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TRANSFORMATION GROUPS OF STRONG CHARACTERISTIC 0

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It is shown that a transformation group (X, T, π) is of strong characteristic 0 if and only if it is of P-strong characteristic 0 for some replete semigroup P in the phase group, provided that all orbit closures are compact. It is shown also that, under certain conditions, (X, T, π) is of P-strong characteristic 0 if and only if $(X \times X, T, \pi \times \pi)$ is Liapunov stable.

By a transformation group we mean a triple (X, T, π), where X is a locally compact Hausdorff space and T is a generative group [5] acting on X by π; that is $\pi : X \times T \to X$ is a continuous map satisfying

(1) $\pi(x, 0) = x$ for every $x \in X$, where 0 denotes the identity of T, and

(2) $\pi\big(\pi(x, s), t\big) = \pi(x, s+t)$ for $x \in X$ and $s, t \in T$.

For brevity $\pi(x, t)$ is denoted by xt.

In 1970 Ahmad [1] introduced the notion of characteristic 0^+ in continuous flows using prolongation sets. In the same year Hajek [6] extended the notions of prolongation to transformation groups. Using Hajek's ideas, Knight [7] was able to define and study transformation groups of characteristic 0. This study was later pursued by Elaydi and Kaul [4], [3]. In an attempt to generalize the unilateral versions of prolongations the author [2] introduced the P-prolongations, where P is

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a replete semigroup in T [5]. Then the property of characteristic 0^+ is generalized to that of P-characteristic 0 [2]. Following these ideas Elaydi and Kaul [3] studied the property of strong characteristic 0, a stronger version of characteristic 0.

In this paper we define the property of P-strong characteristic 0 in a way similar to that of P-characteristic 0.

For the convenience of the reader we give the definitions of the basic notions used. For $x \in X$ and a replete semigroup P in T, we have the following definitions:

(1) the P-limit set of x,

$$L^P(x) = \cap\{\overline{xtP} \mid t \in P\} ;$$

(2) the P-prolongation set of x,

$$D^P(x) = \cap\{\overline{VP} \mid V \text{ is a neighborhood of } x\} .$$

We remark here that $L^P(x)$ is always closed and invariant. The set $D^P(x)$ is closed and P-invariant; that is $D^P(x)t \subset D^P(x)$ for each $t \in P$. Furthermore, $y \in D^P(x)$ if and only if there are nets $\{x_i\}$ in X and $\{p_i\}$ in P with $x_i \to x$ and $x_ip_i \to y$ [2].

Let $X^* = X \cup \{\infty\}$ be the one point compactification of X. Then (X, T, π) can be extended to (X^*, T, π^*), where $\pi^*(x, t) = \pi(x, t)$ for $x \in X$ and $t \in T$ and $\pi^*(\infty, t) = \infty$ for $t \in T$. The P-limit set and the P-prolongation set of $x \in X^*$ in (X^*, T, π^*) are denoted by $L_*^P(x)$ and $D_*^P(x)$, respectively. The closure of a set A in X^* is denoted by $\overline{A}^*$.

A point $x \in X$ is said to be of P-strong characteristic 0 if whenever there are nets $\{x_i\}$ in X and $\{p_i\}$ in P with $x_i \to x$ and $x_ip_i \to y$, then $xp_i \to y$. If in the above definition P is replaced by T, then x is said to be of strong characteristic 0 [3]. As in [2], x is said to be of $\{P\text{-characteristic } 0\}\{\text{characteristic } 0\}$ if $\{D^P(x) = \overline{xP}\}\{D(x) = \overline{xT}\}$. (X, T, π) is said to have the property if every point in X possesses that property. It is clear that if x is of

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P-strong characteristic 0, then it is of P-characteristic 0.

THEOREM 1. *A transformation group* (X, T, π) *is of strong characteristic* 0 *if and only if it is of* P*-strong characteristic* 0, *for some replete semigroup* P *in* T, *provided that* $\overline{xT}$ *is compact for each* $x \in X$.

Proof. The necessity is clear.

The proof of the sufficiency consists of three steps.

(i) We first show that the squared flow $(X \times X, T, \tilde{\pi})$, where $\tilde{\pi}\big((x, y), t\big) = \big(\pi(x, t), \pi(y, t)\big)$, is of P-characteristic 0.

Let $(x, y) \in X \times X$ and let $(a, b) \in D^P(x, y)$. Then there are nets $\{(x_i, y_i)\}$ in $X \times X$ and $\{p_i\}$ in P such that $(x_i, y_i) \to (x, y)$, $(x_i, y_i)p_i \to (a, b)$. This implies that $x_i \to x$, $y_i \to y$, $x_ip_i \to a$ and $y_ip_i \to b$. It follows that $xp_i \to a$ and $yp_i \to b$ and consequently $(x, y)p_i \to (a, b)$. Thus $(a, b) \in \overline{(x, y)P}$. Hence $D^P(x, y) \subset \overline{(x, y)P}$. Since it is always true that $\overline{(x, y)P} \subset D^P(x, y)$, $\overline{(x, y)P} = D^P(x, y)$. Therefore (x, y) is of P-characteristic 0. In fact we have shown that (x, y) is of P-strong characteristic 0.

(ii) In this step we show that $(X \times X, T, \tilde{\pi})$ is of charactistic 0.

We first show that $\overline{(x, y)T}$ is minimal for each $(x, y) \in X \times X$. Since $\overline{(x, y)T} \subset \overline{xT} \times \overline{yT}$ is compact, $L^{P^{-1}}(x, y) \neq \emptyset$, where $P^{-1} = \{p^{-1} \in T \mid p \in P\}$. Let $(c, d) \in L^{P^{-1}}(x, y)$. Then $(c, d) \in \overline{(c, d)T} \subset L^{P^{-1}}(x, y) \subset D^{P^{-1}}(x, y)$. Thus

$$(x, y) \in D^P(c, d) = \overline{(c, d)P} .$$

This implies that

$$\overline{(x, y)T} \subset \overline{(c, d)T} \subset L^{P^{-1}}(x, y) .$$

Let $(e, f) \in \overline{(x, y)T}$. Then $(e, f) \in L^{P^{-1}}(x, y)$. Therefore, as above,

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$\overline{(x, y)T} \subset \overline{(e, f)T}$ and consequently, $\overline{(x, y)T}$ is minimal. Since $L^P(x, y) \neq \emptyset$, it follows that

$$\overline{(x, y)T} = \overline{(x, y)P} = L^P(x, y) .$$

Let $(g, h) \in D(x, y)$. Then there are nets $\{(g_i, h_i)\}$ in $X \times X$ and $\{t_i\}$ in T such that $(g_i, h_i) \to (x, y)$ and $(g_i, h_i)t_i \to (g, h)$. For each i,

$$(g_i, h_i)t_i \in \overline{(g_i, h_i)T} = \overline{(g_i, h_i)P} = D^P(g_i, h_i) .$$

It follows from [2] that $(g, h) \in D^P(x, y) = \overline{(x, y)P} = \overline{(x, y)T}$. Thus $D(x, y) \subset \overline{(x, y)T}$. Hence $D(x, y) = \overline{(x, y)T}$. This shows that $(X \times X, T, \tilde{\pi})$ is of characteristic 0.

(iii) We now show that (X, T, π) is of strong characteristic 0. Assume there is a point $x \in X$ which is not of strong characteristic 0. Then there are nets $\{x_i\}$ in X and $\{t_i\}$ in T such that $x_i \to x$, $x_i t_i \to y \in X$ and $xt_i \not\to y$. Since $\overline{xT}$ is compact, we may assume that $xt_i \to z \in X$. Now $(x_i, x) \to (x, x)$ and $(x_i, x)t_i \to (y, z)$ implies that $(y, z) \in D(x, x)$. From (2) it follows that $(y, z) \in \overline{(x, x)T}$. Consequently, $y = z$ and we thus have a contradiction.

The proof of the theorem is now complete.

We say that a subset M of X is Liapunov stable if for each neighborhood U of M there exists a neighborhood V of M such that $VT \subset U$. A transformation group (X, T, π) is Liapunov stable if $\overline{xT}$ is Liapunov stable for each $x \in X$.

THEOREM 2. *If a transformation group (X, T, π) is of P-strong characteristic, then the squared transformation group $(X \times X, T, \tilde{\pi})$ is Liapunov stable, provided that either X is locally connected or $\overline{(x, y)T}$ is connected for each $(x, y) \in X \times X$. The converse holds whenever $\overline{xT}$ is compact for each $x \in X$.*

Proof. (i) Assume that X is locally connected and suppose that for some $(x, y) \in X \times X$, $\overline{(x, y)T}$ is not Liapunov stable. Since $\overline{(x, y)T}$ is minimal (Theorem 1), it follows that $VT \supset \overline{(x, y)T}$ for every neighborhood V of (x, y). There exists a neighborhood U of $\overline{(x, y)T}$ and a neighborhood filter $\{V_i\}$ of connected open neighborhoods of (x, y) and a net $\{t_i\}$ in T with $V_i t_i \not\subset U$ for each i. Since $V_i t_i$ is also connected, there exists $(x_i, y_i) \in V_i$ such that $(x_i, y_i)t_i \in \partial U$ (the boundary of U). Since ∂U is compact, we may assume that $(x_i, y_i)t_i \to (c, d) \in \partial U$. It follows that $(c, d) \in D(x, y)$. This implies by Theorem 1 that $(c, d) \in \overline{(x, y)T} \subset U$ and we thus have a condition. This shows that $(X \times X, T, \tilde{\pi})$ is Liapunov stable.

(ii) Assume that $\overline{(x, y)T}$ is connected for each $(x, y) \in X \times X$. If $\overline{(x, y)T}$ is not Liapunov stable for some $(x, y) \in X \times X$, then there is a neighborhood U of (x, y) and there exist nets $\{(x_i, y_i)\}$ in U and $\{t_i\}$ in T such that $(x_i, y_i) \to (x, y)$ and $(x_i, y_i)t_i \notin U$ for each i. Since $\overline{(x_i, y_i)T}$ is connected, $\overline{(x_i, y_i)T} \cap \partial U \neq \emptyset$ for each i. Let $(a_i, b_i) \in D(x_i, y_i) \cap \partial U$ for each i. Without loss of generality, we may assume that $(a_i, b_i) \to (a, b) \in \partial U$. Hence $(a, b) \in D(x, y)$ [4, 1.6]. It follows from Theorem 1 that $(a, b) \in \overline{(x, y)T}$ and we thus have a contradiction. Hence $(X \times X, T, \tilde{\pi})$ is Liapunov stable.

To prove the converse under the assumption that $\overline{xT}$ is compact for each $x \in X$ we show first that $X \times X$ is of characteristic 0. If for some $(x, y) \in X \times X$, $D(x, y) \neq \overline{(x, y)T}$, then let $(a, b) \in D(x, y) - \overline{(x, y)T}$. There exists a neighborhood U of $\overline{(x, y)T}$ such that $(a, b) \notin \overline{U}$. Since $\overline{(x, y)T}$ is Liapunov stable, there exists a neighborhood V of $\overline{(x, y)T}$ with $VT \subset U$. Thus $(a, b) \in D(x, y) \subset \overline{VT} \subset \overline{U}$ and we thus have a contradiction. Consequently, $X \times X$ is of characteristic 0. Assume that there exists a point $z \in X$ which is not of P-strong characteristic 0. Then there are nets $\{z_i\}$ in X and $\{p_i\}$ in P such that $z_i \to z$, $z_i p_i \to d$ and $zp_i \not\to d$. Since $\overline{zP}$ is compact, we may assume that $zp_i \to c$. Thus $(c, d) \in D(z, z) = \overline{(z, z)P}$. Thus $c = d$ and we then have a contradiction. This completes the proof of the theorem.

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