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A general intersection formula for Lagrangian cycles

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Abstract

We prove a generalization to the context of real geometry of an intersection formula for the vanishing cycle functor, which in the complex context is due to Dubson, Lê, Ginsburg and Sabbah (after a conjecture of Deligne). It is also a generalization of similar results of Kashiwara and Schapira, where these authors work with a suitable assumption about the micro-support of the corresponding constructible complex of sheaves. We only use a similar assumption about the support of the corresponding characteristic cycle so that our result can be formulated in the language of constructible functions and Lagrangian cycles.

Introduction

In this paper we give a proof (of a generalization to the context of o-minimal structures on the real field, or to analytic (Nash) geometric categories) of the following intersection formula.

Theorem 0.1. *Let M be an m-dimensional real analytic manifold and $f : M \to \mathbb{R}$ a subanalytic C^2-function. Consider on M a bounded subanalytically constructible complex $\mathcal{F}$ of sheaves of vector spaces (over a base-field k), with finite-dimensional stalks $\mathcal{F}_x$ ($x \in M$). Suppose that the intersection of $\sigma_f := \{(x, df_x) \in T^*M \mid x \in M\}$ and the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$ is contained in a compact subanalytic subset $I \subset T^*M$, with $K := \pi(I) \subset \{f = 0\}$. Then one has*

$$\chi(R\Gamma(K, R\Gamma_{\{f \geqslant 0\}}\mathcal{F})) = \sharp([df(M)] \cap [\mathrm{CC}(\mathcal{F})]). \tag{1}$$

Here we use the following notations:

i) χ is the usual *Euler characteristic* (note that $R\Gamma(K, R\Gamma_{\{f\geqslant 0\}}\mathcal{F})$ has finite-dimensional cohomology, since $K := \pi(I)$, with $\pi : T^*M \to M$ the natural projection, is a compact subanalytic subset).

ii) $[\mathrm{CC}(\mathcal{F})] \in H^m_{|\mathrm{CC}(\mathcal{F})|}(T^*M, \pi^{-1}\,\mathrm{or}_M)$ is the *characteristic cycle* of $\mathcal{F}$ (as in [KS90, ch. IX], and see the next section), with or_M the *orientation sheaf* on M.

iii) $[df(M)] \in H^m_{\sigma_f}(T^*M, \mathrm{or}_{T^*M/M})$ corresponds to $1 \in H^0(M, k_M) \simeq H^m_{\sigma_f}(T^*M, \mathrm{or}_{T^*M/M})$ (as in [KS90, Definition 9.3.5], with $\pi^!\mathbb{Z}_M \simeq \mathrm{or}_{T^*M/M}[m]$).

iv) The *intersection number* $\sharp$ is defined by

$$H^m_{\sigma_f}(T^*M, \mathrm{or}_{T^*M/M}) \times H^m_{|\mathrm{CC}(\mathcal{F})|}(T^*M, \pi^{-1}\,\mathrm{or}_M) \xrightarrow{\ \cup\ } H^{2m}_c(T^*M, \mathrm{or}_{T^*M}) \xrightarrow{\ \mathrm{tr}\ } \mathbb{Z}.$$

Here we use $\mathrm{or}_{T^*M/M} \otimes_Z \pi^{-1}\,\mathrm{or}_M = \mathrm{or}_{T^*M}$, and the above cup-product is the composition of the usual cup-product with support and the natural maps

$$H^{2m}_{\sigma_f \cap |\mathrm{CC}(\mathcal{F})|}(T^*M, \mathrm{or}_{T^*M}) \to H^{2m}_I(T^*M, \mathrm{or}_{T^*M}) \to H^{2m}_c(T^*M, \mathrm{or}_{T^*M}).$$

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This intersection formula is an important and useful generalization of [Kas85, Theorem 9.1, p. 207] and [KS90, Theorem 9.5.6, p. 386]. Note that we only assume that the intersection of σ_f and the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$ is contained in a compact subanalytic set, and this is the reason why our formula is a result about Lagrangian cycles. The results of [Kas85] and [KS90] are formulated under suitable assumptions on the micro-support of $\mathcal{F}$, which is in general much bigger than $|\mathrm{CC}(\mathcal{F})|$.

Remark 0.1. The characteristic cycle $\mathrm{CC}(\mathcal{F})$ of $\mathcal{F}$ depends only on the constructible function $\alpha : x \mapsto \chi(\mathcal{F}_x)$ (compare with [KS90, Theorem 9.7.11] and [Sch03, § 5.0.3]). Therefore our results can also be stated in terms of the associated constructible function α. In this context, one can only use the support of the characteristic cycle $\mathrm{CC}(\alpha)$ as an invariant of the constructible function α (the micro-support $\mu\mathrm{supp}(\mathcal{F})$ depends on the sheaf complex $\mathcal{F}$, and *not* (!) only on the corresponding constructible function).

Our proof of Theorem 0.1 is a modification of the proof of [KS90, Theorem 9.5.6]. We use the result of [vDM96, Theorem 1.20 and Theorem D.19] (due in the subanalytic context to Bierstone, Milman and Pawlucki), that $K = \{g = 0\}$ is the zero-set of a non-negative subanalytic C^2-function g. Then there exists a relatively compact open neighborhood U of K in M, and a $\delta_0 > 0$ such that

$$g : \{g < \delta_0\} \cap U \to [0, \delta_0[$$

is proper (since $\{g = 0\} = K$ is compact). After restriction to U we can assume $M = U$. Then

$$\chi(R\Gamma(K, R\Gamma_{\{f\geqslant 0\}}\mathcal{F})) = \chi(H^*(\{g \leqslant \delta\}, M_f^-, \mathcal{F})) = \chi(H^*(\{g \leqslant \delta\}, \mathcal{F})) - \chi(H^*(M_f^-, \mathcal{F})) \qquad (2)$$

calculates for $0 < \epsilon \ll \delta \ll \delta_0$ (i.e. for δ and ϵ sufficiently small, with ϵ small compared to δ) the Euler characteristic of the relative cohomology of a 'tube' $\{g \leqslant \delta\}$ modulo the left Milnor fiber

$$M_f^- := \{g \leqslant \delta, f = -\epsilon\} \quad (0 < \epsilon \ll \delta \ll \delta_0)$$

(compare [Sch03, § 1.1]). In terms of the constructible function α, this Euler characteristic can also be rewritten as (compare [KS90, § 9.7] and [Sch03, § 2.3]):

$$\int_{\{g\leqslant\delta\}} \alpha\, d\chi - \int_{M_f^-} \alpha\, d\chi = \int_K \alpha\, d\chi - \int_{M_f^-} \alpha\, d\chi. \qquad (3)$$

Here we use for a compact subanalytic subset $A \subset M$ the notation

$$\int_A \alpha\, d\chi := \int_M 1_A \cdot \alpha\, d\chi, \quad \text{with} \int_M (\cdot)\, d\chi : CF_c(M, \mathbb{Z}) \to \mathbb{Z}$$

the unique $\mathbb{Z}$-linear map on the group $CF_c(M, \mathbb{Z})$ of $\mathbb{Z}$-valued subanalytically constructible functions with compact support such that

$$\int_M 1_B\, d\chi = \chi(H^*(B, \mathbb{Q}))$$

for $B \subset M$ a compact subanalytic subset.

For the special case $f \equiv 0$ we get $\emptyset = M_f^-$ so that Theorem 0.1 implies the following generalization of the classical Poincaré–Hopf index formula, which corresponds to the case $\alpha = 1_M$ for M a compact manifold (with $\sigma_0 = T_M^*M$ the zero-section of T^*M).

Corollary 0.1 (Global index formula). *Consider a constructible complex $\mathcal{F}$ as in Theorem 0.1 and denote by α the corresponding constructible function $x \mapsto \chi(\mathcal{F}_x)$. Assume $\mathcal{F}$ or α has a compact support. Then*

$$\chi(R\Gamma(M, \mathcal{F})) = \sharp([T_M^*M] \cap [\mathrm{CC}(\mathcal{F})]) \quad \textit{or} \quad \int_M \alpha\, d\chi = \sharp([T_M^*M] \cap [\mathrm{CC}(\alpha)]). \qquad (4)$$

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This global index formula goes back to Dubson [Dub84a, Theorem 2, p. 115] in the complex, and to Kashiwara [Kas85, Theorem 8.1, p. 205] in the real context. Other presentations can be found in [Fu94, Theorem 1.5, p. 832], [Gin86, Theorem 9.1, p. 382], [KS90, Corollary 9.5.2, p. 384] and [Sch03, Equation (5.30), p. 291].

Another important special case of our intersection formula (1) is $I = \{\omega\}$ given by a point $\omega \in T^*M$, with $x := \pi(\omega)$ and $K := \{x\}$. In this case we can use for g the usual distance function (in suitable coordinates $(M, x) \simeq (\mathbb{R}^m, 0)$). Then we get (cf. [KS90, Theorem 9.5.6, p. 386])

$$\chi((R\Gamma_{\{f\geqslant 0\}}\mathcal{F})_x) = \sharp_{df_x}([df(M)] \cap [\mathrm{CC}(\mathcal{F})]) \tag{5}$$

and

$$\alpha(x) - \int_{M_f^-} \alpha\, d\chi = \sharp_{df_x}([df(M)] \cap [\mathrm{CC}(\alpha)]), \tag{6}$$

with $\sharp_\omega$ the corresponding *local intersection number* and M_f^- the *local left Milnor fiber* of f in x. We get in particular that

$$\sigma_f \cap |\mathrm{CC}(\mathcal{F})| = \emptyset \quad \Rightarrow \quad \chi((R\Gamma_{\{f\geqslant 0\}}\mathcal{F})_x) = 0. \tag{7}$$

Remark 0.2. In another paper, we will use the formula (7) (together with a specialization result for Lagrangian cycles) for a proof of the non-characteristic pullback formula [KS90, Proposition 9.4.3, p. 378] under the weaker assumption that the map is only non-characteristic with respect to the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$.

As a further example, let us consider (locally) the case $f = r$ a (real analytic) distance function to the point x, i.e. $r \geqslant 0$ with $\{x\} = \{r = 0\}$. By the curve selection lemma there exists an open neighborhood U of x in M, with

$$dr(U) \cap |\mathrm{CC}(\mathcal{F})| \subset \{dr_x\} \quad \text{and} \quad dr(U) \cap |\mathrm{CC}(\alpha)| \subset \{dr_x\}.$$

This will be explained later on in terms of stratification theory (and compare with [KS90, Proposition 8.3.12, p. 332]). So if we work on the manifold U, then we can apply (5) and (6) with $\emptyset = M_r^-$, as follows.

Corollary 0.2 (Local index formula).

$$\chi(\mathcal{F}_x) = \sharp_{dr_x}([dr(U)] \cap [\mathrm{CC}(\mathcal{F})]) \quad \textit{and} \quad \alpha(x) = \sharp_{dr_x}([dr(U)] \cap [\mathrm{CC}(\alpha)]). \tag{8}$$

So (8) describes an inversion formula for reconstructing the constructible function α out of the Lagrangian cycles $[\mathrm{CC}(\mathcal{F})]$ and $[\mathrm{CC}(\alpha)]$ (compare [Dub84a, Theorem 2, p. 115], [Kas85, Theorem 8.3, p. 205], [KS90, Equation (9.5.8), p. 386] and [Sch03, Equation (5.30), p. 291]).

Theorem 0.1 also gives a purely real proof of the corresponding intersection formula in complex geometry, as follows.

Corollary 0.3. *Let M be an m-dimensional complex analytic manifold and $h : M \to \mathbb{C}$ a holomorphic function. Consider on M a bounded complex analytically constructible complex $\mathcal{F}$ of sheaves of vector spaces, with finite-dimensional stalks $\mathcal{F}_x$ ($x \in M$). Suppose that the intersection of $\sigma_h := \{(x, dh_x) \in T^*M \mid x \in M\}$ and the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$ is contained in a compact analytic subset $I \subset T^*M$, with $K := \pi(I) \subset \{h = 0\}$. Then one has*

$$\chi(R\Gamma(K, \phi_h[-1]\mathcal{F})) = \sharp([dh(M)] \cap [\mathrm{CC}(\mathcal{F})]). \tag{9}$$

Here we use the notation T^*M for the *holomorphic cotangent bundle*, and ϕ_h is the *vanishing cycle functor* of Deligne. The holomorphic section dh of T^*M corresponds under the natural isomorphism $T^*M \simeq T^*(M^R)$ to the section $d(re(h))$ of $T^*(M^R)$, with $T^*(M^R)$ the real cotangent

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bundle of the underlying real manifold M^R. If we use the induced complex orientation of $T^*(M^R)$, then the class

$$[df(M^R)] \in H^{2m}_{\sigma_f}(T^*M, \mathrm{or}_{T^*(M^R)/M^R}) \simeq H^{2m}_{\sigma_h}(T^*M, \mathbb{Z})$$

for $f := re(h)$ corresponds under Poincaré duality to the fundamental class in Borel–Moore homology of the complex manifold σ_h.

Then the corollary follows from Theorem 0.1 and the isomorphism (cf. [Sch03, Lemma 1.3.2, p. 69] and [Sch03, Corollary 1.1.1, p. 31]):

$$R\Gamma(K, \phi_h[-1]\mathcal{F}) \simeq R\Gamma(K, R\Gamma_{\{re(h)\geqslant 0\}}\mathcal{F}). \tag{10}$$

Or in terms of the constructible function α:

$$\int_{M_f^-} \alpha \, d\chi = \int_{M_h} \alpha \, d\chi \tag{11}$$

and

$$\chi(R\Gamma(K, \phi_h[-1]\mathcal{F})) = \int_K \alpha \, d\chi - \int_{M_h} \alpha \, d\chi, \tag{12}$$

with

$$M_h := \{g \leqslant \delta, h = w\} \quad (0 < |w| \ll \delta \ll \delta_0)$$

the Milnor fiber of the holomorphic function h.

This holomorphic intersection formula for the vanishing cycle functor is due to Dubson [Dub84b, Theorem 1, p. 183], Ginsburg [Gin86, Proposition 7.7.1, p. 378], Lê [Lê98, Theorem 4.1.2, p. 242] and Sabbah [Sab85, Theorem 4.5, p. 174]. For a discussion of the history of this holomorphic intersection formula we recommend the paper [Lê98].

But most of these references are in the language of (regular) holonomic D-modules, or perverse sheaves (with respect to middle perversity). So the assumption on the intersection for a holomorphically constructible complex of sheaves corresponds to an assumption on the micro-support [KS90, Theorem 11.3.3, p. 455]. Only the result of Sabbah [Sab85, Theorem 4.5, Remark 4.6, p. 174] is in terms of the underlying (complex analytic) Lagrangian cycle, which corresponds therefore to an assumption on the intersection of σ_h and the support of the corresponding characteristic cycle! Similarly, only this reference [Sab85], and in the real subanalytic context also [Kas85, Theorem 9.1, p. 207] consider the case of an intersection in a compact subset I.

The other references deal only with the case $I = \{\omega\}$ given by a point $\omega \in T^*M$. In this special case we get back a formula conjectured by Deligne (with $x := \pi(\omega)$ and $K := \{x\}$):

$$\chi((\phi_h[-1]\mathcal{F})_x) = \sharp_{dh_x}([dh(M)] \cap [\mathrm{CC}(\mathcal{F})]) \tag{13}$$

and

$$\alpha(x) - \int_{M_h} \alpha \, d\chi = \sharp_{dh_x}([dh(M)] \cap [\mathrm{CC}(\alpha)]), \tag{14}$$

with M_h the local Milnor fiber of h in x. We get in particular that

$$\sigma_h \cap |\mathrm{CC}(\mathcal{F})| = \emptyset \quad \Rightarrow \quad \chi((\phi_h[-1]\mathcal{F})_x) = 0. \tag{15}$$

We used this formula in our paper [Sch02b] for a short proof of a formula of Brasselet, Lê and Seade for the Euler obstruction [BLS00, Theorem 3.1].

As another application of the formula (10) let us consider the following example: $\mathcal{F} = k_N$, with $N \subset M$ a closed complex analytic submanifold. Consider a holomorphic function germ $h : (M, x) \to (\mathbb{C}, 0)$ such that $h|N$ has in $x \in N$ a complex Morse critical point. Then we get $\mathrm{CC}(k_N) = [T^*_N M]$,

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with the orientation convention of [KS90, ch. IX]. If we denote by

$$[T^*_N M]_c \in H^{2m}_{T^*_N M}(T^*M, \mathbb{Z})$$

the class that corresponds under Poincaré duality to the fundamental class of the complex manifold $T^*_N M$, then we get of course

$$\sharp_{dh_x}([dh(M)] \cap [T^*_N M]_c) = 1.$$

But $re(h)|N$ has in x a real Morse critical point of index $n := \dim_C(N)$. We therefore get by [KS90, Equation (9.5.18), p. 388] for $f := re(h)$ that

$$\sharp_{df_x}([df(M)] \cap [\mathrm{CC}(k_N)]) = (-1)^n,$$

and this implies

$$[T^*_N M]_c = (-1)^n \cdot [T^*_N M] \in H^{2m}_{T^*_N M}(T^*M, \mathbb{Z}).$$

So one has to be very careful about orientation conventions and the definition of characteristic cycles that one uses. Here we follow the notations and conventions of [KS90]. Note that there are many approaches to this subject, often using quite different techniques and conventions. We recall in the next section a detailed comparison, which is worked out in [Sch03, § 5.0.3].

Let us only recall that the theory of characteristic cycles has its origin in the theory of holonomic D-modules and the local index formula of Kashiwara [Kas73, Theorem on p. 804] (compare [Kas83, Theorem 6.3.1, p. 127], [BDK81, Theorem 2, p. 574] and [Gin86, Theorem 11.7, p. 393]). This corresponds to Corollary 0.2 for

$$\mathcal{F} = R\mathrm{hom}_{D_M}(\mathcal{M}, \mathcal{O}_X) =: \mathcal{S}\mathrm{ol}(\mathcal{M})$$

the *solution complex* of a holonomic D_M-module $\mathcal{M}$.

Later on, the theory of characteristic cycles was extended to constructible functions and sheaves in the context of real geometry. First in the subanalytic context by Kashiwara [Kas83, KS90], and independently also by Fu [Fu94]. A simple approach in the semialgebraic context is sketched in [GrM99], and the extension to o-minimal structures and analytic geometric categories has been worked out in [SV96]. All these approaches in real geometry are based on a suitable Morse theory, e.g. the micro-local sheaf theory of Kashiwara and Schapira [KS90], or the stratified Morse theory of Goresky and MacPherson [GoM88].

For a detailed comparison and translation of these two different theories in the framework of Morse theory for constructible sheaves, including a geometric introduction to characteristic cycles of constructible functions and sheaves, we refer to our book [Sch03, ch. 5]. This language will be used in the proof of Theorem 0.1. Moreover, we use the following important result about the behavior of the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$ under a suitable intersection.

Theorem 0.2. *Let M be a real analytic manifold and $g : M \to \mathbb{R}$ a subanalytic C^2-function. Consider on M a bounded subanalytically constructible complex $\mathcal{F}$ of sheaves of vector spaces, with finite-dimensional stalks $\mathcal{F}_x$ ($x \in M$). Assume $\mathcal{F}$ is constructible with respect to a subanalytic Whitney b-regular stratification (i.e. the cohomology sheaves of $\mathcal{F}$ are locally constant on the subanalytic strata S) such that the set of non-degenerate covectors (with respect to this stratification) is dense in all fibers of the projection $T^*_S M \to S$ (for all strata S). Let δ be a regular value of g such that $\{g = \delta\}$ is transversal to all strata S. Then one has for the open inclusion $j : \{g < \delta\} \to M$ the following estimate for the support $|\mathrm{CC}(Rj_*j^*\mathcal{F})|$ of the characteristic cycle of $Rj_*j^*\mathcal{F}$:*

$$|\mathrm{CC}(Rj_*j^*\mathcal{F})| \cap \pi^{-1}(\{g = \delta\}) \subset \{\omega + \lambda \cdot dg_x \in T^*M \mid \lambda \leqslant 0, \pi(\omega) = x, g(x) = \delta, \omega \in |\mathrm{CC}(\mathcal{F})|\}. \quad (16)$$

Remark 0.3. The assumption on our stratification is for example satisfied for a subanalytic μ-stratification in the sense of [KS90, Definition 8.3.19, p. 334] (see [KS90, Corollary 8.3.24, p. 336]).

This μ-condition is by [Tro89] equivalent to the w-condition of Verdier, and such a w-regular stratification can always be used in the context of 'geometric categories' as in the last section of this paper (cf. [Ta98]). But it is not known (at least to the author) if the w-condition implies the assumption on our stratification (used in the theorem) in this more general context.

Kashiwara and Schapira used in their proof of [KS90, Theorem 9.5.6, p. 386] a similar result in terms of the micro-support of the corresponding constructible complex of sheaves (cf. [KS90, Proposition 8.4.1, p. 338] and [KS90, Proposition 5.4.8(a), p. 233]).

We will give a proof based on our results on Morse theory for constructible sheaves [Sch03, ch. 5], especially on an explicit result about suitable 'stratified spaces with boundary' [Sch03, Theorem 5.0.2, p. 279]. This approach works also if one uses the supports of the corresponding characteristic cycles. It implies a similar estimate for

$$|\mathrm{CC}(Rj_! j^* \mathcal{F})| \cap \pi^{-1}(\{g = \delta\})$$

(with $\lambda \geqslant 0$), from which we deduce (16) by duality.

1. Characteristic cycles

In this section we explain the construction of the characteristic cycles. As an application we give a proof of Theorem 0.2 in a much more general context, which applies especially to complexes of sheaves constructible with respect to suitable 'geometric categories'.

Let X be a closed subset of a smooth manifold M. In this paper, a smooth manifold M has by definition a countable topology. We also assume (for simplicity) that the dimension of the connected components of M is bounded. A *stratification* of X is a filtration $X_\bullet$ of X:

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

by closed subsets such that $X^i := X_i \backslash X_{i-1}$ $(i = 0, \dots, n)$ is a smooth submanifold of M. The connected components of the X^i are by definition the *strata* of this stratification.

Here we fix a degree $k = 1, \dots, \infty, \omega$ of smoothness (with ω = real analytic). (Sub)manifold and smooth map or function always means a C^k-(sub)manifold and C^k-smooth map or function. We assume that the stratification $X_\bullet$ is *Whitney b-regular*: If $x_n \in X^j$ (for $i < j$) and $y_n \in X^i$ are sequences converging to $x \in X^i$ such that the tangent planes $T_{x_n} X^j$ converge to some limiting plane τ, and the secant lines $l_n = \overline{x_i, y_i}$ (with respect to some local coordinates) converge to some limiting line l, then $l \subset \tau$.

Consider on X a bounded complex $\mathcal{F}$ of sheaves of vector spaces (over a base-field k), with finite-dimensional stalks $\mathcal{F}_x$ $(x \in X)$. We assume that $\mathcal{F}$ is constructible with respect to the stratification $X_\bullet$, i.e. the cohomology sheaves of $\mathcal{F}$ are locally constant on all X^i.

From now on we assume $X = M$ and $k \geqslant 2$. Then we have by [Sch03, Corollary 4.0.3, p. 212] and [Sch03, Proposition 4.1.2, p. 222] the following estimate for the micro-support $\mu\mathrm{supp}(\mathcal{F})$ of $\mathcal{F}$ in the sense of [KS90, Definition 5.1.2, p. 221]:

$$\mu\mathrm{supp}(\mathcal{F}) \subset \Lambda := \bigcup_i T^*_{X^i} X \subset T^*X. \tag{17}$$

Here $T^*_S X$ denotes the conormal bundle of a locally closed submanifold S of X. By Whitney regularity, Λ is a closed subset of T^*X (this is in fact equivalent to a-regularity).

Moreover, we have by [Sch03, Proposition 4.0.2, p. 214] that $\mathcal{F}$ is cohomologically constructible in the sense of [KS90, Definition 3.4.1, p. 158]. Therefore we can define as in [KS90, p. 377] the following chain of morphisms:

$$\begin{aligned} R\mathrm{hom}(\mathcal{F},\mathcal{F}) &\simeq R\pi_* \,\mu\mathrm{hom}(\mathcal{F},\mathcal{F}) \simeq R\pi_* R\Gamma_\Lambda \,\mu\mathrm{hom}(\mathcal{F},\mathcal{F}) \\ &\simeq R\pi_* R\Gamma_\Lambda \mu_\triangle(\mathcal{F}\boxtimes D(\mathcal{F})) \to R\pi_* R\Gamma_\Lambda \mu_\triangle \delta_*(\mathcal{F}\otimes D(\mathcal{F})) \\ &\to R\pi_* R\Gamma_\Lambda \mu_\triangle \delta_* \omega_X \simeq R\pi_* R\Gamma_\Lambda(\pi^{-1}\omega_X), \end{aligned}$$

with $\pi : T^*X \to X$ the projection, $\delta : X \to X \times X$ the diagonal embedding and ω_X the dualizing complex on X. Since X is a manifold, one has by [KS90, Proposition 3.3.2, p. 152] that

$$\omega_X \simeq \mathrm{or}_X[d], \quad \text{with } d := \dim(X).$$

Here we assume that X is pure-dimensional (or one should interpret $d := \dim(X)$ as a locally constant function).

For the definition of the above chain of morphisms we use the following properties:

i) $R\mathrm{hom}(\mathcal{F},\mathcal{F}) \simeq R\pi_* \,\mu\mathrm{hom}(\mathcal{F},\mathcal{F})$ (see [KS90, Proposition 4.4.2(i), p. 202]).

ii) $\mu\mathrm{hom}(\mathcal{F},\mathcal{F})$ has its support in $\mu\mathrm{supp}(\mathcal{F})$ (see [KS90, Corollary 5.4.10(ii), p. 234]), and therefore by (17) also in Λ.

iii) $\mu\mathrm{hom}(\mathcal{F},\mathcal{F}) \simeq \mu_\triangle(\mathcal{F}\boxtimes D(\mathcal{F}))$ (see [KS90, Definition 4.4.1(iii), p. 202] and [KS90, Proposition 3.4.4, p. 159]). Here we can apply [KS90, Proposition 3.4.4], since $\mathcal{F}$ is cohomologically constructible.

iv) We finally use the natural morphisms $\mathcal{F}\boxtimes D(\mathcal{F}) \to \delta_*\delta^*(\mathcal{F}\boxtimes D(\mathcal{F})) \simeq \delta_*(\mathcal{F}\otimes D(\mathcal{F}))$, and $\mathcal{F}\otimes D(\mathcal{F}) \to \omega_X$, together with $\mu_\triangle \delta_* \omega_X \simeq \pi^{-1}\omega_X$.

Definition 1.1. The image of $\mathrm{id} \in \mathrm{Hom}(\mathcal{F},\mathcal{F}) \simeq H^0(X, R\mathrm{hom}(\mathcal{F},\mathcal{F}))$ in $H^0_\Lambda(T^*X, \pi^{-1}\omega_X) \simeq H^d_\Lambda(T^*X, \pi^{-1}\mathrm{or}_X)$ is called the *characteristic cycle* $\mathrm{CC}(\mathcal{F})$ of $\mathcal{F}$.

Remark 1.1. This definition extends [KS90, Definition 9.4.1, p. 377] to our context. The same construction works for a Whitney a-regular stratification $X_\bullet$, if we have the estimate (17) and the property that $\mathcal{F}$ is cohomologically constructible, e.g. $X_\bullet$ is (locally) C-regular in the sense of Bekka [Bek91] (compare with [Sch03, ch. 4]).

We now explain the calculation of $\mathrm{CC}(\mathcal{F})$ in terms of our (stratified) Morse theory for constructible sheaves [Sch03, ch. 5].

Consider the set of non-degenerate covectors with respect to our Whitney b-regular stratification $X_\bullet$ (see [Sch03, Definition 5.1.2, p. 302] and [GoM88, Definition 1.8, p. 44]):

$$\Lambda' := \bigcup_i \left(T^*_{X^i}X \Big\backslash \bigcup_{i\neq j} \mathrm{cl}(T^*_{X^j}X) \right). \tag{18}$$

Choose a smooth function germ $f : (X,x) \to (\mathbb{R},0)$, with $x \in X^s$ and $df_x \in \Lambda'$. Take a normal slice N at x, i.e. a locally closed submanifold N of X, with $N \cap X^s = \{x\}$ such that N intersects X^s transversally in x (cf. [Sch03, Definition 5.0.2(2), p. 271] and [GoM88, Definition 1.4, p. 41]). Then the isomorphism-class of

$$\mathrm{NMD}(\mathcal{F},f,x) := (R\Gamma_{\{f|N \geqslant 0\}}(\mathcal{F}|N))_x \tag{19}$$

is the (sheaf theoretic) 'normal Morse datum' of f in x with respect to $\mathcal{F}$, i.e. it is the cohomological counterpart of the corresponding normal Morse data of Goresky and MacPherson [GoM88, Definition 3.6.1, p. 65]. By [Sch03, Theorem 5.0.1(2), p. 272], this isomorphism-class depends only on df_x and is locally constant on Λ'. By [Sch03, Proposition 5.0.2, p. 279] we have a distinguished triangle

$$R\Gamma(l^-_X, \mathcal{F})[-1] \longrightarrow \mathrm{NMD}(\mathcal{F},f,x) \longrightarrow \mathcal{F}_x \xrightarrow{[1]} \tag{20}$$

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with l_X^- the 'lower halflink' of f in x (see [GoM88, Definition 3.9.1, p. 66]). In the notation of this paper, this l_X^- is just the intersection of X with a local left Milnor fiber of $f|N$ in x. But $\mathcal{F}|l_X^-$ is constructible with respect to an induced Whitney b-regular stratification of l_X^-. Then

$$R\Gamma(l_X^-, \mathcal{F})$$

has finite-dimensional cohomology, since l_X^- is compact (cf. [Sch03, Remark 4.2.2, p. 240] and [Bor84, Theorem 3.5, p. 70]). By the above distinguished triangle, this is also true for $\mathrm{NMD}(\mathcal{F}, f, x)$, with

$$\chi(\mathrm{NMD}(\mathcal{F}, f, x)) = \chi(\mathcal{F}_x) - \chi(R\Gamma(l_X^-, \mathcal{F})). \tag{21}$$

So we can associate to a connected component Λ'_j of Λ' the integer (for $df_x \in \Lambda'_j$):

$$m_j := \chi(\mathrm{NMD}(\mathcal{F}, f, x)) := \chi((R\Gamma_{\{f|N \geqslant 0\}}(\mathcal{F}|N))_x). \tag{22}$$

If we work in the context of 'geometric categories' as in the next section, with f a definable function germ, then this multiplicity m_j can also be expressed in terms of the constructible function $\alpha : x \mapsto \chi(\mathcal{F}_x)$ (compare [Sch03, Equation (5.16), p. 284]):

$$m_j := \chi(\mathrm{NMD}(\alpha, f, x)) := \alpha(x) - \int_{l_X^-} \alpha \, d\chi. \tag{23}$$

Now take an open subset Ω in T^*X, with $\Omega \cap \Lambda = \Lambda'$ (e.g. the complement of the set $\Upsilon := \Lambda \backslash \Lambda'$ of degenerate covectors). Then one gets for the image of $\mathrm{CC}(\mathcal{F})$ under the natural map $H^d_\Lambda(T^*X, \pi^{-1} \mathrm{or}_X) \to H^d_{\Lambda'}(\Omega, \pi^{-1} \mathrm{or}_X)$ the formula

$$\mathrm{im}(\mathrm{CC}(\mathcal{F})) = \prod m_j \cdot [\Lambda'_j]. \tag{24}$$

Here $[\Lambda'_j]$ is defined as in [KS90, ch. 9.4], i.e. it is the image of the class $[T^*_{X^s}X] \in H^d_{T^*_{X^s}X}(\Omega, \pi^{-1} \mathrm{or}_X)$ of the reference (with $\pi(\Lambda'_j) \subset X^s$) under the projection

$$H^d_{T^*_{X^s}X}(\Omega, \pi^{-1} \mathrm{or}_X) \simeq \prod H^d_{\Lambda'_k}(\Omega, \pi^{-1} \mathrm{or}_X) \to H^d_{\Lambda'_j}(\Omega, \pi^{-1} \mathrm{or}_X),$$

where the product is over all k with $\pi(\Lambda'_k) \subset X^s$.

This follows as in [KS90, p. 382] from our identification [Sch03, Equation (5.52), p. 311] of the 'normal Morse datum' $\mathrm{NMD}(\mathcal{F}, f, x)$ of f in x with the 'local type of $\mathcal{F}$ in df_x' in the sense of Kashiwara and Schapira (compare [KS90, Proposition 6.6.1(ii), p. 274] and [KS90, Definition 7.5.4, p. 311]).

Assume $H^d_\Upsilon(T^*X, \pi^{-1} \mathrm{or}_X) = 0$ so that the above map of local cohomology groups is injective. Then the characteristic cycle $\mathrm{CC}(\mathcal{F})$ is uniquely determined by the Euler characteristics m_j of the 'normal Morse data' of $\mathcal{F}$. This applies, for example, if Υ has a stratification with all strata of dimension $< d = \dim(X)$ (e.g. in the context of 'geometric categories' as in the next section). It would also be enough (compare with the proof of [KS90, Proposition 9.2.2(i), p. 367]) that Υ has this property locally (i.e. each point in Υ has an open neighborhood U in T^*X such that $U \cap \Upsilon$ has such a stratification).

Remark 1.2. By (21) and (23) we get in this case an easy geometric description of the characteristic cycle $\mathrm{CC}(\cdot)$, or more precisely, of $\mathrm{im}(\mathrm{CC}(\cdot))$. One should ask if one can use this description as a definition of the characteristic cycle $\mathrm{CC}(\cdot)$. The main problem is then to show that this is a cycle coming from

$$H^d_\Lambda(T^*X, \pi^{-1} \mathrm{or}_X) \hookrightarrow H^d_{\Lambda'}(\Omega, \pi^{-1} \mathrm{or}_X).$$

The sophisticated Definition 1.1 seems to be the only one working for a general Whitney b-regular stratification. But in the subanalytic context, other more geometric approaches to this question are due to Kashiwara [Kas85, Theorem 4.1, p. 199] and Fu (compare with [Fu94, Definition 4.1, p. 856] and [Fu94, Theorem 4.7, p. 865]), where [Fu94] uses the language of 'geometric measure theory'. A 'translation' of the last approach into 'geometric categories' and a specialization result in homology is worked out in [Sch03, Theorem 5.0.3, p. 289] and [Sch03, § 5.2.2].

Suppose that Υ has (locally) such a stratification, and that the set Λ' of non-degenerate covectors is dense in Λ. Then one gets the following 'explicit' description of the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$:

$$|\mathrm{CC}(\mathcal{F})| = \bigcup_{m_j \neq 0} \mathrm{cl}(\Lambda'_j). \tag{25}$$

Remark 1.3. The micro-support $\mu\mathrm{supp}(\mathcal{F})$ contains by definition the closure $\mathrm{cl}(\Lambda'_j)$ of all Λ'_j such that the corresponding 'normal Morse datum'

$$\mathrm{NMD}(\mathcal{F}, f, x) \simeq (R\Gamma_{\{f|N \geqslant 0\}}(\mathcal{F}|N))_x \quad (df_x \in \Lambda'_j)$$

is not isomorphic to 0. Moreover, in the context of 'geometric categories', $\mu\mathrm{supp}(\mathcal{F})$ is exactly given as the union of these $\mathrm{cl}(\Lambda'_j)$ (see [Sch03, Proposition 5.0.1, p. 273]). We see, in particular, that in general $|\mathrm{CC}(\mathcal{F})|$ is much smaller than $\mu\mathrm{supp}(\mathcal{F})$.

Suppose that the set Λ' of non-degenerate covectors is dense in Λ, and that Υ has (locally) a stratification with all strata of dimension $< d = \dim(X)$ such that the frontier $\partial\Lambda'_j := \mathrm{cl}(\Lambda'_j)\backslash\Lambda'_j$ of Λ'_j is a union of strata of Υ (for all j). Then all morphisms in the following commutative diagram are injective.

$$\begin{array}{ccc} H^d_{|\mathrm{CC}(\mathcal{F})|}(T^*X, \pi^{-1}\,\mathrm{or}_X) & \longrightarrow & H^d_{\Omega\cap|\mathrm{CC}(\mathcal{F})|}(\Omega, \pi^{-1}\,\mathrm{or}_X) \\ \downarrow & & \downarrow \\ H^d_{\Lambda}(T^*X, \pi^{-1}\,\mathrm{or}_X) & \longrightarrow & H^d_{\Lambda'}(\Omega, \pi^{-1}\,\mathrm{or}_X) \end{array}$$

Moreover, there exists a unique class $[\mathrm{CC}(\mathcal{F})] \in H^d_{|\mathrm{CC}(\mathcal{F})|}(T^*X, \pi^{-1}\,\mathrm{or}_X)$, with image $\mathrm{CC}(\mathcal{F}) \in H^d_{\Lambda}(T^*X, \pi^{-1}\,\mathrm{or}_X)$, and image

$$\prod_{m_j \neq 0} m_j \cdot [\Lambda'_j] \in H^d_{\Omega\cap|\mathrm{CC}(\mathcal{F})|}(\Omega, \pi^{-1}\,\mathrm{or}_X).$$

This is the cohomology class that we used in the Introduction (see the next section, and compare with [SV96]). For the convenience of the reader, we also compare our definition of the characteristic cycle $\mathrm{CC}(\cdot)$ with the other conventions used in the literature (compare with [Sch03, § 5.0.3] for the details).

For simplicity, we assume that M is oriented (as in most of the following references), i.e. an isomorphism $\mathrm{or}_M \simeq \mathbb{Z}_M$ has been chosen. Moreover, we orient T^*M by using first this orientation of the base M, and then the induced (real dual) orientation of the fibers $T^*_xM = \hom_{\mathbb{R}}(T_xM, \mathbb{R})$ (as in [Fu94, GrM99, SV96, Sch03]). This differs by the factor $(-1)^{m(m+1)/2}$ from the symplectic orientation of T^*M (used in [Kas85]). In particular, for M a complex analytic manifold of complex dimension m, this agrees only up to the factor $(-1)^m$ with the complex orientation of T^*M! Let $[T^*M]$ be the fundamental class of this oriented manifold in Borel–Moore homology, and denote by $a : T^*M \to T^*M$ the antipodal map (i.e. multiplication by -1 on the fibers).

i) By definition $\mathrm{CC}(\mathcal{F})$ corresponds to the characteristic cycle used in [KS90].

ii) $\mathrm{CC}(\mathcal{F}) \cap (-1)^m \cdot [T^*M]$ corresponds to the characteristic cycle used in [Kas85, SV96, Sch03].

iii) $a_*(\mathrm{CC}(\alpha) \cap (-1)^m \cdot [T^*M])$ corresponds to the characteristic cycle used in [Fu94, GrM99].

Assume M is a complex analytic manifold of complex dimension m, with $[T^*M]_c$ the fundamental class of the complex manifold T^*M. Let $\mathcal{F}$ or α be complex analytically constructible.

i) $\mathrm{CC}(\mathcal{F}) \cap (-1)^m \cdot [T^*M]_c$ corresponds to the characteristic cycle used in [Dub84a, Dub84b, Lê98].

ii) $\mathrm{CC}(\alpha) \cap [T^*M]_c$ corresponds to the characteristic cycle used in [Sab85].

iii) $\mathrm{CC}(\mathcal{F}) \cap (-1)^m \cdot [T^*M]_c$, with $\mathcal{F} := \mathcal{S}\mathrm{ol}(\mathcal{M})$ or $\mathcal{F} := DR(\mathcal{M}) := R\mathrm{hom}_{D_M}(\mathcal{O}_M, \mathcal{M})$, corresponds to the characteristic cycle of the holonomic D-module $\mathcal{M}$ used in [BDK81, Dub84a, Lê98].

iv) $\mathrm{CC}(\mathcal{F}) \cap [T^*M]_c$, with $\mathcal{F} := DR(\mathcal{M}) := R\mathrm{hom}_{D_M}(\mathcal{O}_M, \mathcal{M})[m]$, corresponds to the characteristic cycle of the holonomic D-module $\mathcal{M}$ used in [Gin86].

Now we prove the first main result of this section.

Theorem 1.1. *Let $g : X \to \mathbb{R}$ be a C^2-function on the smooth manifold X. Consider a Whitney b-regular stratification $X_\bullet$ of X such that the set of non-degenerate covectors (with respect to this stratification) is dense in all fibers of the projection $T^*_S X \to S$ (for all strata S). Let δ be a regular value of g such that $\{g = \delta\}$ is transversal to all strata S. Assume that the set of degenerate covectors of $X_\bullet$ and also of the induced Whitney b-regular stratification of $\{g \leqslant \delta\}$ has (locally) a stratification with all strata of dimension $< d = \dim(X)$.*

Consider on X a bounded complex $\mathcal{F}$ of sheaves of vector spaces (over a base-field k), with finite-dimensional stalks $\mathcal{F}_x$ ($x \in X$). Assume $\mathcal{F}$ is constructible with respect to $X_\bullet$. Then one has for the open inclusion $j : \{g < \delta\} \to X$ the following estimate for the support of the characteristic cycles of $Rj_!j^*\mathcal{F}$ and $Rj_*j^*\mathcal{F}$:

$$|\mathrm{CC}(Rj_!j^*\mathcal{F})| \cap \pi^{-1}(\{g = \delta\}) \subset \{\omega + \lambda \cdot dg_x \in T^*M \mid \lambda \geqslant 0, \pi(\omega) = x, g(x) = \delta, \omega \in |\mathrm{CC}(\mathcal{F})|\}, \tag{26}$$

$$|\mathrm{CC}(Rj_*j^*\mathcal{F})| \cap \pi^{-1}(\{g = \delta\}) \subset \{\omega + \lambda \cdot dg_x \in T^*M \mid \lambda \leqslant 0, \pi(\omega) = x, g(x) = \delta, \omega \in |\mathrm{CC}(\mathcal{F})|\}. \tag{27}$$

Proof. 1) By transversality, $\{g \leqslant \delta\}$ gets an induced Whitney b-regular stratification with strata S' of the form $\{g < \delta\} \cap S$, $\{g = \delta\} \cap S$ (for S a stratum of $X_\bullet$). The assumptions imply that also the set of non-degenerate covectors with respect to this induced stratification of $\{g \leqslant \delta\}$ is dense in all fibers of the projection $T^*_{S'}X \to S'$ (for all strata S', compare with the proof of [Sch03, Theorem 5.0.2, p. 279]).

2) $Rj_!j^*\mathcal{F}$ and $Rj_*j^*\mathcal{F}$ have also finite-dimensional stalks, and are constructible with respect to the induced stratification of $\{g \leqslant \delta\}$ (see [Sch03, Proposition 4.0.2, p. 214]). More precisely, they have their support in $\{g \leqslant \delta\}$, and their restrictions to $\{g \leqslant \delta\}$ are constructible with respect to the induced stratification. Therefore the characteristic cycles of these complexes of sheaves are defined, and we can apply the description (25) for their support.

3) First we consider $Rj_!j^*\mathcal{F}$. Let

$$\omega \in T^*_{S'}X \subset |\mathrm{CC}(Rj_!j^*\mathcal{F})| \cap \pi^{-1}(\{g = \delta\})$$

be given, with ω non-degenerate with respect to the induced stratification of $\{g \leqslant \delta\}$, and $x := \pi(\omega)$ a point in a 'boundary' stratum $S' = S \cap \{g = \delta\}$. We can approximate ω by a covector $\lambda \cdot dg_x + \omega' \in T^*_{S'}X$ (with $\lambda \neq 0$ and $x = \pi(\omega')$) such that $\omega' \in T^*_S X$ is non-degenerate with respect to $X_\bullet$, and the following properties hold (see for instance [Sch03, Theorem 5.0.2, p. 279]):

a) $\lambda < 0$ implies that the 'normal Morse datum' with respect to $Rj_!j^*\mathcal{F}$ at the covector ω is isomorphic to 0.

b) $\lambda > 0$ implies that the 'normal Morse datum' with respect to $Rj_!j^*\mathcal{F}$ at the covector ω is isomorphic to the 'normal Morse datum' with respect to $\mathcal{F}$ at the covector ω', up to a shift by $[-1]$.

Compare with [Sch03, p. 280] for the details (and note that there we used the function $\delta - g$ for the description of $\{g \leqslant \delta\} = \{\delta - g \geqslant 0\}$). These results can also be deduced from [KS90, Proposition 6.1.9(ii), p. 256] and [KS90, Proposition 7.5.10, p. 314]. So if the first 'normal Morse datum' (at ω) has a non-zero Euler characteristic, then the same is true for the second 'normal Morse datum' (at ω'). By the description (25), this implies our claim for $Rj_!j^*\mathcal{F}$ (since $Rj_!j^*\mathcal{F}$ and $\mathcal{F}$ have the same 'normal Morse data' for the strata S' of the form $\{g < \delta\} \cap S$).

4) The claim for $Rj_*j^*\mathcal{F}$ follows from the case before by the duality isomorphism (with $D(\cdot)$ the duality functor as in [KS90, Definition 3.1.16(ii), p. 148]):

$$D(Rj_!j^*\mathcal{F}) \simeq Rj_*j^*(D(\mathcal{F})),$$

because $|\mathrm{CC}(D(\mathcal{G}))|$ (for $\mathcal{G} = \mathcal{F}$, $Rj_!j^*\mathcal{F}$) is equal to the image of $|\mathrm{CC}(\mathcal{G})|$ under the antipodal map $a : T^*X \to T^*X$ (i.e. multiplication by -1 on the fibers of π). This follows from the description (25) together with [Sch03, Equation (5.25), p. 290] (compare also with [KS90, Proposition 9.4.4, p. 380]). □

Remark 1.4. Suppose that we are in the context of 'geometric categories' (as in the next section). Then we can use [Sch03, Proposition 5.0.1, p. 273] for the description of the micro-support (i.e. the description of Remark 1.3), and the above proof gives also the corresponding estimate in terms of the micro-support (instead of the support of the characteristic cycles):

$$\mu\mathrm{supp}(Rj_!j^*\mathcal{F}) \cap \pi^{-1}(\{g = \delta\}) \subset \{\omega + \lambda \cdot dg_x \in T^*M \mid \lambda \geqslant 0, \pi(\omega) = x, g(x) = \delta, \omega \in \mu\mathrm{supp}(\mathcal{F})\},$$
$$\mu\mathrm{supp}(Rj_*j^*\mathcal{F}) \cap \pi^{-1}(\{g = \delta\}) \subset \{\omega + \lambda \cdot dg_x \in T^*M \mid \lambda \leqslant 0, \pi(\omega) = x, g(x) = \delta, \omega \in \mu\mathrm{supp}(\mathcal{F})\}.$$

In this way one can get in particular a proof of [Mas01, Theorem 5.3, p. 293] in terms of our (stratified) Morse theory for constructible sheaves [Sch03, ch. 5], without the use of the general micro-local theory of Kashiwara and Schapira.

For the proof of the definable counterpart of Theorem 0.1, we also need in the next section (the second part of) the following result (compare with [Kas85, Theorem 4.2, Theorem 4.3, p. 199] and [KS90, Theorem 9.5.3, Corollary 9.5.4, p. 385]).

Theorem 1.2. *Let $f : X \to [a, b[\subset \mathbb{R}$ be a C^2-function on the smooth manifold X (with $b \in \mathbb{R} \cup \{\infty\}$). Consider a Whitney b-regular stratification $X_\bullet$ of X such that the set Λ' of non-degenerate covectors (with respect to this stratification) is dense in Λ. Assume that the set $\Upsilon := \Lambda \backslash \Lambda'$ of degenerate covectors of $X_\bullet$ has (locally) a stratification with all strata of dimension $< d = \dim(X)$ such that the frontier $\partial\Lambda'_j := \mathrm{cl}(\Lambda'_j) \backslash \Lambda'_j$ is a union of strata of Υ (for all connected components Λ'_j of Λ').*

Let $\mathcal{F}$ be a bounded complex of sheaves of vector spaces on X, with finite-dimensional stalks $\mathcal{F}_x$ ($x \in X$). Assume that $\mathcal{F}$ is constructible with respect to $X_\bullet$, and that $f|\,\mathrm{supp}(\mathcal{F})$ is proper.

i) *Suppose that $df(X) \cap \mu\mathrm{supp}(\mathcal{F})$ is compact. Then $R\Gamma(X, \mathcal{F})$ is a cohomologically bounded complex with finite-dimensional cohomology, and*

$$\chi(R\Gamma(X, \mathcal{F})) = \sharp([df(X)] \cap [\mathrm{CC}(\mathcal{F})]).$$

ii) *Suppose that $-df(X) \cap \mu\mathrm{supp}(\mathcal{F})$ is compact. Then $R\Gamma_c(X, \mathcal{F})$ is a cohomologically bounded complex with finite-dimensional cohomology, and*

$$\chi(R\Gamma_c(X, \mathcal{F})) = \sharp([-df(X)] \cap [\mathrm{CC}(\mathcal{F})]).$$

Here we use the definition of the intersection product of the introduction, with $df(X) := \sigma_f := \{(x, df_x) \in T^*X \mid x \in X\}$. Note that we have the inclusion $|\mathrm{CC}(\mathcal{F})| \subset \mu\mathrm{supp}(\mathcal{F})$.

Proof. The proof is by Morse theory, and is almost the same as the proof of [Sch03, Theorem 5.0.4, p. 290] given in [Sch03, p. 321] (compare also with [Kas85] and the proof of [KS90, Theorem 10.3.8, p. 429]).

By approximation (which does not change the intersection number by a homotopy argument), we can assume that f is a proper stratified Morse function, i.e. all critical points of f with respect to $X_\bullet$ are stratified Morse critical points in the sense of [Sch03, Definition 5.0.2, p. 271] and [GoM88, Definition 2.1, p. 52]: if $x \in X^s$ is a critical point of $f|X^s$, then df_x is a non-degenerate covector and $f|X^s$ has in x a classical Morse critical point (i.e. its Hessian is non-degenerate).

By Remark 1.3 and the assumption on $df(X)$ (or $-df(X)$), there are then only finitely many such critical points x, whose 'normal Morse datum' with respect to $\mathcal{F}$ are non-trivial at df_x (or $-df_x$). By [Sch03, Lemma 5.1.1, p. 296] and [Sch03, Lemma 5.1.2, p. 300] we get for $r \in [a, b[$ big enough (by a Mittag–Leffler argument as in [Sch03, Corollary 6.1.2, p. 423]):

$$R\Gamma(X, \mathcal{F}) \simeq R\Gamma(\{f \leqslant r\}, \mathcal{F})$$

in the first case, and

$$R\Gamma_c(\{f < r\}, \mathcal{F}) \simeq R\Gamma_c(X, \mathcal{F})$$

in the second case. By induction, it is enough to consider the case that f has at most one critical point $x \in \{f \leqslant r\}$, with $x \in X^s$, and $f(x) < r$. Then we get by [Sch03, Theorem 5.0.1(1), p. 272] and [Sch03, Lemma 5.1.2, p. 300] that

$$R\Gamma(\{f \leqslant r\}, \mathcal{F}) \simeq \mathrm{NMD}(\mathcal{F}, f, x)[-\tau] \quad \text{or} \quad R\Gamma_c(\{f < r\}, \mathcal{F}) \simeq \mathrm{NMD}(\mathcal{F}, -f, x)[-\tau],$$

with τ the Morse index of $\pm f|X^s$ (i.e. its Hessian in x has τ negative eigenvalues). But the corresponding 'normal Morse datum' $\mathrm{NMD}(\mathcal{F}, \pm f, x)$ is finite dimensional (as explained before). Therefore our claim follows from the description (24) for $\mathrm{CC}(\mathcal{F})$, and the local intersection formula (see [KS90, Equation (9.5.18), p. 388] and [Sch03, Equation (5.20), p. 286]):

$$\sharp_{df_x}([\pm df(X)] \cap [T^*_{X^s}X]) = (-1)^\tau. \qquad \square$$

Remark 1.5. We used in the above proof the fact that we can approximate f by a stratified Morse function. This follows from our assumption about the (local) existence of a suitable stratification of the set of degenerate covectors. By using a proper C^2-embedding $X \hookrightarrow \mathbb{R}^N$, we can assume $X = \mathbb{R}^N$. Then our claim follows for example from [Orr87, Theorem 1, Theorem 3].

2. Intersection formula

In this section we work in one of the following 'geometric categories'.

i) X is an affine space $\mathbb{R}^n$, and $\mathcal{S}$ is an *o*-minimal structure on the real field $(\mathbb{R}, +, \cdot)$ (see [vDr98, vDM96], e.g. the structure of semialgebraic subsets of real affine spaces). We could also assume, that X is a real analytic $\mathcal{S}$-manifold as in [vDM96, pp. 507–508] (e.g. a real analytic Nash manifold).

ii) X is a real analytic manifold, and $\mathcal{S}$ is an analytic geometric category ([vDM96, SV96], e.g. the structure of subanalytic subsets of real analytic manifolds).

iii) X is a real analytic Nash manifold, and $\mathcal{S}$ is a Nash geometric category ([Sch03, ch. 2] and [Sch02a], e.g. the structure of locally semialgebraic subsets of real analytic Nash manifolds).

The subsets of $\mathcal{S}(X)$ are by definition the definable subsets of X. A *definable map* $f : A \to B$ between definable subsets $A \subset X$ and $B \subset X'$ (with X, X' ambient manifolds as in the above cases) is a continuous map with definable graph. A complex of sheaves $\mathcal{F}$ on X is called $\mathcal{S}$-*constructible* if it is constructible with respect to a filtration $X_\bullet$:

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n = X$$

by closed definable subsets of X (i.e. the cohomology sheaves of $\mathcal{F}$ are locally constant on all $X^i := X_i \backslash X_{i-1}$). By [vDM96, Theorem 1.19 and Theorem 4.8] we can then assume that $\mathcal{F}$ is constructible with respect to a definable Whitney b-regular C^p-stratification, with $1 \leqslant p < \infty$ (i.e. a definable filtration as before, with all X^i C^p-submanifolds of X, which is b-regular).

Remark 2.1. For the following discussion, the reader should also compare with [SV96, § 10]. For the basic results about definable sets and maps, we refer to [vDM96]. Note that their results about analytic geometric categories are also true in the context of Nash geometric categories (with the obvious modifications, as explained in [Sch03, ch. 2], where we developed the basic theory of $\mathcal{S}$-constructible complexes of sheaves).

Let $X_\bullet$ be a definable Whitney stratification. Then

$$\Lambda' := \bigcup_i \Big(T^*_{X^i}X \Big\backslash \bigcup_{i \neq j} \mathrm{cl}(T^*_{X^j}X) \Big) \subset \Lambda := \bigcup_i T^*_{X^i}X$$

are definable subsets of T^*X, and the dimension of the set $\Upsilon := \Lambda \backslash \Lambda'$ of degenerate covectors is $< d := \dim(X)$. Therefore, the set of non-degenerate covectors Λ' is dense in Λ, and we can find [vDM96, Theorem 1.19 and Theorem 4.8] a Whitney b-regular stratification $\Lambda_\bullet$ of Λ of the form

$$\emptyset = \Lambda_{-1} \subset \Lambda_0 \subset \cdots \subset \Lambda_{d-1} = \Upsilon \subset \Lambda_d = \Lambda.$$

In particular, the frontier $\partial \Lambda'_j := \mathrm{cl}(\Lambda'_j) \backslash \Lambda'_j$ of each connected component Λ'_j of Λ' is a union of strata of Υ (see [Sch03, Proposition 4.0.2(1), p. 214]).

Therefore we can apply the description of the first section for $|\mathrm{CC}(\mathcal{F})|$, $\mu\mathrm{supp}(\mathcal{F})$ and $[\mathrm{CC}(\mathcal{F})]$, and Theorem 1.1 implies Theorem 0.2.

Now we are ready for the proof of the main theorem of this paper.

Theorem 2.1. *Let $f : X \to \mathbb{R}$ be a definable C^2-function. Consider on X a bounded $\mathcal{S}$-constructible complex $\mathcal{F}$ of sheaves of vector spaces, with finite-dimensional stalks $\mathcal{F}_x$ ($x \in X$). Suppose that the intersection of σ_f and the support $|\mathrm{CC}(\mathcal{F})|$ of the characteristic cycle of $\mathcal{F}$ is contained in a compact definable subset $I \subset T^*M$, with $K := \pi(I) \subset \{f = 0\}$. Then one has*

$$\chi(R\Gamma(K, R\Gamma_{\{f \geqslant 0\}}\mathcal{F})) = \sharp([df(X)] \cap [\mathrm{CC}(\mathcal{F})]). \tag{28}$$

Note that $R\Gamma(K, R\Gamma_{\{f \geqslant 0\}}\mathcal{F})$ has finite-dimensional cohomology, since K is a compact definable subset (compare [Sch03, ch. 2]).

Proof. 1) Since K is a compact definable subset of X, there exists by [vDM96, Theorems 1.20, 4.22 and D.19] a definable C^2-function $g : X \to \mathbb{R}$, with $\{g = 0\} = K$. We can also assume that g is non-negative (otherwise use g^2).

Choose a definable Whitney b-regular C^2-stratification $X_\bullet$ of X with the following properties ([vDM96, Theorem 1.19 and Theorem 4.8]):

a) $\mathcal{F}$ is constructible with respect to $X_\bullet$;
b) the definable sets $\{f = 0\}$ and K are unions of strata;
c) the set of non-degenerate covectors (with respect to this stratification) is dense in all fibers of the projection $\pi' : T^*_S M \to S$ (for all strata S).

The last property can be achieved inductively by [vDM96, Theorem 1.19 and Theorem 4.8] and the fact that the dimension of the set of degenerate covectors Υ is $< d := \dim(X)$. Note that the set of non-degenerate covectors is dense in such a fiber $\{\pi' = x\}$ if and only if $\dim(\Upsilon \cap \{\pi' = x\}) < d - \dim(S)$. But the set

$$\{x \in S \mid \dim(\Upsilon \cap \{\pi' = x\}) \geqslant d - \dim(S)\}$$

is a definable subset of S with dimension $< \dim(S)$ (compare [vDr98, Proposition 1.5, p. 65]).

2) There exists a relatively compact open neighborhood U of K in M, and a $\delta_0 > 0$ with the following properties.

a′) $g : \{g < \delta_0\} \cap U \to [0, \delta_0[$ is proper (since $\{g = 0\} = K$ is compact). After restriction to U we can and will assume $M = U$.

b′) g has no critical values in $\{0 < g < \delta_0\}$ with respect to $X_\bullet$ (this follows from property a′ and the C^1-version of the curve selection lemma [vDM96, Lemma 1.17]).

c′) The natural morphism

$$R\Gamma(\{g < \delta\}, R\Gamma_{\{f \geqslant 0\}}\mathcal{F}) \to R\Gamma(K, R\Gamma_{\{f \geqslant 0\}}\mathcal{F})$$

is for all $0 < \delta < \delta_0$ an isomorphism. This follows from the fact that the restriction of $Rg_*(\mathcal{F} \mid \{g < \delta_0\})$ to $[0, \delta_0[$, with δ_0 as in property a′, is $\mathcal{S}$-constructible (see [Sch03, Theorem 2.0.1, p. 83] and [Sch03, Corollary 2.2.1, p. 102]). One can also use property b′ and (a cohomological version of) the first isotopy lemma of Thom (as in [Sch03, § 4.1.1]).

d′) $k \geqslant 0,\ 0 < g(x) < \delta_0,\ f(x) > 0 \Rightarrow k \cdot dg_x + df_x \notin \Lambda$.

One shows the last property indirectly (compare [KS90, pp. 386–387]), and note that

$$\{(x, k) \mid k \geqslant 0, 0 < g(x) < \delta_0, f(x) > 0, k \cdot dg_x + df_x \in \Lambda\}$$

is a definable subset of $X \times \mathbb{R}$:

Otherwise there exists [vDM96, Lemma 1.17] a stratum X^s and a C^1-curve $\gamma : [0, \delta_0[\to X \times \mathbb{R}^2$, with $\gamma(t) =: (x(t), \alpha(t), \beta(t))$ such that $\alpha(t) \geqslant 0$, $\beta(t) > 0$, $g(x(0)) = 0$, $f(x(t)) > 0$ for $t > 0$, and

$$\alpha(t) \cdot dg_{x(t)} + \beta(t) \cdot df_{x(t)} \in T^*_{X^s}X \quad \text{for } t > 0.$$

But this implies $f(x(0)) = 0$, since $\{g = 0\} = K \subset \{f = 0\}$, and

$$\alpha(t) \cdot d/dt\,(g(x(t))) + \beta(t) \cdot d/dt\,(f(x(t))) \equiv 0,$$

since $T^*_{X^s}X$ is a conic isotropic submanifold of T^*X (cf. [KS90, p. 483]).

But by the monotonicity theorem [vDM96, Theorem 4.1] we have that $f \circ x$ and $g \circ x$ are constant or strictly monotonic on $]0, a[$ for $a > 0$ small enough. By the assumptions, this implies the contradiction $d/dt\,(g(x(t))) \geqslant 0$ and $d/dt\,(f(x(t))) > 0$ for $0 < t < a$.

3) Fix a δ with $0 < \delta < \delta_0$, and consider the inclusion $j : \{g < \delta\} \to X$. By the choice of $X_\bullet$ and property b′ we can apply Theorem 1.1. Let $\mathcal{F}' := Rj_*j^*\mathcal{F}$, which is constructible with respect to the induced definable b-regular stratification of $\{g \leqslant \delta\}$. By the curve selection lemma we can find $\epsilon > 0$ so that f has in $\{0 < |f| \leqslant \epsilon\}$ no critical points with respect to this induced stratification.

Note that $\mathrm{supp}(\mathcal{F}') \subset \{g \leqslant \delta\}$ is compact. Therefore we get by [Sch03, Lemma 5.0.1, p. 270] and [Sch03, Lemma 5.1.1(2), p. 296] (compare also with [KS90, Corollary 5.4.19(ii), p. 239]) that

$$R\Gamma(X, R\Gamma_{\{f \geqslant 0\}}\mathcal{F}') \simeq R\Gamma_c(\{f > -\epsilon\}, \mathcal{F}'). \tag{29}$$

Together with property c′ and the general isomorphism

$$R\Gamma(X, R\Gamma_{\{f \geqslant 0\}}\mathcal{F}') \simeq R\Gamma(\{g < \delta\}, R\Gamma_{\{f \geqslant 0\}}\mathcal{F}),$$

this implies the isomorphism

$$R\Gamma(K, R\Gamma_{\{f \geqslant 0\}}\mathcal{F}) \simeq R\Gamma_c(\{f > -\epsilon\}, \mathcal{F}'). \tag{30}$$

Choose an $a \in \mathbb{R}$ with $\mathrm{supp}(\mathcal{F}') \subset \{f < a\}$, and consider $h := -f : U := \{-\epsilon < f < a\} \to [-a, \epsilon[$. Note that $h|\mathrm{supp}(\mathcal{F}'|U)$ is proper. Since f has in $\{0 < |f| \leqslant \epsilon\}$ no critical points with respect to the induced stratification of $\{g \leqslant \delta\}$, we get

$$df(U) \cap \mu\mathrm{supp}(\mathcal{F}'|U) \subset T^*U|_{\{f \geqslant 0\}}. \tag{31}$$

In particular, $df(U) \cap \mu\mathrm{supp}(\mathcal{F}'|U) \subset df(\mathrm{supp}(\mathcal{F}') \cap \{f \geqslant 0\})$ is a compact subset of T^*U. Therefore we can apply Theorem 1.2, part ii to the manifold U, the function h and the sheaf complex $\mathcal{F}'|U$, and get the intersection formula

$$\chi(R\Gamma_c(\{f > -\epsilon\}, \mathcal{F}')) = \sharp([df(U)] \cap [\mathrm{CC}(\mathcal{F}'|U)]). \tag{32}$$

By Equation (30), the proof of the theorem is complete if we show that

$$|\mathrm{CC}(\mathcal{F}')| \cap \{df_x \mid g(x) = \delta, f(x) > -\epsilon\} = \emptyset.$$

For $-\epsilon < f(x) < 0$, we get $df_x \notin |\mathrm{CC}(\mathcal{F}')|$ by (31).

Assume $df_x \in |\mathrm{CC}(\mathcal{F}')|$, with $f(x) \geqslant 0$ and $g(x) = \delta$. By Theorem 1.1 we get $df_x = \omega - c \cdot dg_x$, with $c \geqslant 0$, $\pi(\omega) = x$ and $\omega \in |\mathrm{CC}(\mathcal{F})|$. In the case $f(x) > 0$ we get a contradiction to property d′:

$$c \cdot dg_x + df_x = \omega \in |\mathrm{CC}(\mathcal{F})| \subset \Lambda.$$

So we can assume $f(x) = 0$. In the case $c = 0$ we get

$$df_x = \omega \in |\mathrm{CC}(\mathcal{F})|.$$

But by $g(x) = \delta > 0$ this is impossible, since the intersection of σ_f and $|\mathrm{CC}(\mathcal{F})|$ is contained in the compact subset $I \subset T^*M$, with $\{g = 0\} = K := \pi(I)$. In the case $c > 0$ we get

$$dg_x = (\omega - df_x)/c \in \Lambda,$$

since $|\mathrm{CC}(\mathcal{F})| \subset \Lambda$, and $df_x \in \Lambda$ for $f(x) = 0$ (because by property b $\{f = 0\}$ is a union of strata of $X_\bullet$). But this is a contradiction to property b′. □

References

BDK81 J. L. Brylinski, A. Dubson and M. Kashiwara, *Formule d'indice pour les modules holonomes et obstruction d'Euler locale*, C. R. Acad. Sci. Paris **293** (1981), 573–576.

Bek91 K. Bekka, *C-régularité et trivialité topologique*, in *Warwick Singularity Symposium (1989)*, Lecture Notes in Mathematics, vol. 1462 (Springer, 1991), pp. 42–62.

BLS00 J. P. Brasselet, D. T. Lê and J. Seade, *Euler obstruction and indices of vector fields*, Topology **39** (2000), 1193–1208.

Bor84 A. Borel, *et al.*, *Intersection cohomology*, Progress in Mathematics, vol. 50 (Birkhäuser, Basel, 1984).

Dub84a A. Dubson, *Formule pour l'indice des complexes constructibles et D-modules holonomes*, C. R. Acad. Sci. Paris **298** (1984), 113–116.

Dub84b A. Dubson, *Formule pour les cycles évanescents*, C. R. Acad. Sci. Paris **299** (1984), 181–184.

Fu94 J. G. H. Fu, *Curvature measures of subanalytic sets*, Amer. J. Math. **116** (1994), 819–880.

Gin86 V. Ginsburg, *Characteristic cycles and vanishing cycles*, Invent. Math. **84** (1986), 327–402.

GoM88 M. Goresky and R. MacPherson, *Stratified Morse theory* (Springer, Berlin, 1988).

GrM99 M. Grinberg and R. MacPherson, *Euler characteristics and Lagrangian intersections*, in *Symplectic geometry and topology*, IAS/Park City Math. Ser., vol. 7 (American Mathematical Society, Providence, RI, 1999), 265–293.

Kas73 M. Kashiwara, *Index theorem for maximally overdetermined systems of linear differential equations*, Proc. Japan Acad. **49** (1973), 803–804.

Kas83 M. Kashiwara, *Systems of microdifferential equations*, Progress in Mathematics, vol. 34 (Birkhäuser, Basel, 1983).

Kas85 M. Kashiwara, *Index theorem for constructible sheaves*, Astérisque **130** (1985), 193–209.

KS90 M. Kashiwara and P. Schapira, *Sheaves on manifolds* (Springer, Berlin, 1990).

Lê98 D. T. Lê, *Morsification of D-modules*, Bol. Soc. Mat. Mexicana **3** (1998), 229–248.

Mas01 D. Massey, *A little microlocal Morse theory*, Math. Ann. **321** (2001), 275–294.

Orr87 P. Orro, *Espaces conormaux et densité des fonctions de Morse*, C. R. Acad. Sci. Paris **305** (1987), 269–272.

Sab85 C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Astérisque **130** (1985), 161–192.

Sch02a J. Schürmann, *On the comparison of different notions of geometric categories*, Preprint (2002), math.AG/0201283.

Sch02b J. Schürmann, *A short proof of a formula of Brasselet, Lê and Seade for the Euler obstruction*, Preprint (2002), math.AG/0201316.

Sch03 J. Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne (New Series), vol. 63 (Birkhäuser, Basel, 2003).

SV96 W. Schmid and K. Vilonen, *Characteristic cycles for constructible sheaves*, Invent. Math. **124** (1996), 451–502.

Ta98 L. L. Ta, *Verdier and strict Thom stratifications in o-minimal structures*, Illinois J. Math. **42** (1998), 347–356.

Tro89 D. Trotman, *Une version microlocale de la condition (w) de Verdier*, Ann. Inst. Fourier (Grenoble) **39** (1989), 825–829.

vDM96 L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

vDr98 L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Notes Series, vol. 248 (Cambridge University Press, 1998).

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