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SINGULAR DIRECTIONS IN VEECH SURFACES

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Abstract

Singular directions in a Veech surface are shown to be exactly the directions of its saddle connections.

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1. Introduction

A real d-dimensional vector $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ is singular if for any $\delta > 0$, the system of inequalities

$$\begin{cases} \max_{1\le k\le d} |q\alpha_k - p_k| < \delta/T^{1/d} \\ 0 < q < T \end{cases} \tag{1.1}$$

admits solutions $(p_1 \dots, p_d, q) \in \mathbb{Z}^{d+1}$, provided that T is large enough. When $\delta = 1$, the classical Dirichlet theorem (see [11]) states that (1.1) admits solutions $(p, q) \in \mathbb{Z}^{d+1}$ for T large enough. It is well known that a real number $\alpha \in \mathbb{R}$ is singular if and only if α is rational. However, Cheung and Chevallier [4, 5] showed that when $d \ge 2$, the set of singular d-dimensional vectors has Hausdorff dimension $d^2/(d + 1)$.

A translation surface is a closed Riemann surface associated with a nonzero holomorphic one-form. The lattice $\mathbb{Z}^2$ makes a genus one translation surface, the flat torus T^2, such that each integer vector corresponds to the holonomy vector of some closed geodesic. Conversely, it is classical that any genus one translation surface corresponds to a unique rank-2 lattice in the plane $\mathbb{R}^2$. In general, a translation surface determines a closed, discrete and centro-symmetric subset of the plane $\mathbb{R}^2$, the set of holonomy vectors of oriented saddle connections in the translation surface (see [9, 14]). Hence, it is interesting to investigate Diophantine approximation of real numbers in the context of higher genus translation surfaces (see [1, 2]).

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Let S^1 be the set of unit vectors in the plane $\mathbb{R}^2$. Denote by $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^2$ and by $\angle uv \in [0, \pi]$ the angle between two nonzero vectors u and v in $\mathbb{R}^2$.

DEFINITION 1.1. Let (X, ω) be a translation surface and $\mathcal{Z}$ the set of holonomy vectors of oriented saddle connections. A unit vector $\theta \in S^1$ is a singular direction of (X, ω) if for any $\delta > 0$, the system of inequalities

$$\begin{cases} \|v\|\,|\sin \angle v\theta| < \delta/T \\ \|v\|\,|\cos \angle v\theta| < T \end{cases} \tag{1.2}$$

admits solutions $v \in \mathcal{Z}$, provided that T is large enough.

Hubert and Schmidt [8] showed that the set of holonomy vectors of saddle connections in a translation surface always has a finite Minkowski constant. As a result, there exists $\delta > 0$ such that (1.2) admits solutions $v = (p, q)$ in this set for T large enough.

A Veech surface has a large group of affine symmetries. In a celebrated paper [13], Veech showed that the directional flows of such translation surfaces satisfy the dichotomy: each directional flow is either uniquely ergodic or completely periodic. McMullen [10] completed the classification of genus two Veech surfaces, and Smillie and Weiss [12] gave various geometric characterisations of such translation surfaces.

The main result of this paper is the following theorem.

THEOREM 1.2. *A direction of a Veech surface is singular if and only if it is parallel to some saddle connection in the Veech surface.*

2. Background

2.1. Cheung's $\mathcal{Z}$-expansion. Cheung's $\mathcal{Z}$-expansion theory generalises the geometric interpretation of the classical continued fraction (for details, see [3, 6]).

Let $\mathcal{Z}$ be a discrete, closed and centro-symmetric subset of the plane $\mathbb{R}^2$ and suppose that $\mathcal{Z}$ does not contain the origin. The Minkowski constant $\mu(\mathcal{Z})$ of $\mathcal{Z}$ is the supremum of areas of bounded, convex and centro-symmetric subsets of the plane $\mathbb{R}^2$ which are disjoint from $\mathcal{Z}$. Assume that $\mathcal{Z}$ has a finite Minkowski constant $\mu(\mathcal{Z})$. Then for any $\delta > \mu(\mathcal{Z})/4$, (1.2) has solutions in $\mathcal{Z}$, provided that T is large enough.

DEFINITION 2.1. An element $v \in \mathcal{Z}$ is said to be a $\mathcal{Z}$-convergent to a unit vector $\theta \in S^1$, if $\cos \angle v\theta \geq 0$ and, for any $u \in \mathcal{Z}$ with $\cos \angle u\theta \geq 0$:

(1) $\|v\| \cos \angle v\theta \leq \|u\| \cos \angle u\theta$ implies $\|v\| \sin \angle v\theta \leq \|u\| \sin \angle u\theta$ and
(2) $\|v\| \cos \angle v\theta < \|u\| \cos \angle u\theta$ implies $\|v\| \sin \angle v\theta < \|u\| \sin \angle u\theta$.

The sequence $\{v_k\}$ of $\mathcal{Z}$-convergents to θ is ordered so that $\|v_k\| \cos \angle v_k\theta$ is increasing strictly as k increases or, equivalently, $\|v_k\| \sin \angle v_k\theta$ is decreasing strictly as k increases.

REMARK 2.2. For convenience, we also arrange the sequence of $\mathcal{Z}$-convergents to a unit vector θ so that the zeroth convergent v_0 realises the minimal $\|v\| \cos \angle v\theta$ with v

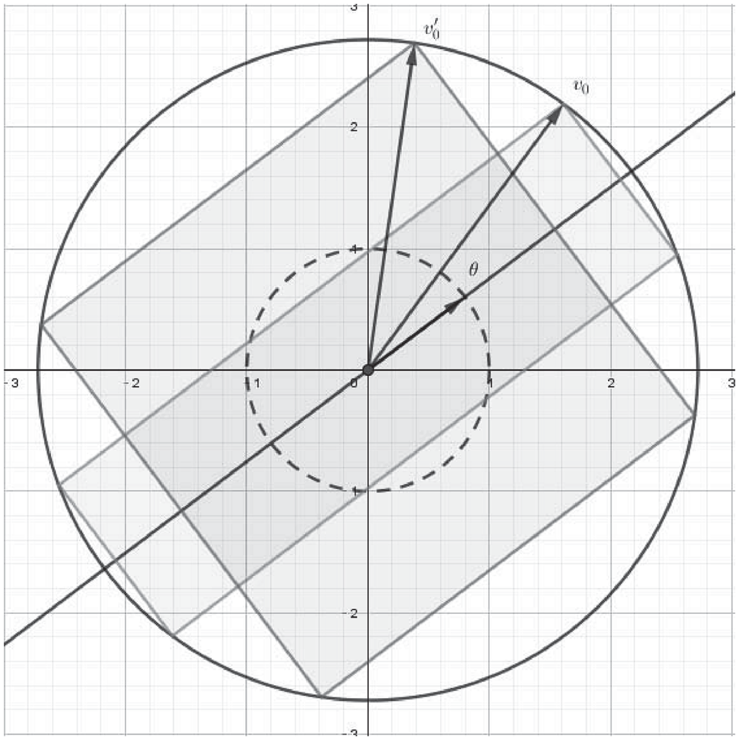

FIGURE 1. The zeroth $\mathcal{Z}$-convergent.

taken over all $\mathcal{Z}$-convergents. This makes sense because any vector of minimal length in $\mathcal{Z}$ is a $\mathcal{Z}$-convergent to any unit vector. In other words, the zeroth convergent v_0 is just one of the vectors v of minimal length in $\mathcal{Z}$ such that $\|v\| \cos \angle v\theta$ is minimal (see Figure 1). Under our convention, there may be two choices of the kth convergent for each $k \in \mathbb{Z}$.

2.2. Translation surface. A translation surface (X, ω) is a closed Riemann surface X associated with a nonzero holomorphic one-form ω (see [9, 15] for details). The integral of the holomorphic one-form ω induces a translation structure, that is, an atlas of coordinates on the underlying surface between which the transition functions are locally translations. Pulling back the Euclidean metric on the plane by these coordinates induces a flat metric on the underlying surface, which has trivial holonomy and zero Gauss curvature, except at a finite number of conical singular points corresponding to zeros of the one-form.

Let S^1 be the circle in the complex plane consisting of unit complex numbers. For any $\theta \in S^1$, the line segments on the plane parallel to θ induce a foliation $\mathcal{F}_\theta$ of the surface, called the flow in the direction θ, whose leaves, named θ-trajectories, are geodesics with respect to the flat metric. A saddle connection of a translation surface is a geodesic segment, which has conical points as end points and contains no conical points in its interior. The holonomy vector of an oriented saddle connection γ is the integral of ω along γ. A direction of (X, ω) is minimal if there is no leaf of the directional flow which is a saddle connection.

DEFINITION 2.3. Let (X, ω) be a translation surface and let $\mathcal{Z}$ be the set of holonomy vectors of its oriented saddle connections. A direction θ in (X, ω) is singular if θ is $\mathcal{Z}$-singular.

Two translation surfaces are affinely isomorphic if there is a homeomorphism between them, given by an affine isomorphism, which preserves singular points and is represented by affine maps under their translation structures. By the connectivity, the linear part of these affine representatives is unique; it is called the derivative of the affine isomorphism. When the derivative is the identity matrix, the affine isomorphism is an isomorphism and the two translation surfaces are isomorphic. The Veech group of a translation surface is defined as the group of derivatives of all its affine self-isomorphisms. It is well known that a Veech group is a discrete sub-group of $\mathrm{SL}(2, \mathbb{R})$ (see [7, 14]).

DEFINITION 2.4. A translation surface is called a Veech surface if its Veech group is a lattice in $\mathrm{SL}(2, \mathbb{R})$.

LEMMA 2.5 [12]. *A translation surface is a lattice surface if and only if it has no visual triangle, that is, the set $\mathcal{Z}$ of holonomy vectors of its saddle connections satisfies*

$$\inf\{|u \times v| \neq 0 : u, v \in \mathcal{Z}\} > 0.$$

3. Proof of Theorem 1.2

In the remainder of this paper, we assume that (X, ω) is a Veech surface and $\mathcal{Z}$ is the set of holonomy vectors of oriented saddle connections in (X, ω). Recall that the Minkowski constant $\mu(\mathcal{Z})$ is finite.

A unit vector θ is said to be $\mathcal{Z}$-minimal if the kth $\mathcal{Z}$-convergent exists for any positive integer $k \in \mathbb{Z}_{>0}$.

LEMMA 3.1. *A unit vector θ is $\mathcal{Z}$-minimal if and only if it is not parallel to some vector in $\mathcal{Z}$.*

PROOF. *Necessity.* By contradiction, assume that θ is parallel to an element w in $\mathcal{Z}$. Recall that $\mathcal{Z}$ is discrete and centro-symmetric and does not contain the origin. Hence, there exists a unique element $w_0 \in \mathcal{Z}$, with $\angle w_0\theta = 0$, which is parallel to w and has minimal length. As a result, $\|w_0\| \sin \angle w_0\theta = 0$ so that w_0 is a $\mathcal{Z}$-convergent to θ. It is sufficient to show that

$$\|v\| \cos \angle v\theta \leq \|w_0\|$$

for any $\mathcal{Z}$-convergent v to θ, which contradicts the assumption that θ is $\mathcal{Z}$-minimal. In fact, if there is a $\mathcal{Z}$-convergent u to θ such that $\|u\| \cos \angle u\theta > \|w_0\|$, then the definition

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of $\mathcal{Z}$-convergents implies that

$$\|u\| \sin \angle u\theta < \|w_0\| \sin \angle w_0\theta = 0,$$

which cannot happen.

Sufficiency. Assume by contradiction that θ is not $\mathcal{Z}$-minimal. Then there is a $\mathcal{Z}$-convergent v to θ such that

$$\|v\| \cos \angle v\theta \geq \|u\| \cos \angle u\theta$$

for any convergent u to θ. Since $\mathcal{Z}$ has a finite Minkowski constant, $\|v\| \sin \angle v\theta = 0$. Otherwise, the infinite strip

$$\{w \in \mathbb{R}^2 : \|w\| \sin \angle w\theta < \|v\| \sin \angle v\theta\}$$

contains no point in $\mathcal{Z}$ and has infinite area. As a result, θ is parallel to $v \in \mathcal{Z}$. □

LEMMA 3.2. *Let v_k and v_{k+1} be a pair of consecutive $\mathcal{Z}$-convergents to θ. Then*

$$2 \sin \angle v_k v_{k+1} < \sin \angle v_k\theta \cos \angle v_{k+1}\theta \leq \mu(\mathcal{Z})/\|v_k\|\|v_{k+1}\|. \tag{3.1}$$

PROOF. From the definition of $\mathcal{Z}$-convergents, for any $k \in \mathbb{Z}$, the system of inequalities

$$\begin{cases} \|v\|\,|\sin \angle v\theta| < \|v_k\| \sin \angle v_k\theta \\ \|v\|\,|\cos \angle v\theta| < \|v_k\| \cos \angle v_k\theta \end{cases}$$

admits no solution in $\mathcal{Z}$. Since v_{k+1} is the next convergent after v_k,

$$\|v\| \sin \angle v\theta \geq \|v_k\| \sin \angle v_k\theta$$

for any $v \in \mathcal{Z}$ satisfying

$$\|v_k\| \cos \angle v_k\theta < \|v\| \cos \angle v\theta < \|v_{k+1}\| \cos \angle v_{k+1}\theta.$$

Therefore, the rectangle

$$\{w \in \mathbb{R}^2 : \|w\|\,|\cos \angle w\theta| < \|v_{k+1}\|\,|\cos \angle v_{k+1}\theta|,\ \|w\|\,|\sin \angle w\theta| < \|v_k\|\,|\sin \angle v_k\theta|\}$$

contains no point in $\mathcal{Z}$ and has area

$$\|v_k\| \sin \angle v_k\theta\, \|v_{k+1}\| \cos \angle v_{k+1}\theta$$

(see Figure 2). Together with the definition of the Minkowski constant $\mu(\mathcal{Z})$, this gives the right-hand inequality of (3.1). The left-hand inequality of (3.1) follows from the fact that the rectangle with four vertices $\pm v_k$ and $\pm v_{k+1}$ has area

$$2|v_k \times v_{k+1}| = 2\|v_k\|\,\|v_{k+1}\| \sin \angle v_k v_{k+1}$$

and is contained in the rectangle constructed above. □

THEOREM 3.3. *A $\mathcal{Z}$-minimal direction θ is $\mathcal{Z}$-singular if and only if its kth $\mathcal{Z}$-convergent v_k satisfies*

$$\lim_{k \to +\infty} \|v_{k+1}\| \cos \angle v_{k+1}\theta\, \|v_k\| \sin \angle v_k\theta = 0.$$

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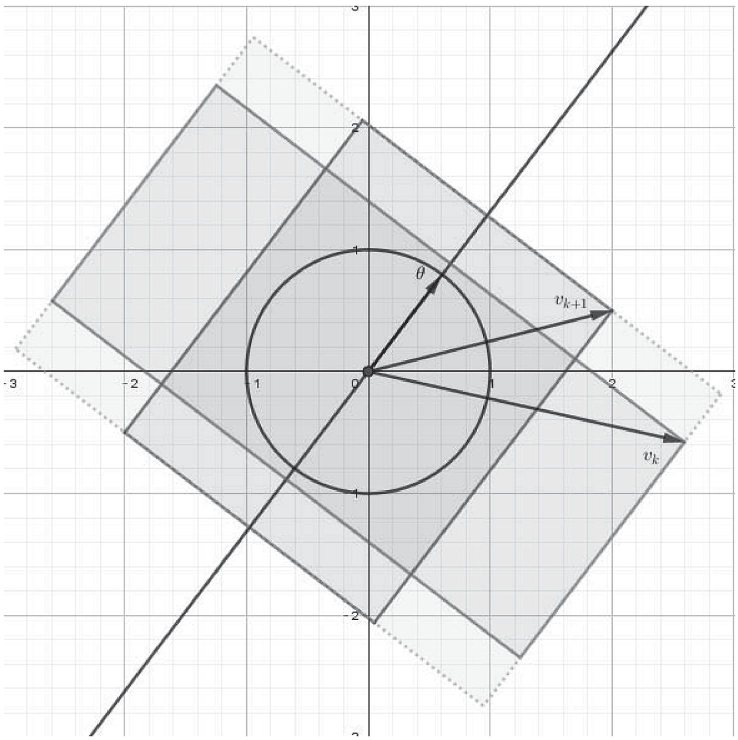

FIGURE 2. A pair of consecutive $\mathcal{Z}$-convergents.

PROOF. As θ is $\mathcal{Z}$-minimal, the height of its kth $\mathcal{Z}$-convergent strictly increases to infinity as k tends to infinity.

Sufficiency. Assume that

$$\lim_{k\to+\infty} \|v_{k+1}\| \cos \angle v_{k+1}\theta \, \|v_k\| \sin \angle v_k\theta = 0.$$

Then for any $\delta > 0$, there exists k_0 such that

$$\|v_{k+1}\| \cos \angle v_{k+1}\theta \, \|v_k\| \sin \angle v_k\theta < \delta$$

for any $k \geq k_0$. Since $\|v_k\| \cos \angle v_k\theta$ strictly increases to infinity as k increases to infinity, for any $T > \|v_{k_0}\| \cos \angle v_{k_0}\theta$, there is a unique $k \geq k_0$ such that

$$\|v_k\| \cos \angle v_k\theta < T \leq v_{k+1} \cos \angle v_{k+1}\theta.$$

As a result,

$$\|v_k\| \sin v_k\theta < \frac{\delta}{\|v_{k+1}\| \cos \angle v_{k+1}\theta} \leq \delta/T,$$

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which means that v_k is a solution of the system of inequalities

$$\begin{cases} \|v\| \sin \angle v\theta < \delta/T \\ 0 < \|v\| \cos \angle v\theta < T \end{cases}$$

for $T > v_{k_0}$.

Necessity. By the definition of ω-singularity, for any $\delta > 0$, there exists k_0 such that for any $k \geq k_0$, the system of inequalities

$$\begin{cases} \|v\| \sin \angle v\theta < \delta/(\|v_{k+1}\| \cos \angle v_{k+1}\theta) \\ 0 < \|v\| \cos \angle v\theta < \|v_{k+1}\| \cos \angle v_{k+1}\theta \end{cases}$$

has solutions $v \in \mathcal{Z}$. Either

$$0 < \|v\| \cos \angle v\theta < \|v_k\| \cos \angle v_k\theta$$

or

$$\|v_k\| \cos \angle v_k\theta \leq \|v\| \cos \angle v\theta < \|v_{k+1}\| \cos \angle v_{k+1}\theta.$$

The definition of the consecutive pair of $|v_k|$ and $|v_{k+1}|$ implies that

$$\|v_k\| \sin \angle v_k\theta \leq \|v\| \sin \angle v\theta$$

for both of the above cases. Therefore,

$$\|v_{k+1}\| \cos \angle v_{k+1}\theta \, \|v_k\| \sin \angle v_k\theta < \delta$$

for any $k \geq k_0$. Since $\delta > 0$ is arbitrary,

$$\lim_{k\to+\infty} \|v_{k+1}\| \cos \angle v_{k+1}\theta \, \|v_k\| \sin \angle v_k\theta = 0. \qquad \square$$

Proof of Theorem 1.2 To prove the theorem, we first note that the sufficiency is obvious. For the necessity, by contradiction, we assume that θ is a singular and a minimal direction of (X, ω). Combining Lemma 3.2 and Theorem 3.3,

$$\lim_{k\to+\infty} |v_k \times v_{k+1}| = 0.$$

Since $|v_k \times v_{k+1}| \neq 0$, this yields

$$\inf\{|u \times v| \neq 0 : u, v \in \mathcal{Z}\} = 0.$$

Recall that (X, ω) is a Veech surface. By Lemma 2.5, we get the contradiction. □

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