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A Larger Class of Ornstein Transformations with Mixing Property

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Abstract. We prove that Ornstein transformations are almost surely totally ergodic provided only that the cutting parameter is not bounded. We thus obtain a larger class of Ornstein transformations with the mixing property.

1 Introduction

In [O] D. S. Ornstein, using a random procedure, introduced a class of rank one transformations and by a probabilistic technique showed that some class of these transformations are almost surely mixing. Precisely, Ornstein shows the existence of a deterministic sequence x in $\{1, 2, 3, 4\}^{\mathbb{N}}$ for which we have almost surely the mixing property for $T_{\omega,x}$. We show in this note that for any sequence of non negative integers, the associated Ornstein transformations have almost surely the mixing property. We will assume that the reader is familiar with the method of cutting and stacking for constructing rank one transformation.

2 Rank one transformation by construction

Using the cutting and stacking method described in [Fr 1], [Fr 2], one can define inductively a family of measure preserving transformations, called rank one transformations, as follows:

Let B_0 be the unit interval equipped with the Lebesgue measure. At stage one we divide B_0 into p_0 equal parts, add spacers and form a stack of height h_1 in the usual fashion. At the k-th stage we divide the stack obtained at the $(k-1)$-st stage into p_{k-1} equal columns, add spacers and obtain a new stack of height h_k. If during the k-th stage of our construction the number of spacers put above the j-th column of the $(k-1)$-st stack is $a_j^{(k-1)}$, $0 \leq a_j^{(k-1)} < \infty$, $1 \leq j \leq p_k$, then we have

$$h_k = p_{k-1}h_{k-1} + \sum_{j=1}^{p_{k-1}} a_j^{(k-1)}.$$

Proceeding thus we get a rank one transformation T on a certain measure space $(X, \mathcal{B}, \nu)$ which may be finite or σ-finite depending on the number of spacers added.

The construction of any rank one transformation thus needs two parameters, *viz.*, $(p_k)_{k=0}^{\infty}$: (parameter of cutting and stacking), and $\big((a_j^{(k)})_{j=1}^{p_k}\big)_{k=0}^{\infty}$: (parameter of spacers). We have by definition

$$T =_{def} T_{(p_k,(a_j^{(k)})_{j=1}^{p_k})_{k=0}^{\infty}}.$$

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3 Ornstein's class of transformations

In Ornstein's construction, the p_k's are rapidly increasing, and the number of spacers, $a_i^{(k)}$, $1 \leq i \leq p_k - 1$, are chosen stochastically as follows: we choose independently, using the uniform distribution on the set $X_k = \{\frac{-h_{k-1}}{2}, \ldots, \frac{h_{k-1}}{2}\}$, the numbers $(x_{k,i})_{i=1}^{p_k-1}$, and x_{k,p_k} is chosen deterministically in X_k. We put, for $1 \leq i \leq p_k$,

$$a_i^{(k)} = h_{k-1} + x_{k,i} - x_{k,i-1}, \quad \text{with} \quad x_{k,0} = 0,$$

one sees that

$$h_{k+1} = p_k(h_k + h_{k-1}) + x_{k,p_k}.$$

So the deterministic sequence of positive integers $(p_k)_{k=0}^{\infty}$ and $(x_{k,p_k})_{k=0}^{\infty}$ completely determine the sequence of heights $(h_k)_{k=1}^{\infty}$. The total measure of the resulting measure space is finite if $\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty$. We will assume that this requirement is satisfied.

We thus have a probability space of Ornstein transformations $\prod_{l=1}^{\infty} X_l^{p_l-1}$ equipped with the natural probability measure $\bigotimes_{l=1}^{\infty} P_l$, where $P_l = \bigotimes_{l=1}^{p_l-1} \mathcal{U}_l$; $\mathcal{U}_l$ is the uniform probability on X_l. We denote this space by $(\Omega, \mathcal{A}, \mathbb{P})$. So $x_{k,i}$, $1 \leq i \leq p_k - 1$, is a projection from Ω onto the i-th co-ordinate space of Ω_k, $1 \leq i \leq p_k - 1$. Naturally each point $\omega = \Big(\omega_k = \big(x_{k,i}(\omega)\big)_{i=1}^{p_k-1}\Big)_{k=1}^{\infty}$ in Ω define the spacers and so a rank one transformation which we denote by $T_{\omega,x}$ where $x = (x_{k,p_k})_{k=1}^{\infty}$ is, say, admissible, *i.e.*,

(i) for each k, $2x_{k,p_k} \leq h_{k-1}$.
(ii) $\sum_{k=1}^{\infty} \frac{x_{k,p_k}}{p_k h_k} < \infty$.

The definition above gives a any general definition of random construction due to Ornstein.

Now, let recall that automorphism is said to be totally ergodic if all its powers are ergodic. It is any easy exercise to see that a measure preserving automorphism is totally ergodic if and only if no root of unity is its eigenvalue. We mention also that in the particular case of Ornstein transformations construted in [O] the construction of the Ornstein probabilistic space Ω is done step by step, continuing for that the construction of associated rank one transformations. Precisely, in order to construct the $(k+1)$-st stage we apply the arithmetical lemma (see [N]) to get the cutting parameter p_k and the set of good spacers parameter at this stage which is the subset of $X_k^{p_k-1}$. Finally, in order to exhibit a mixing rank one transformation, Ornstein proved using a combinatorial argument that one can choose the deterministic sequence $x = (x_{k,p_k})_{k=1}^{\infty}$ in the space $\{1, 2, 3, 4\}^{\mathbb{N}}$ so that a.e. $T_{\omega,x}$ is totally ergodic. We will show that this argument is not necessary by proving the following:

Lemma of Total Ergodicity *If the cutting parameter $(p_k)_{k=1}^{\infty}$ is not bounded then the associated Ornstein's transformations are almost surely totally ergodic for any fixed admissible sequence $x = (x_{k,p_k})_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$.*

Proof We need the following characterization of the eigenvalues of a rank one transformation due to Choksi and Nadkarni [C-N, Theorem 4]:

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Let T be a rank one transformation with parameters $\big(p_k, (a_i^{(k)})_{i=1}^{p_k}\big)_{k=0}^{\infty}$. Let

$$P_k(z) = \sum_{j=0}^{p_k-1} z^{jh_k + \sum_{i=0}^{j} a_i^{(k)}} \quad \text{with} \quad a_0^{(k)} = 0.$$

Then z is an eigenvalue of T if and only if

$$\sum_{k=1}^{\infty} \Big(1 - \frac{1}{p_k^2} |P_k(z)|^2\Big) < \infty.$$

We apply this criterion to the setting on hand. Here

$$P_k(z) = \sum_{j=0}^{p_k-1} z^{j(h_k+h_{k-1})+x_{k,j}},$$

so, for $z \neq 1$,

$$\frac{1}{p_k^2}|P_k(z)|^2 = \frac{1}{p_k} + \frac{1}{p_k^2} \sum_{p\neq q} z^{(p-q)(h_k+h_{k-1})} z^{x_{k,p}-x_{k,q}}.$$

Integrating with respect to $\mathbb{P}$ and using the independence of $x_{k,p}$ and $x_{k,q}$ when p is different from q we have

$$\int_\Omega \frac{1}{p_k^2}|P_k(z)|^2\, dP = \frac{1}{p_k} + \frac{1}{p_k^2} \sum_{p\neq q} z^{(p-q)(h_k+h_{k-1})} \frac{1}{(h_{k-1}+1)^2} \Big| \sum_{s=-\frac{h_{k-1}}{2}}^{\frac{h_{k-1}}{2}} z^s \Big|^2.$$

Now

$$\frac{1}{(h_{k-1}+1)^2} \Big| \sum_{s=-\frac{h_{k-1}}{2}}^{\frac{h_{k-1}}{2}} z^s \Big|^2 \leq \frac{1}{(h_{k-1}+1)^2} \frac{|1 - z^{h_{k-1}+1}|^2}{|1-z|^2} \leq \frac{1}{h_{k-1}^2} \frac{4}{|1-z|^2}.$$

Thus

$$\int_\Omega \frac{1}{p_k^2}|P_k(z)|^2\, dP \leq \Big(\frac{1}{p_k} + \frac{4}{h_{k-1}^2 |1-z|^2}\Big) \to 0$$

as $k \to \infty$. So we can extract a subsequence which converges almost surely to 0:

$$\frac{1}{p_{k_n}} \Big| \sum_{j=0}^{p_{k_n}-1} z^{j(h_{k_n}+h_{k_n-1})+x_{k_n,j}} \Big| \to 0$$

$\mathbb{P}$ a.s. Thus for $z \neq 1$

$$\sum_{k=1}^{\infty} \Big(1 - \frac{1}{p_k^2}|P_k(z)|^2\Big) = \infty$$

$\mathbb{P}$ a.s. So we have

$$\mathbb{P}\{\omega : z \text{ is an eigenvalue of } T_\omega\} = 0.$$

Now

$$\{\omega : T_\omega \text{ is not totally ergodic}\} = \bigcup_{\{z: z \text{ a root of unity}\}} \{\omega : z \text{ is an eigenvalue of } T_\omega\},$$

a union over a countable set. So

$$\mathbb{P}\{\omega : T_\omega \text{ is not totally ergodic}\} = 0$$

and the lemma is proved.

So we have proved this more general theorem of Ornstein :

Ornstein's Theorem *There exists a sequence $(p_k)_{k\geq 0}$, where $p_k > 10^k$, such that, for all admissible $x \in \mathbb{N}^{\mathbb{N}}$, we have:*

$$\mathbb{P}\{\omega : T_{\omega,x} \textit{ is mixing}\} = 1.$$

A more approachable proof of classical Ornstein theorem can be founded in [N].

Ornstein showed that each member of this family of transformations commutes only with its powers, *i.e.*, has trivial centralizer and further that it has no non-trivial factors, *i.e.*, it is prime. Both these properties are also shared by the much simpler Chacon's transformation. One can prove these facts using the subsequent work of J. King. A well known theorem of J. King [K1] says that any $\frac{1}{2}$-partial mixing rank one has a minimal self joinings so by [R, p. 117] the centralizer of T is trivial and T is prime.

It seems natural to ask if whether in the above construction there is mixing or at least weak mixing almost surely if we take $p_k = 10^k$?

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