Bull. Aust. Math. Soc. **92** (2015), 405–419
doi:10.1017/S0004972715000738

FRACTAL BASES FOR BANACH SPACES OF SMOOTH FUNCTIONS

M. A. NAVASCUÉS, P. VISWANATHAN✉, A. K. B. CHAND,
M. V. SEBASTIÁN and S. K. KATIYAR

(Received 8 March 2015; accepted 9 May 2015; first published online 30 July 2015)

Abstract

This article explores the properties of fractal interpolation functions with variable scaling parameters, in the context of smooth fractal functions. The first part extends the Barnsley–Harrington theorem for differentiability of fractal functions and the fractal analogue of Hermite interpolation to the present setting. The general result is applied on a special class of iterated function systems in order to develop differentiability of the so-called α-fractal functions. This leads to a bounded linear map on the space $C^k(I)$ which is exploited to prove the existence of a Schauder basis for $C^k(I)$ consisting of smooth fractal functions.

2010 *Mathematics subject classification*: primary 28A80; secondary 41A05, 46B15.

Keywords and phrases: smooth fractal function, Hermite fractal interpolation function, fractal operator, topological isomorphism, Schauder basis, Markushevich basis.

1. Introduction

Barnsley [1] proposed the concept of a fractal interpolation function (FIF), which initiated a new research field in approximation theory. By a fractal function, we mean a function whose graph is the attractor of an iterated function system (IFS). Fractal functions not only form the basis of a constructive approximation for nondifferentiable functions but also subsume various traditional smooth approximation techniques. For certain problems, fractal functions provide better approximants than their classical nonrecursive counterparts (see, for instance, [2, 12]).

In almost all cases, the FIFs are generated from IFSs whose free parameters, the so-called scaling factors, are constant. To provide more flexibility and to fit complicated curves that show less self-similarity, FIFs with variable scaling factors (function scalings) have been introduced and their analytical properties, such as smoothness, stability and sensitivity, investigated (see [18]). In the first part of this article, we seek conditions on elements of the IFS with variable scaling factors so that the corresponding FIF is C^k-continuous. Towards this end, we extend the theorem on differentiability of FIFs due to Barnsley and Harrington [3] to the present setting of function scalings. Thus, this part of the article may be viewed as a sequel to [18]. This leads naturally to the construction of Hermite FIFs, which were introduced in [11], but now with the setting of variable scalings, and provides a solution procedure with more flexibility to the classical Hermite interpolation problem using differentiable fractal functions.

Barnsley [1] and Navascués [7] observed that by a suitable choice of IFS whose elements are selected in terms of a prescribed continuous function f, an entire family of fractal functions f^α, called the α-fractal functions, can be constructed to interpolate and approximate f. We apply our theorem on differentiability of FIFs with variable scalings to obtain conditions for fractal functions f^α to be k-times continuously differentiable whenever the source function f is. This gives a correspondence $f \mapsto f^\alpha$ on the space $C^k(I)$ of all k-times continuously differentiable functions on a real compact interval I. We establish properties of this map which eventually helps to prove that $C^k(I)$ admits a Schauder basis consisting of smooth fractal functions with variable scaling parameters. Overall, the current article demonstrates anew that the fractal functions are not as strange as they may appear at first glance. On the contrary, they enjoy interesting connections with other branches of mathematics, including approximation theory, numerical analysis, functional analysis and operator theory.

2. Notation and preliminaries

We reserve the notation $C(I)$ for the Banach space of real-valued continuous functions defined on a real compact interval I with the norm

$$\|f\|_\infty := \max_{x\in I} |f(x)|.$$

Further, we denote by $C^k(I)$ the Banach space of real-valued functions having k continuous derivatives with the norm

$$\|f\|_k := \max\{\|f^{(r)}\|_\infty : r = 0, 1, \dots, k\},$$

although, as proved in [8], all the results obtained can be generalised to the complex field. For any $r \in \mathbb{N}$, let $\mathbb{N}_r = \{1, 2, \dots, r\}$ and $\mathbb{N}_r^0 := \mathbb{N}_r \cup \{0\}$. As usual, we shall denote by X^* the topological dual of a normed linear space X. For any $m \in \mathbb{N}$ and for an arbitrary set A, A^m denotes the cartesian product $A \times A \times \cdots \times A$ (m times). For a bounded linear operator $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$, the operator norm is defined as

$$\|T\| = \sup\{\|Tx\|_Y : x \in X, \|x\|_X \le 1\}.$$

Let (X, d) be a complete metric space. Suppose that m continuous maps $f_1, f_2, \dots, f_m$ on X are given. We set $\mathbf{F} = \{f_1, f_2, \dots, f_m\}$. The pair $\{X; \mathbf{F}\}$ is called an Iterated Function System (IFS). A subset A of X is said to be invariant with respect to $\mathbf{F}$ if

$$A = \bigcup_{i=1}^{m} f_i(A).$$

For a given set $\mathbf{F}$ of contractions, it can be shown that there is a unique nonempty compact subset A of X which is invariant with respect to $\mathbf{F}$. This invariant set A is referred to as an attractor or a deterministic fractal. Barnsley [1] explored the notion of IFS to define a function which interpolates a prescribed data set and whose graph is a fractal, as follows.

Consider a data set $\{(x_i, y_i) \in \mathbb{R}^2 : i \in \mathbb{N}_N\}$, $N > 2$, with strictly increasing abscissae. Let $I = [x_1, x_N]$, $I_i = [x_i, x_{i+1}]$ for $i \in \mathbb{N}_{N-1}$ and $L_i : I \to I_i$ be contraction homeomorphisms such that

$$L_i(x_1) = x_i, \quad L_i(x_N) = x_{i+1}. \tag{2.1}$$

Suppose $0 \le r_i < 1$ for $i \in \mathbb{N}_{N-1}$ and consider continuous maps $F_i : I \times \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{gathered} F_i(x_1, y_1) = y_i, \quad F_i(x_N, y_N) = y_{i+1}, \\ |F_i(x, y) - F_i(x, y^*)| \le r_i |y - y^*| \quad x \in I, \ y, y^* \in \mathbb{R}. \end{gathered} \tag{2.2}$$

Now one can define functions $W_i : I \times \mathbb{R} \to I_i \times \mathbb{R} \subseteq I \times \mathbb{R}$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)) \quad \forall i \in \mathbb{N}_{N-1}.$$

Let $\mathbf{W} := \{W_i : i \in \mathbb{N}_{N-1}\}$.

THEOREM 2.1 [1]. *The IFS $\{I \times \mathbb{R}; \mathbf{W}\}$ has a unique attractor G which is the graph of a continuous function $g : I \to \mathbb{R}$ satisfying $g(x_i) = y_i$ for all $i \in \mathbb{N}_N$. Furthermore, if $C_{y_1, y_N}(I) := \{h \in C(I) : h(x_1) = y_1, h(x_N) = y_N\}$ is endowed with the uniform metric and $T : C_{y_1,y_N}(I) \to C_{y_1,y_N}(I)$ is defined by $Th(x) = F_i(L_i^{-1}(x), h(L_i^{-1}(x)))$, $x \in I_i$, $i \in \mathbb{N}_{N-1}$ then the function g is the unique fixed point of T.*

The function g appearing in the preceding theorem is a FIF; g satisfies the functional equation

$$g(L_i(x)) = F_i(x, g(x)) \quad x \in I, \ i \in \mathbb{N}_{N-1}.$$

A common IFS in the study of FIFs emanates from the maps

$$L_i(x) = a_i x + b_i, \quad F_i(x, y) = \alpha_i y + q_i(x), \tag{2.3}$$

where $\{\alpha_i : \alpha_i \in (-1, 1), \ i \in \mathbb{N}_{N-1}\}$ acts as a family of parameters termed vertical scaling factors, and for each $i \in \mathbb{N}_{N-1}$, $q_i : I \to \mathbb{R}$ is a continuous function satisfying

$$q_i(x_1) = y_i - \alpha_i y_1, \quad q_i(x_N) = y_{i+1} - \alpha_i y_N.$$

On account of Theorem 2.1, the FIF defined through the IFS with the maps in (2.3) satisfies the functional equation

$$g(x) = \alpha_i g(L_i^{-1}(x)) + q_i(L_i^{-1}(x)) \quad x \in I_i, \ i \in \mathbb{N}_{N-1}. \tag{2.4}$$

Barnsley pointed out that the special class of IFSs in (2.3) can be used to associate a family of continuous fractal functions with a prescribed $f \in C(I)$. In (2.3), take

$$q_i(x) = f(L_i(x)) - \alpha_i b(x),$$

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

where $b \in C(I)$ interpolates f at the extremes of the interval I. In view of Theorem 2.1, the IFS under consideration provides a FIF which we denote by $f^{\alpha}_{\Delta,b} = f^{\alpha}$, where $\Delta = \{x_1, x_2, \ldots, x_N\}$ is a chosen partition of I and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (-1, 1)^{N-1}$ is called a scale vector. Note that $f^{\mathbf{0}} = f$ and

$$f^{\alpha}(x) = f(x) + \alpha_i (f^{\alpha} - b)(L_i^{-1}(x)) \quad x \in I_i,\ i \in \mathbb{N}_{N-1}.$$

Navascués [7] observed that if the 'base function' b depends linearly on f, say $b = Lf$, where $L : C(I) \to C(I)$ is a bounded linear map, then the correspondence $f \mapsto f^{\alpha}$ determines a bounded linear (fractal) operator $\mathcal{F}^{\alpha}$ on $C(I)$. This operator connects fractal functions with fields such as functional analysis, operator theory and approximation theory (see, for instance, [8, 9, 16]). In [17], suitable values of the scaling factors are identified so that f^{α} preserves fundamental shape properties such as positivity, monotonicity and convexity inherent in the function f.

Recently, Wang and Yu [18] extended the class of FIFs appearing in (2.3) by considering variable scaling parameters (function scaling factors) $\alpha_i \in C(I)$ satisfying $\|\alpha_i\|_\infty := \sup\{|\alpha_i(x)| : x \in I\} < 1$ instead of constant scaling factors $\alpha_i \in (-1, 1)$, $i \in \mathbb{N}_{N-1}$. The corresponding FIF satisfies

$$g(x) = \alpha_i(L_i^{-1}(x)) g(L_i^{-1}(x)) + q_i(L_i^{-1}(x)) \quad x \in I_i,\ i \in \mathbb{N}_{N-1} \tag{2.5}$$

and the $\boldsymbol{\alpha}$-fractal function f^{α} satisfies

$$f^{\alpha}(x) = f(x) + \alpha_i(L_i^{-1}(x))[f^{\alpha}(L_i^{-1}(x)) - b(L_i^{-1}(x))] \quad x \in I_i,\ i \in \mathbb{N}_{N-1}. \tag{2.6}$$

In [18], analytical properties such as Hölder continuity, stability and sensitivity of the FIF g (cf. (2.5)) are studied. In the next section, we shall investigate differentiability of the fractal function g and related issues.

3. Smooth FIFs with variable scalings

The Barnsley–Harrington (BH) theorem [3], which is stated below, gives conditions on the parameters for the C^k-continuity of the FIF g (cf. (2.4)).

THEOREM 3.1 [3]. *Let $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ be a given data set with strictly increasing abscissae. Let $L_i(x) = a_i x + b_i$ and $F_i(x, y) = \alpha_i y + q_i(x)$ satisfy (2.1) and (2.2) respectively for $i \in \mathbb{N}_{N-1}$. Suppose that for some integer $k \geq 0$, $|\alpha_i| < a_i^k$ and $q_i \in C^k(I)$ for $i \in \mathbb{N}_{N-1}$. Let*

$$F_{i,p}(x, y) = \frac{\alpha_i y + q_i^{(p)}(x)}{a_i^p}, \quad y_{1,p} = \frac{q_1^{(p)}(x_1)}{a_1^p - \alpha_1}, \quad y_{N,p} = \frac{q_{N-1}^{(p)}(x_N)}{a_{N-1}^p - \alpha_{N-1}} \quad p \in \mathbb{N}_k.$$

If $F_{i-1,p}(x_N, y_{N,p}) = F_{i,p}(x_1, y_{1,p})$ for $i = 2, 3, \ldots, N-1$ and $p \in \mathbb{N}_k$ then the IFS $\{I \times \mathbb{R}; (L_i(x), F_i(x, y)) : i \in \mathbb{N}_{N-1}\}$ determines a FIF $g \in C^k(I)$. Further, $g^{(p)}$ is the FIF determined by $\{I \times \mathbb{R}; (L_i(x), F_{i,p}(x, y)) : i \in \mathbb{N}_{N-1}\}$ for $p \in \mathbb{N}_k$.

In [3], the construction of a smooth fractal function starts with the fact that the indefinite integral of a C^0-FIF is again a FIF, albeit for a different IFS. However, due to the appearance of the variable scaling functions, it is not clear whether the integral of g appearing in (2.5) is a FIF. Therefore, in contrast to the 'integral approach' of [3], we obtain k-times continuously differentiable fractal functions with variable scalings as the fixed point of a suitable operator. This approach unifies various methods for constructing fractal splines (see, for instance, [4, 13]) in the more general setting of function scalings. We now state the main theorem of this section.

THEOREM 3.2. *Let $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ be a given set of interpolation data with strictly increasing abscissae. For $i \in \mathbb{N}_{N-1}$, let $L_i(x) = a_i x + b_i$ and $F_i(x, y) = F_{i,0}(x, y) = \alpha_i(x)y + q_i(x)$ satisfy (2.1) and (2.2) respectively. Suppose $y_{1,p}$ and $y_{N,p}$, $p \in \mathbb{N}_k^0$, are arbitrarily chosen real numbers except that $y_{1,0} = y_1$ and $y_{N,0} = y_N$. For $i \in \mathbb{N}_{N-1}$, assume that there exist functions α_i and q_i in $C^k(I)$ such that $\|\alpha_i\|_k < (a_i/2)^k$ and for $i = 2, 3, \dots, N-1$, $p \in \mathbb{N}_k$,*

$$\frac{\sum_{j=0}^{p} \binom{p}{j} y_{1,j}\, \alpha_i^{(p-j)}(x_1) + q_i^{(p)}(x_1)}{a_i^p} = \frac{\sum_{j=0}^{p} \binom{p}{j} y_{N,j}\, \alpha_{i-1}^{(p-j)}(x_N) + q_{i-1}^{(p)}(x_N)}{a_{i-1}^p}, \tag{3.1}$$

$$\begin{aligned} q_1^{(p)}(x_1) &= y_{1,p}\, a_1^p - \sum_{j=0}^{p} \binom{p}{j} y_{1,j}\, \alpha_1^{(p-j)}(x_1), \\ q_{N-1}^{(p)}(x_N) &= y_{N,p}\, a_{N-1}^p - \sum_{j=0}^{p} \binom{p}{j} y_{N,j}\, \alpha_{N-1}^{(p-j)}(x_N). \end{aligned} \tag{3.2}$$

Then the corresponding FIF $g \in C^k(I)$ and, for $p \in \mathbb{N}_k$,

$$g^{(p)}(L_i(x)) = a_i^{-p}\Big[\sum_{j=0}^{p} \binom{p}{j} \alpha_i^{(p-j)}(x) g^{(j)}(x) + q_i^{(p)}(x)\Big] \quad x \in I,\ i \in \mathbb{N}_{N-1}.$$

PROOF. We begin by noting that

$$\mathcal{D}^k(I) := \{h \in C^k(I) : h^{(p)}(x_1) = y_{1,p},\ h^{(p)}(x_N) = y_{N,p},\ p \in \mathbb{N}_k^0\}$$

endowed with the norm $\|\cdot\|_k$ is a complete metric space. Define $T : \mathcal{D}^k(I) \to \mathcal{D}^k(I)$ via

$$(Th)(x) = \alpha_i(L_i^{-1}(x))h(L_i^{-1}(x)) + q_i(L_i^{-1}(x)) \quad x \in I_i,\ i \in \mathbb{N}_{N-1}. \tag{3.3}$$

Since the functions h, α_i and q_i are in $C^k(I)$, Th is k-times continuously differentiable on (x_i, x_{i+1}) for each $i \in \mathbb{N}_{N-1}$. Bearing in mind that $L_i : I \to [x_i, x_{i+1}]$ for $i \in \mathbb{N}_{N-1}$ satisfies $L_i(x_1) = L_{i-1}(x_N) = x_i$,

$$\begin{aligned} (Th)^{(p)}(x_i^+)\, a_i^p &= \sum_{j=0}^{p} \binom{p}{j} h^{(j)}(x_1)\alpha_i^{(p-j)}(x_1) + q_i^{(p)}(x_1), \\ (Th)^{(p)}(x_i^-)\, a_{i-1}^p &= \sum_{j=0}^{p} \binom{p}{j} h^{(j)}(x_N)\alpha_{i-1}^{(p-j)}(x_N) + q_{i-1}^{(p)}(x_N). \end{aligned}$$

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

In conjunction with (3.1), this yields

$$(Th)^{(p)}(x_i^+) = (Th)^{(p)}(x_i^-) \quad i = 2, 3, \dots, N-1, \ p \in \mathbb{N}_k^0.$$

From the conditions prescribed in (3.2),

$$(Th)^{(p)}(x_1) = y_{1,p}, \quad (Th)^{(p)}(x_N) = y_{N,p} \quad p \in \mathbb{N}_k^0.$$

Therefore, Th is well defined and it is an element of $\mathcal{D}^k(I)$. By successive differentiation on (3.3), we observe that for h, h^* in $\mathcal{D}^k(I)$ and $x \in I_i$,

$$\begin{aligned}|(Th)^{(p)}(x) - (Th^*)^{(p)}(x)| &= a_i^{-p}\Big|\sum_{j=0}^{p}\binom{p}{j}\alpha_i^{(p-j)}(L_i^{-1}(x))(h-h^*)^{(j)}(L_i^{-1}(x))\Big| \\ &\le a_i^{-p}\|\alpha_i\|_p\|h-h^*\|_p\sum_{j=0}^{p}\binom{p}{j}.\end{aligned} \tag{3.4}$$

For $p \in \mathbb{N}_k^0$, using the assumption on the scaling functions α_i,

$$\|(Th)^{(p)} - (Th^*)^{(p)}\|_\infty \le \max\Big\{\frac{2^k}{a_i^k}\|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\Big\}\|h-h^*\|_k < \|h-h^*\|_k.$$

Therefore, $\|Th - Th^*\|_k < \|h-h^*\|_k$ and T is a contraction on $\mathcal{D}^k(I)$ and has a unique fixed point (by the Banach fixed point theorem) $g \in \mathcal{D}^k(I) \subset C^k(I)$. Further, $g^{(p)}$, $p \in \mathbb{N}_k$, satisfies the functional equation stated in the theorem. □

Remark 3.3. Theorem 3.2 includes the Barnsley–Harrington theorem as a special case. Consider $\alpha_i(x) = \alpha_i$ for all $x \in I$. Then only one term (which corresponds to $j = p$) in the summation appearing in the right-hand side of (3.4) is nonzero, and consequently

$$|(Th)^{(p)}(x) - (Th^*)^{(p)}(x)| < a_i^{-p}|\alpha_i|\,\|h-h^*\|_p.$$

As in the theorem, T is a contraction if the scaling factors satisfy $|\alpha_i| < a_i^k$ for all $i \in \mathbb{N}_{N-1}$. Note also that (3.1) reduces to

$$a_i^{-p}[\alpha_i y_{1,p} + q_i^{(p)}(x_1)] = a_{i-1}^{-p}[\alpha_{i-1}y_{N,p} + q_{i-1}^{(p)}(x_N)],$$

that is, $F_{i-1,p}(x_N, y_{N,p}) = F_{i,p}(x_1, y_{1,p})$, and the equations in (3.2) reduce to

$$y_{1,p} = \frac{q_1^{(p)}(x_1)}{a_1^p - \alpha_1}, \quad y_{N,p} = \frac{q_{N-1}^{(p)}(x_N)}{a_{N-1}^p - \alpha_{N-1}},$$

which coincide with the conditions prescribed in Theorem 3.1.

The following theorem generalises a result in [17] to the setting of variable scaling.

Theorem 3.4. *Let $f \in C^k(I)$ be a prescribed function. If the scaling functions α_i and the base function b in $C^k(I)$ fulfil the conditions $\|\alpha_i\|_k < (a_i/2)^k$ for $i \in \mathbb{N}_{N-1}$ and*

$$b^{(p)}(x_1) = f^{(p)}(x_1), \quad b^{(p)}(x_N) = f^{(p)}(x_N) \quad p \in \mathbb{N}_k^0$$

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

then the corresponding $\boldsymbol{\alpha}$-fractal function f^{α} (cf. (2.6)) belongs to $C^k(I)$. Further, $(f^{\alpha})^{(p)}(x_i) = f^{(p)}(x_i)$ for $p \in \mathbb{N}_k^0$, $i \in \mathbb{N}_k$, and $(f^{\alpha})^{(p)}$ satisfies

$$(f^{\alpha})^{(p)}(L_i(x)) = f^{(p)}(L_i(x)) + a_i^{-p}\Big(\sum_{j=0}^{p}\binom{p}{j}\alpha_i^{(p-j)}(x)(f^{\alpha}-b)^{(j)}(x)\Big).$$

Proof. The result follows from Theorem 3.2 by considering the particular choice $q_i(x) := f(L_i(x)) - \alpha_i(x)b(x)$ and observing that the constraints imposed on b yield the required conditions on q_i. □

Remark 3.5. Note that the function $b = H_f$, the two-point Hermite interpolant of contact k for f with knots at x_1 and x_N, satisfies the conditions prescribed in the preceding theorem. Further, this b depends linearly on f.

4. Hermite FIF with variable scaling factors

A natural question arises in connection with Theorem 3.2 about the existence of function tuples $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$ with $\|\alpha_i\|_k < (a_i/2)^k$ for all $i \in \mathbb{N}_{N-1}$ and $\boldsymbol{q} := (q_1, q_2, \ldots, q_{N-1})$ in $(C^k(I))^{N-1}$ such that the system governed by (3.1)–(3.2) is solvable for arbitrarily chosen points $y_{1,p}$ and $y_{N,p}$. Theorem 3.4 addresses this only for a special choice of q_i. In practice, one is less likely to want a C^k-FIF than to construct an interpolant passing through specified points with specified slopes. The following theorem establishes the existence of α_i and q_i in $C^k(I)$ so that the corresponding FIF g not only belongs to $C^k(I)$ but possesses prescribed $k+1$ derivative values (including function values) at the abscissae. That is, for a prescribed set of data $\{(x_i, y_{i,p}) : i \in \mathbb{N}_N,\ p \in \mathbb{N}_k^0\}$, where $y_{i,p}$ are arbitrary, g solves a Hermite interpolation problem $g^{(p)}(x_i) = y_{i,p}$ for $i \in \mathbb{N}_N$ and $p \in \mathbb{N}_k^0$.

Theorem 4.1. *For arbitrary real numbers $x_1 < x_2 < \cdots < x_N$ and $y_{i,p}$, $i \in \mathbb{N}_N$, $p \in \mathbb{N}_k^0$, there exist scaling functions α_i and functions q_i in $C^k(I)$ such that the FIF g determined through the maps $L_i(x) = a_i x + b_i$ satisfying (2.1) and the maps $F_i(x,y) = \alpha_i(x)y + q_i(x)$ satisfying (2.2) is C^k-continuous. Furthermore, $g^{(p)}(x_i) = y_{i,p}$ for $i \in \mathbb{N}_N$ and $p \in \mathbb{N}_k^0$.*

Proof. Let $a_i^{-p}\{\sum_{j=0}^{p}\binom{p}{j}y_{1,j}\,\alpha_i^{(p-j)}(x_1) + q_i^{(p)}(x_1)\} = y_{i,p}$ for $i \in \mathbb{N}_{N-1}$ and $p \in \mathbb{N}_k$. Then (3.1)–(3.2), prescribed in Theorem 3.2, can be recast as

$$a_i^{-p}\Big\{\sum_{j=0}^{p}\binom{p}{j}y_{1,j}\,\alpha_i^{(p-j)}(x_1) + q_i^{(p)}(x_1)\Big\} = y_{i,p} \quad i \in \mathbb{N}_{N-1},$$

$$a_{i-1}^{-p}\Big\{\sum_{j=0}^{p}\binom{p}{j}y_{N,j}\,\alpha_{i-1}^{(p-j)}(x_N) + q_{i-1}^{(p)}(x_N)\Big\} = y_{i,p} \quad i = 2, 3, \ldots, N,\ p \in \mathbb{N}_k.$$

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

The above set of equations is equivalent to

$$\begin{aligned} &a_i^{-p}\Big\{\sum_{j=0}^{p}\binom{p}{j} y_{1,j}\,\alpha_i^{(p-j)}(x_1)+q_i^{(p)}(x_1)\Big\}=y_{i,p},\\ &a_i^{-p}\Big\{\sum_{j=0}^{p}\binom{p}{j} y_{N,j}\,\alpha_i^{(p-j)}(x_N)+q_i^{(p)}(x_N)\Big\}=y_{i+1,p}\quad i\in\mathbb{N}_{N-1},\ p\in\mathbb{N}_k. \end{aligned}\tag{4.1}$$

The interpolation and continuity conditions imposed on the maps F_i,

$$\begin{aligned} F_i(x_1,y_1)&=\alpha_i(x_1)y_1+q_i(x_1)=y_i=y_{i,0},\\ F_i(x_N,y_N)&=\alpha_i(x_N)y_N+q_i(x_N)=y_{i+1}=y_{i+1,0}\quad i\in\mathbb{N}_{N-1}, \end{aligned}$$

complete the equations (cf. (4.1)) for the case $p=0$. We choose scaling functions $\alpha_i\in C^k(I)$, $i\in\mathbb{N}_{N-1}$ arbitrarily except for $\|\alpha_i^{(p)}\|_\infty<(a_i/2)^k$ for all $p\in\mathbb{N}_k^0$ and then select $q_i\in C^k(I)$ such that

$$\begin{aligned} &q_i^{(p)}(x_1)=y_{i,p}a_i^p-\sum_{j=0}^{p}\binom{p}{j} y_{1,j}\,\alpha_i^{(p-j)}(x_1),\\ &q_i^{(p)}(x_N)=y_{i+1,p}a_i^p-\sum_{j=0}^{p}\binom{p}{j} y_{N,j}\,\alpha_i^{(p-j)}(x_N)\quad p\in\mathbb{N}_k^0. \end{aligned}\tag{4.2}$$

These requirements ensure the conditions of Theorem 3.2. Thus the solution of the system (3.1) and (3.2) reduces to the existence of C^k-continuous functions q_i fulfilling (4.2) which is is evident. For instance, q_i can be taken as a two-point Hermite interpolant of degree $2k+1$. Therefore there exist $(\alpha_1,\alpha_2,\dots,\alpha_{N-1})$ and $(q_1,q_2,\dots,q_{N-1})$ in $(C^k(I))^{N-1}$ satisfying the hypotheses of Theorem 3.2 and consequently the corresponding FIF g is C^k-continuous. Finally, for $p\in\mathbb{N}_k^0$ and $i\in\mathbb{N}_N$,

$$g^{(p)}(x_i)=g^{(p)}(L_i(x_1))=a_i^{-p}\Big[\sum_{j=0}^{p}\binom{p}{j}\alpha_i^{(p-j)}(x_1)g^{(j)}(x_1)+q_i^{(p)}(x_1)\Big]=y_{i,p}.$$

This concludes the proof. □

The proof relies on the solution of the Hermite interpolation problem, a classical problem in numerical analysis. However, for the sake of completeness, we provide an explicit expression for the maps q_i of the IFS (see also [15]). For arbitrary numbers $x_0<x_1<\cdots<x_m$ and $f_{i,p}$, $i\in\mathbb{N}_m^0$, $p\in\mathbb{N}_{n_i-1}$, there exists precisely one polynomial $P\in\mathcal{P}_n$, $n+1:=\sum_{i=0}^m n_i$ which satisfies $P^{(p)}(x_i)=f_{i,p}$ for $i\in\mathbb{N}_m^0$, $p\in\mathbb{N}_{n_i-1}$. The Hermite interpolating polynomial $P\in\mathcal{P}_n$ can be given explicitly in the form

$$P(x)=\sum_{i=0}^{m}\sum_{p=0}^{n_i-1} f_{i,p}L_{i,p}(x),$$

where the polynomials $L_{i,p} \in \mathcal{P}_n$ are generalised Lagrange polynomials defined as follows. Starting with the auxiliary polynomials

$$l_{i,p}(x) = \frac{(x-x_i)^p}{p!} \prod_{j=0, j\neq i}^{m} \left(\frac{x-x_j}{x_i-x_j}\right)^{n_j} \quad 0 \le i \le m,\ 0 \le p \le n_i - 1,$$

put $L_{i,n_i-1}(x) = l_{i,n_i-1}(x)$ for $i = 0, 1, \ldots, m$ and, recursively for $p = n_i - 2, n_i - 3, \ldots, 1, 0,$

$$L_{i,p}(x) := l_{i,p}(x) - \sum_{\nu=p+1}^{n_i-1} l_{i,p}^{(\nu)}(x_i) L_{i,\nu}(x).$$

Since $n_i - 1 = k$ for the case under consideration,

$$\begin{aligned} q_i(x) = &\sum_{j=0}^{k} \Big[y_{i,j} a_i^j - \sum_{r=0}^{j} \binom{j}{r} y_{1,r}\, \alpha_i^{(j-r)}(x_1) \Big] L_{1,j}(x) \\ &+ \sum_{j=0}^{k} \Big[y_{i+1,j} a_i^j - \sum_{r=0}^{j} \binom{j}{r} y_{N,r}\, \alpha_i^{(j-r)}(x_N) \Big] L_{N,j}(x), \end{aligned}$$

where

$$\begin{aligned} &L_{1,k}(x) = l_{1,k}(x), \quad L_{N,k}(x) = l_{N,k}(x), \\ &L_{1,j}(x) = l_{1,j}(x) - \sum_{\nu=j+1}^{k} l_{1,j}^{(\nu)}(x_1) L_{1,\nu}(x) \quad j = k-1, k-2, \ldots, 1, 0, \\ &L_{N,j}(x) = l_{N,j}(x) - \sum_{\nu=j+1}^{k} l_{N,j}^{(\nu)}(x_N) L_{N,\nu}(x) \quad j = k-1, k-2, \ldots, 1, 0, \\ &l_{1,j}(x) = \frac{(x-x_1)^j}{j!} \left(\frac{x-x_N}{x_1-x_N}\right)^{k+1}, \\ &l_{N,j}(x) = \frac{(x-x_N)^j}{j!} \left(\frac{x-x_1}{x_N-x_1}\right)^{k+1} \quad j \in \mathbb{N}_k^0. \end{aligned}$$

Let us denote the resulting FIF $g \in C^k(I)$ that satisfies the conditions $g^{(p)}(x_i) = y_{i,p}$ for $i \in \mathbb{N}_N$ and $p \in \mathbb{N}_k^0$ by H, or, more explicitly, by H^α (to make evident the dependence of H on the scaling (function) vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1})$). If $\alpha_i \equiv 0$ for all $i \in \mathbb{N}_{N-1}$, then $H(x) = q_i(L_i^{-1}(x))$ for $x \in I_i = [x_i, x_{i+1}]$. Therefore, H is a polynomial of degree not exceeding $2k+1$ on I_i and consequently H is a Hermite function, that is, a function in the space $\mathcal{H}^{k+1} := \{\phi : I \to \mathbb{R} | \phi \in C^k(I), \phi|_{I_i} \in \mathcal{P}_{2k+1}\}$. We call H a *Hermite fractal interpolation function* (HFIF) with variable scaling factors. It generalises the HFIF with constant scaling factors studied in [11].

REMARK 4.2. Note that the conditions prescribed involve only the values of the scaling functions on the extremes of the interval. This fact provides flexibility, since the remaining α_i can be arbitrarily chosen, subject to the restrictions on their magnitudes.

5. A fractal operator with function scalings

In this section we shall establish some basic properties of a fractal operator which are implicit in Theorem 3.4. For a fixed set of scaling functions α_i and a base function b satisfying the conditions in Theorem 3.4, define

$$\mathcal{D}^{\alpha}_{\Delta,b} : C^k(I) \to C^k(I), \quad \mathcal{D}^{\alpha}_{\Delta,b}(f) = f^{\alpha}.$$

Even though f^{α} is k-times continuously differentiable, we refer to it as a fractal function because its graph is a union of transformed copies of itself, that is,

$$G(f^{\alpha}) = \bigcup_{i \in \mathbb{N}_{N-1}} W_i(G(f^{\alpha})).$$

The corresponding operator $\mathcal{D}^{\alpha}_{\Delta,b}$ is termed a fractal operator. We assume further that $b = Lf$, where L is a bounded linear operator on $C^k(I)$ and I_d is the identity operator on $C^k(I)$. Note that due to the conditions required on b, we need L to be a linear operator satisfying $Lf^{(p)}(x_j) = f^{(p)}(x_j)$ for $p \in \mathbb{N}^0_k$ and $j = 1, N$. The corresponding fractal operator is denoted by $\mathcal{D}^{\alpha}_{\Delta,L}$. The construction has connections with [8], but we have a new set of conditions on the scaling and a different domain space.

Theorem 5.1. *The operator $\mathcal{D}^{\alpha}_{\Delta,L}$ is a bounded linear operator on $C^k(I)$.*

Proof. The linearity of L and the unicity of the fixed point of the functional equation for $\boldsymbol{\alpha}$-fractal functions give $(\lambda f + \mu g)^{\alpha} = \lambda f^{\alpha} + \mu g^{\alpha}$ for any $\lambda, \mu \in \mathbb{R}$ and any functions f, g in $C^k(I)$, so the map $\mathcal{D}^{\alpha}_{\Delta,L}$ is linear. Arguing as in Theorem 3.2,

$$\|f^{\alpha} - f\|_k \le \max\{(2/a_i)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\} \|f^{\alpha} - Lf\|_k \le \frac{K}{1-K}\|f - Lf\|_k, \tag{5.1}$$

where $K = \max\{(2/a_i)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\}$. A little manipulation yields

$$\|\mathcal{D}^{\alpha}_{\Delta,L}(f)\|_k = \|f^{\alpha}\|_k \le \left(\frac{K\|I_d - L\|}{1-K} + 1\right)\|f\|_k. \tag{5.2}$$

Consequently, $\mathcal{D}^{\alpha}_{\Delta,L}$ is a bounded operator and $\|\mathcal{D}^{\alpha}_{\Delta,L}\| \le (K\|I_d - L\|/(1-K) + 1)$. □

Theorem 5.2. *For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (C^k(I))^{N-1}$ with $\|\alpha_i\|_k < \|L\|^{-1}(a_i/2)^k$ for all $i \in \mathbb{N}_{N-1}$, the fractal operator $\mathcal{D}^{\alpha}_{\Delta,L}$ is injective and bounded below. In particular, the range of $\mathcal{D}^{\alpha}_{\Delta,L}$, denoted by $Rg(\mathcal{D}^{\alpha}_{\Delta,L})$, is a closed subspace of $C^k(I)$.*

Proof. From the proof of the previous theorem,

$$\|f^{\alpha} - f\|_k \le K\|f^{\alpha} - Lf\|_k \tag{5.3}$$

where $K := \max\{(2/a_i)^k \|\alpha_i\|_k : i \in \mathbb{N}_{N-1}\} < \|L\|^{-1}$. Let $\mathcal{D}^{\alpha}_{\Delta,L}(f) = f^{\alpha} = 0$. From (5.3),

$$\|f\|_k \le K\|L\| \, \|f\|_k.$$

Since $K\|L\| < 1$, we deduce that $f = 0$, establishing the injectivity of the linear operator $\mathcal{D}^{\alpha}_{\Delta,L}$. From (5.3) we also see that

$$\|f\|_k - \|f^{\alpha}\|_k \le \|f - f^{\alpha}\|_k \le K\|f^{\alpha} - Lf\| \le K(\|f^{\alpha}\|_k + \|L\|\,\|f\|_k),$$

and hence

$$\|f\|_k \le \frac{K+1}{1-K\|L\|}\|f^{\alpha}\|_k. \tag{5.4}$$

Consequently, the operator $\mathcal{D}^{\alpha}_{\Delta,L}$ is bounded below. In fact, from (5.2) and (5.4),

$$\frac{1-K\|L\|}{K+1}\|f\|_k \le \|\mathcal{D}^{\alpha}_{\Delta,L}(f)\| \le \left(\frac{K\|I_d - L\|}{1-K} + 1\right)\|f\|_k.$$

Now it is a routine exercise to show that $\mathrm{Rg}(\mathcal{D}^{\alpha}_{\Delta,L})$ is closed. Let $g \in \overline{\mathrm{Rg}(\mathcal{D}^{\alpha}_{\Delta,L})}$. Then, there exists a sequence (f_n) in $C^k(I)$ such that $\mathcal{D}^{\alpha}_{\Delta,L}(f_n) \xrightarrow{\|\cdot\|_k} g$ as $n \to \infty$. In particular, $(\mathcal{D}^{\alpha}_{\Delta,L}(f_n)) = (f_n^{\alpha})$ is a Cauchy sequence. Using (5.4) we note that

$$\|f_n - f_m\|_k \le \frac{K+1}{1-K\|L\|}\|f_n^{\alpha} - f_m^{\alpha}\|_k.$$

Therefore, the sequence (f_n) is a Cauchy sequence in the Banach space $C^k(I)$ and hence convergent, say, $f_n \xrightarrow{\|\cdot\|_k} f$ in $C^k(I)$. Boundedness of the map $\mathcal{D}^{\alpha}_{\Delta,L}$ implies that $\mathcal{D}^{\alpha}_{\Delta,L}(f_n) \xrightarrow{\|\cdot\|_k} \mathcal{D}^{\alpha}_{\Delta,L}(f)$ and hence that $g = \mathcal{D}^{\alpha}_{\Delta,L}(f)$. Thus, $g \in \mathrm{Rg}(\mathcal{D}^{\alpha}_{\Delta,L})$, demonstrating that $\mathrm{Rg}(\mathcal{D}^{\alpha}_{\Delta,L})$ is closed. □

As a prelude to our next theorem, we recall the following fundamental result from operator theory (see, for instance, [6]).

Lemma 5.3. *If T is a bounded linear operator from a Banach space into itself such that $\|T\| < 1$, then $I - T$ has a bounded inverse and the Neumann series $\sum_{k=0}^{\infty} T^k$ converges in the operator norm to $(I - T)^{-1}$.*

Theorem 5.4. *If $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (C^k(I))^{N-1}$ and $\|\alpha_i\|_k < (a_i/2)^k(1 + \|I_d - L\|)^{-1}$ for all $i \in \mathbb{N}_{N-1}$ then $\mathcal{D}^{\alpha}_{\Delta,L}$ is an isomorphism (linear, bijective and bicontinuous map).*

Proof. From Theorem 5.2, $\mathcal{D}^{\alpha}_{\Delta,L}$ is bounded, linear and injective. From (5.1),

$$\|f^{\alpha} - f\|_k \le \frac{K}{1-K}\|f - Lf\|_k$$

where $K < (1 + \|I_d - L\|)^{-1}$ by the assumption on the scaling functions. Thus,

$$\|\mathcal{D}^{\alpha}_{\Delta,L} - I_d\| \le \frac{K\|I_d - L\|}{1-K} < 1.$$

From Lemma 5.3, $\mathcal{D}^{\alpha}_{\Delta,L}$ has a bounded inverse, and the result follows. □

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

6. Fractal basis for $C^k(I)$

Let us now recall some definitions and results relating to Schauder bases.

DEFINITION 6.1. We call $(\{x_n\}, \{\beta_n\})$ (or simply $\{x_n\}$) a Schauder basis (or simply a basis) for a Banach space X if for each $x \in X$ there exist unique scalars $c_i = \beta_i(x)$ such that $x = \sum_{i=1}^{\infty} c_i x_i$ (that is, the sequence of partial sums $\sum_{i=1}^{M} c_i x_i$ converges to x in norm as M tends to infinity). Note that each β_n is an element of the dual space X^*. The partial sum operators, or natural projections, associated with the basis $(\{x_n\}, \{\beta_n\})$ are the mappings $S_N : X \to X$ defined by $S_N(x) = \sum_{i=1}^{N} \beta_i(x) x_i$. A basis $(\{x_n\}, \{\beta_n\})$ is said to be interpolatory with nodes $\{t_n\}$ if, for each $x \in X$, its Nth approximation $S_N x$ coincides with x at the nodes $t_1, t_2, \ldots, t_N$.

DEFINITION 6.2. A countable family $(e_n, e_n^*)_{n\in\mathbb{N}} \subset X \times X^*$ is said to be (i) biorthogonal if $e_n^*(e_m) = \delta_{m,n}$ for all $m, n \in \mathbb{N}$, (ii) fundamental or complete if $\mathrm{span}\{e_n : n \in \mathbb{N}\}$ is dense in X, and (iii) total if for each $x \in X$ with $e_n^*(x) = 0$ for all $n \in \mathbb{N}$ it follows that $x = 0$. A biorthogonal, fundamental and total sequence $(e_n, e_n^*)_{n\in\mathbb{N}}$ is called a Markushevich basis or simply an M-basis.

REMARK 6.3. Every Schauder basis of a Banach space X is an M-basis for X.

THEOREM 6.4 [5]. *The space $C(I)$ endowed with the supremum norm possesses a Schauder basis of polygonal (piecewise linear) functions.*

A system of polygonal functions $\{g_m\}_{m=0}^{\infty}$ forming a basis for $C(I)$, called the Faber–Schauder system, is constructed as follows. Consider an arbitrary countable sequence of points $\{a = x_0, c = x_1, x_2, \ldots\}$ that is everywhere dense in the interval $I = [a, c]$. Set $g_0(x) \equiv 1$ and $g_1(x) = (x - a)/(c - a)$ on $[a, c]$. For $m > 1$, divide the interval into $m - 1$ parts by the points $x_0, x_1, \ldots, x_{m-1}$ and choose the interval $[x_i, x_j]$ that contains x_m. Set $g_m(x_i) = g_m(x_j) = 0$, $g_m(x_m) = 1$ and extend $g_m(x)$ linearly to $[x_i, x_m]$ and $[x_m, x_j]$. Outside $[x_i, x_j]$, set g_m to be zero. Note that for $f \in C(I)$, the partial sum projection $S_N f$ of f coincides with the piecewise linear continuous function on $[a, c]$ with nodes $x_0, x_1, \ldots, x_N$ interpolating f at those nodes. Thus, the Faber–Schauder system is an interpolatory Schauder basis for $C(I)$ with the node points $\{a = x_0, c = x_1, x_2, \ldots\}$.

The following theorem demonstrates the existence of a Schauder basis of fractal functions for the space $C(I)$.

THEOREM 6.5 [10]. *Consider a sequence of scale vectors (α^m) with $\sum_{m=0}^{\infty} |\alpha^m|_\infty < \infty$. The system $(g_m^{\alpha^m})_{m=0}^{\infty}$, where $g_m^{\alpha^m}$ is the affine fractal interpolation function with respect to the same interpolation data as that of g_m and scale vectors (α^m), is a Schauder basis for $C(I)$.*

REMARK 6.6. Since $g_m^{\alpha^m}$ interpolates g_m at chosen partition points and $\{g_m\}_{m=0}^{\infty}$ is an interpolatory Schauder basis for $C(I)$, the system $\{g_m^{\alpha^m}\}_{m=0}^{\infty}$ is an interpolatory Schauder basis of fractal functions for $C(I)$ which we refer to as a fractal Faber–Schauder system.

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

REMARK 6.7. The following easy and elegant construction of bases for $C^k(I)$ seems to be less widely known than it deserves. In fact, any Schauder basis in $C(I)$ gives a corresponding basis in the space $C^k(I)$ (see [14] for details).

For simplicity, let us consider $I = [0, 1]$. Let $(\{\phi_n\}, \{\mu_n\})$ be any basis for $C(I)$. Let $f_1(x) = 1$, $\gamma_1(f) = f(0)$ and for $n \geq 1$ let $f_n(x) = \int_0^x \phi_{n-1}(t)\,dt$ and $\gamma_n(f) = \mu_{n-1}(f')$ for $f \in C^1(I)$, $x \in I$. Then $(\{f_n\}, \{\gamma_n\})$ is a basis for $C^1(I)$. One can obtain a basis for $C^k(I)$ by repeating the above process k times. Let

$$f_1(x) = 1,$$
$$f_2(x) = x,$$
$$f_k(x) = x^{k-1}/(k-1)!,$$
$$f_n(x) = \int_0^x \int_0^{t_{k-1}} \cdots \int_0^{t_2} \int_0^{t_1} \phi_{n-k}(t)\,dt\,dt_1 \ldots dt_{k-1}$$

for $n = k+1, k+2, \ldots$ and

$$\gamma_1(f) = f(0),$$
$$\gamma_2(f) = f^{(1)}(0),$$
$$\gamma_k(f) = f^{(k-1)}(0),$$
$$\gamma_n(f) = \mu_{n-k}(f^{(k)})$$

for $n = k+1, k+2, \ldots$ and $f \in C^k(I)$, $x \in I$. Then $(\{f_n\}, \{\gamma_n\})$ is a basis for $C^k(I)$.

The next theorem is a direct consequence of Theorem 5.1 and the fact that the bases are preserved by topological isomorphisms.

THEOREM 6.8. *The space $C^k(I)$ admits a Schauder basis, and hence in particular a Markushevich basis, consisting of C^k-continuous fractal functions.*

PROOF. Let $\{f_n\}$ be a Schauder basis for $C^k(I)$ with associated coefficient functionals $\{\beta_n\}$. Consider a partition Δ of the interval, the scaling functions α_i, $i \in \mathbb{N}_{N-1}$ and an operator L satisfying the conditions of Theorem 5.4 so that $\mathcal{D}^\alpha_{\Delta,L}$ is a topological isomorphism on $C^k(I)$. Let $f \in C^k(I)$. Then $(\mathcal{D}^\alpha_{\Delta,L})^{-1}(f) \in C^k(I)$ and for suitable linear functionals $\{\beta_n\}$,

$$(\mathcal{D}^\alpha_{\Delta,L})^{-1}(f) = \sum_{n=1}^{\infty} \beta_n((\mathcal{D}^\alpha_{\Delta,L})^{-1}(f)) f_n.$$

By the continuity of $\mathcal{D}^\alpha_{\Delta,L}$,

$$f = \sum_{n=1}^{\infty} \beta_n((\mathcal{D}^\alpha_{\Delta,L})^{-1}(f)) f_n^\alpha.$$

To prove the unicity of the representation, let $f = \sum_{n=1}^{\infty} \gamma_n f_n^\alpha$ be another representation of f. From the continuity of $(\mathcal{D}^\alpha_{\Delta,L})^{-1}$ we see that $(\mathcal{D}^\alpha_{\Delta,L})^{-1}(f) = \sum_{n=1}^{\infty} \gamma_n f_n$ and hence $\gamma_n = \beta_n((\mathcal{D}^\alpha_{\Delta,L})^{-1}(f))$ for all $n \in \mathbb{N}$. Thus, $\{f_n^\alpha\}$ is a Schauder basis for $C^k(I)$. It can be further noted that if $\{\beta_n\}$ are the coefficient functionals associated with the Schauder basis $\{f_n\}$ then the coefficient functionals associated with the corresponding fractal Schauder basis $\{f_n^\alpha\}$ are given by $\{\beta_n \circ (\mathcal{D}^\alpha_{\Delta,L})^{-1}\}$. □

Remark 6.9. The existence of a Schauder basis $\{g_m^\alpha\}_{m=0}^\infty$ consisting of fractal functions for $C(I)$ is reported in Theorem 6.5. Therefore, it might be possible to prove Theorem 6.8 by applying the 'repeated integration' method in Remark 6.7 to the Schauder basis $\{g_m^\alpha\}_{m=0}^\infty$, keeping in mind that the integral of a FIF obtained from an IFS with constant scalings is again a FIF. We adopted a different method because it is not known whether the primitive of a fractal function with variable scalings is a fractal function.

Proposition 6.10. *Let $(g_n, g_n^*)_{n\in\mathbb{N}}$ be an M-basis for $C^k(I)$. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \in (C^k(I))^{N-1}$ with $\|\alpha_i\|_k < \|L\|^{-1}(a_i/2)^k$ for all $i \in \mathbb{N}_{N-1}$, the system $(g_n^\alpha, h_n^*)_{n\in\mathbb{N}}$, where $h_n^* = g_n^* \circ (\mathcal{D}^\alpha_{\Delta,L})^{-1}$, is an M-basis for $\mathcal{D}^\alpha_{\Delta,L}(C^k(I))$.*

Proof. First, note that with the stated assumption on $\boldsymbol{\alpha}$, $\mathcal{D}^\alpha_{\Delta,L}(C^k(I))$ is a closed subspace of the Banach space $C^k(I)$ (cf. Theorem 5.2) and hence a Banach space. Furthermore, $\mathcal{D}^\alpha_{\Delta,L}$ is injective. For $m, n \in \mathbb{N}$,

$$h_n^*(g_m^\alpha) = (g_n^* \circ (\mathcal{D}^\alpha_{\Delta,L})^{-1})(g_m^\alpha) = g_n^*(g_m) = \delta_{m,n}.$$

Therefore, $(g_n^\alpha, h_n^*)_{n\in\mathbb{N}}$ is biorthogonal. Let $h_n^*(g^\alpha) = 0$ for all $n \in \mathbb{N}$. This implies $g_n^*(g) = 0$ for all $n \in \mathbb{N}$, which in conjunction with the fact that $(g_n, g_n^*)_{n\in\mathbb{N}}$ is an M-basis gives $g = 0$, which in turn implies $g^\alpha = 0$. This shows that $(g_n^\alpha, h_n^*)_{n\in\mathbb{N}}$ is total. Let $g \in \mathcal{D}^\alpha_{\Delta,L}(C^k(I))$. Then there exists $f \in C^k(I)$ such that $g = \mathcal{D}^\alpha_{\Delta,L}(f)$. Since $(g_n, g_n^*)_{n\in\mathbb{N}}$ is an M-basis for $C^k(I)$,

$$g = \mathcal{D}^\alpha_{\Delta,L}(f) = \mathcal{D}^\alpha_{\Delta,L}\Big(\lim_{N\to\infty} \sum_{i=1}^N k_i g_i\Big) = \lim_{N\to\infty} \sum_{i=1}^N k_i g_i^\alpha,$$

implying that $\overline{\operatorname{span}\{g_n^\alpha : n \in \mathbb{N}\}} = \mathcal{D}^\alpha_{\Delta,L}(C^k(I))$. We conclude that

$$(g_n^\alpha, h_n^*)_{n\in\mathbb{N}} \subset \mathcal{D}^\alpha_{\Delta,L}(C^k(I)) \times (\mathcal{D}^\alpha_{\Delta,L}(C^k(I)))^*$$

is an M-basis for $\mathcal{D}^\alpha_{\Delta,L}(C^k(I))$, completing the proof. □

References

[1] M. F. Barnsley, 'Fractal functions and interpolation', *Constr. Approx.* **2**(1) (1986), 303–329.

[2] M. F. Barnsley, *Fractals Everywhere* (Academic Press, San Diego, 1988).

[3] M. F. Barnsley and A. N. Harrington, 'The calculus of fractal interpolation functions', *J. Approx. Theory* **57**(1) (1989), 14–34.

[4] A. K. B. Chand and P. Viswanathan, 'A constructive approach to cubic Hermite fractal interpolation function and its constrained aspects', *BIT* **53**(4) (2013), 841–865.

[5] E. W. Cheney, *Approximation Theory* (AMS Chelsea Publishing Company, Providence, RI, 1966).

[6] J. B. Conway, *A Course in Functional Analysis*, 2nd edn (Springer, New York, 1996).

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press

[7] M. A. Navascués, 'Fractal polynomial interpolation', *Z. Anal. Anwend.* **24**(2) (2005), 1–20.
[8] M. A. Navascués, 'Fractal approximation', *Complex Anal. Oper. Theory* **4**(4) (2010), 953–974.
[9] M. A. Navascués, 'Fractal bases of L_p spaces', *Fractals* **20** (2012), 141–148.
[10] M. A. Navascués, 'Affine fractal functions as bases of continuous functions', *Quaest. Math.* **37** (2014), 1–14.
[11] M. A. Navascués and M. V. Sebastián, 'Generalization of Hermite functions by fractal interpolation', *J. Approx. Theory* **131**(1) (2004), 19–29.
[12] M. A. Navascués and M. V. Sebastián, 'Fitting curves by fractal interpolation: an application to electroencephalographic processing', in: *Thinking in Patterns: Fractals and Related Phenomena in Nature* (ed. M. M. Novak) (World Scientific Publishing, Singapore City, 2004), 143–154.
[13] M. A. Navascués and M. V. Sebastián, 'Smooth fractal interpolation', *J. Inequal. Appl.* **2006**(78734) (2006), 1–20.
[14] S. Schonefeld, 'Schauder bases in spaces of differentiable functions', *Bull. Amer. Math. Soc. (N.S.)* **75** (1969), 586–590.
[15] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis* (Springer, NewYork, 1980).
[16] P. Viswanathan and A. K. B. Chand, 'Fractal rational functions and their approximation properties', *J. Approx. Theory* **185** (2014), 31–50.
[17] P. Viswanathan, A. K. B. Chand and M.A. Navascués, 'Fractal perturbation preserving fundamental shapes: bounds on the scale factors', *J. Math. Anal. Appl.* **419**(2) (2014), 804–817.
[18] H. Y. Wang and J. S. Yu, 'Fractal interpolation functions with variable parameters and their analytical properties', *J. Approx. Theory* **175** (2013), 1–18.

M. A. NAVASCUÉS, Departmento de Matemática Aplicada,
Escuela de Ingeniería y Arquitectura, Universidad de Zaragoza,
C/- María de Luna 3, Zaragoza 50018, Spain
e-mail: manavas@unizar.es

P. VISWANATHAN, Mathematical Sciences Institute,
Australian National University, Canberra, Australia
e-mail: amritaviswa@gmail.com

A. K. B. CHAND, Department of Mathematics,
Indian Institute of Technology Madras, Chennai, India
e-mail: chand@iitm.ac.in

M. V. SEBASTIÁN, Centro Universitario de la Defensa de Zaragoza,
Academia General Militar, Zaragoza, Spain
e-mail: msebasti@unizar.es

S. K. KATIYAR, Department of Mathematics,
Indian Institute of Technology Madras, Chennai, India
e-mail: sbhkatiyar@gmail.com

https://doi.org/10.1017/S0004972715000738 Published online by Cambridge University Press