phi(B^n))) \sqcup (W_2 \backslash \mathrm{int}(\psi(B^{n+2})), F_2 \backslash \mathrm{int}(\psi(B^n))))/\sim.$$

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The equivalence relation identifies $\phi(\theta)$ with $\psi(\theta)$ for $\theta \in S^{n+1}$. The following result is straighforward.

THEOREM 3.1. *Let* $\overline{\phi}$ *and* $\overline{\psi}$ *be the restrictions of* ϕ *and* ψ *to* R^n. *There is a natural inclusion* $F_1 \#_{\overline{\phi},\overline{\psi}} F_2 \subset W_1 \#_{\phi,\psi} W_2$ *as a locally flat submanifold. Via this inclusion there is a homeomorphism*

$$(W_1, F_1) \#_{\phi,\psi} (W_2, F_2) \cong (W_1 \#_{\phi,\psi} W_2, F_1 \#_{\overline{\phi},\overline{\psi}} F_2)$$

Our main result is the following.

THEOREM 3.2. *Given pairs of embeddings,* (ϕ_1, ψ_1) *and* (ϕ_2, ψ_2), *the manifold pairs* $(W_1, F_1) \#_{\phi_1,\psi_1} (W_2, F_2)$ *and* $(W_1, F_1) \#_{\phi_2,\psi_2} (W_2, F_2)$ *are oriented homeomorphic. A homeomorphism can be chosen so that it restricts to be a homeomorphism of the splitting* $(n+1)$*–spheres.*

The proof follows readily from three lemmas. The first is elementary. The second is the deepest, depending on the existence and uniqueness theorems for normal bundles of codimension two submanifolds. The third, though slightly technical, is elementary. In the second two, we change our perspective, viewing $(\mathbb{R}^{n+2}, \mathbb{R}^n)$ as the pair $(\mathbb{R}^n \times \mathbb{R}^2, \mathbb{R}^n \times \{0\})$.

LEMMA 3.3. *Let* $F \subset W$ *be a connected, codimension-two, locally flat submanifold and let* $\phi\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (W, F)$ *and* $\phi'\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (W, F)$ *be embeddings. Then there is an orientation preserving self-homeomorphism of* (W, F) *that carries* ϕ' *to an embedding* $\phi''\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (W, F)$ *for which* $\phi''((B^{n+2}, B^n)) \subset Int(\phi((B^{n+2}, B^n)))$.

Proof. The proof follows readily from the next two observations.

(1) Let a and b be points on F. Then there is an orientation preserving homeomorphism $h\colon (W, F) \to (W, F)$ for which $h(a) = b$. To prove this, consider the set

$$B = \{x \in F \mid \text{there exists an } h\colon (W, F) \to (W, F) \text{ for which } h(a) = x\}.$$

Working locally, one can prove that B is both open and closed.

(2) To ensure that $\phi''((B^{n+2}, B^n)) \subset \mathrm{Int}(\phi((B^{n+2}, B^n)))$ we can again work locally, using the following observation. Let U be an arbitrary neighbourhood of $0 \in \mathbb{R}^{n+2}$. Then there is a homeomorphism $h\colon (\mathbb{R}^{n+2}, \mathbb{R}^n) \to (\mathbb{R}^{n+2}, \mathbb{R}^n)$ for which $h(B^{n+2}) \subset U$ and for which $h(x) = x$ for all x with $\|x\| \geqslant 2$. □

LEMMA 3.4. *Let* $\phi\colon (B^n \times B^2, B^n \times \{0\}) \to Int((B^n \times B^2, B^n \times \{0\}))$ *be an embedding satisfying* $\phi^{-1}(B^n \times \{0\}) = B^n \times \{0\}$. *Assume that* ϕ *extends to an embedding of an open neighbourhood of* $B^n \times B^2 \subset \mathbb{R}^{n+2}$. *Then there is an ambient isotopy of* $(\mathbb{R}^n \times \mathbb{R}^2, \mathbb{R}^n \times \{0\})$ *carrying* ϕ *to an embedding* ϕ' *such that* $\phi'((B^n \times B^2, B^n \times \{0\})) = (B^n \times B^2, B^n \times \{0\})$. *Furthermore, the isotopy can be chosen so that* ϕ' *is of the form* $\phi'(x, y) = (\phi_1(x), \phi_2(x, y))$.

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Proof. The Annulus Conjecture in dimension n implies that the image $\phi(B^n \times \{0\}) \subset \mathrm{Int}(B^n \times \{0\})$ is isotopic (in $\mathbb{R}^n \times \{0\}$) to $B^n \times \{0\}$. We can extend this to an isotopy of $\mathbb{R}^n \times \{0\}$ and then use the product structure to extend this isotopy to $\mathbb{R}^{n+2}$. With this, we can assume that $\phi(B^n \times \{0\}) = B^n \times \{0\}$. Notice that after the isotopy, it is not necessarily the case that $\phi(B^n \times B^2) \subset B^n \times B^2$.

The condition that ϕ has an extension to a neighbourhood in $\mathbb{R}^{n+2}$ then ensures that the image $\phi(B^n \times B^2)$ forms a normal bundle over $B^n \times \{0\}$, which, by the extension theorem for bundles, is a sub-bundle of a normal bundle to $\mathbb{R}^n \times \{0\}$ in $\mathbb{R}^{n+2}$. By the uniqueness theorem for normal bundles, there is a fibre preserving ambient isotopy carrying one bundle to the other. Restricting to the image of ϕ gives the desired result.

The existence and uniqueness results for normal bundles that we used above appear in Kirby-Siebenmann [**15**] for the higher dimensional case and in Freedman-Quinn [**11**, § 9.3] for dimension 4. The necessary isotopy extension result was proved by Edwards-Kirby [**8**, corollaries 1.3 and 1.4]. □

We now assume that $\phi\colon (B^n \times B^2, B^n \times \{0\}) \to (B^n \times B^2, B^n \times \{0\})$ is an orientation preserving homeomorphism of pairs that preserves the product structure in the sense that ϕ can be decomposed as $\phi(x, y) = (\phi_1(x), \phi_2(x, y))$ for functions ϕ_1 and ϕ_2.

LEMMA 3.5. *The map ϕ is isotopic to the identity as a map of pairs.*

Proof. Consider $\phi_1 = \phi\big|_{B^n}$. This is an orientation preserving homeomorphism of B^n. As described in the proof of theorem 2.2, the restriction to the boundary S^{n-1} is isotopic to the identity. By the Alexander trick, this isotopy extends to B^n. The product structure permits us to extend this isotopy to $B^n \times B^2$, and thus we can assume that ϕ_1 is the identity and ϕ is of the form

$$\phi(x, y) = (x, \phi_2(x, y)).$$

The function ϕ_2 defines a map from B^n to the orientation preserving homeomorphism group of the 2–ball fixing the origin, $\psi\colon B^n \to \mathrm{Homeo}_+(B^2, 0)$; that is, $\psi(x)(y) = \phi_2(x, y)$.

A coning construction defines an injection

$$\mathrm{Homeo}_+(S^1) \rightarrowtail \mathrm{Homeo}_+(B^2, 0) \subset \mathrm{Homeo}_+(B^2).$$

The Alexander trick provides a deformation retraction from $\mathrm{Homeo}_+(B^2)$ to the image of $\mathrm{Homeo}_+(S^1)$, and a check of its proof shows that the deformation preserves $\mathrm{Homeo}_+(B^2, 0)$. The space $\mathrm{Homeo}_+(S^1)$ deformation retracts to $SO(2)$ (see [**12**, 4.2] for a proof) and so $\mathrm{Homeo}_+(B^2, 0)$ is path connected. Thus ψ is homotopic to the constant map for which $\psi(x)$ is the identity for all x. This homotopy provides the desired isotopy of ϕ, completing the proof. □

4. Problems

Problem 4.1. For $i = 1$ and 2, let $F_i^n \subset W_i^{n+k}$ be connected oriented locally flat codimension k submanifolds of oriented topological manifolds. Prove that the connect sum $(W_1, F_1) \# (W_2, F_2)$ is well-defined up to homeomorphism. More precisely, prove the higher codimension analogue of theorem 2.2.

There are two relevant observations. The first is Stallings's result [21] that topological knots in S^n of codimension greater than 2 are unknotted. In particular, the connected sum of knotted spheres in S^{n+k} is trivially well-defined if $k \geqslant 3$. On the other hand, normal bundles do not exist in general for higher codimension (see, for instance, [13, 20]) so the proofs presented here cannot be generalized.

Problem 4.2 Relative Annulus Conjecture. Prove the following conjecture. Suppose that $f, g\colon (S^{n+k}, S^n) \to (\mathbb{R}^{n+k+1}, \mathbb{R}^{n+1})$ are disjoint locally flat embeddings for which $f(S^{n+k})$ is in the bounded component of $\mathbb{R}^{n+k+1} \backslash g(S^{n+k})$. Then the submanifold of $(\mathbb{R}^{n+k+1}, \mathbb{R}^{n+1})$ that is bounded by $f(S^{n+k}, S^n)$ and $g(S^{n+k}, S^n)$ is homeomorphic to $(S^{n+k}, S^n) \times [0, 1]$.

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Appendix A. Connected sums of submanifolds in the smooth category

Here we summarize the essential ingredient of the proof of the smooth version of theorem 3.2. The following result is a special case of Theorem [18, lemma 5.1].

Lemma A.1. *Suppose that $U \subset \mathbb{R}^n$ is open set containing the origin and that $\psi\colon U \to \mathbb{R}^n$ is a differentiable map. If $\psi(0) = 0$ and the derivative at 0 satisfies $D\psi_0 = \mathrm{Id}$, then there is a neighbourhood $V \subset U$ of 0 and an isotopy $s \to \phi^s$ of maps of $\mathbb{R}^n$ for which: (1) ϕ^0 is the identify; (2) ϕ^s is the identity off of U for all s; and (3) $\phi^1\big|_V = \psi\big|_V$.*

This is a consequence of [18, lemma 5.2]. Here is a statement with the notation slightly simplified.

Lemma A.2. *Let G a differentiable function from a neighbourhood of the origin in $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying $G(0) = 0$ and with the derivative at 0 satisfying $DG_0 = \mathrm{Id}$. Then for every sufficiently small $r > 0$ there is a differentiable mapping $F\colon \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ for which*

(1) *$F^s\colon x \to F(x, s)$ is a diffeomorphism for all s.*

(2) *F^0 is the identity map.*

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(3) $F^s(x) = x$ *if* $\|x\| > 2r$ *and* $0 \leqslant s \leqslant 1$.

(4) $F^1(x) = G(x)$ *if* $\|x\| \leqslant r$.

(5) $s \to F^s$ *is an isotopy.*

The proof of this result is an explicit construction. First, the function $A(x)$ is defined by $G(x) = x + A(x)$. Then a suitable smooth family of real-valued functions $\sigma_r(x)$ on $\mathbb{R}^n$ is defined. The isotopy $F(x, s)$ is given by the following.

$$F(x, s) = x + s\sigma_r(\|x\|^2)A(x) \quad \text{for } \|x\| \leqslant 2r$$
$$F(x, s) = x \quad \text{for } \|x\| \geqslant 2r$$

The proof that connected sums of submanifolds (of arbitrary codimension) is well-defined depends on relative versions of the previous two lemmas. That is, if ϕ and G in the two statements are function defined on pairs in $(\mathbb{R}^n, \mathbb{R}^k)$, then an isotopy of maps of pairs is required. In this setting the isotopy $F(x, s)$ above provides such a relative isotopy.

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