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AN ANALOGUE OF KOLMOGOROV'S LAW OF THE ITERATED LOGARITHM FOR ARRAYS

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This paper is concerned with the almost sure convergence for arrays of independent, but not necessarily identically distributed, random variables. We show that Kolmogorov's law of the iterated logarithm does not hold for arrays and obtain an analogue of Kolmogorov's law.

1. INTRODUCTION

Assume that $\{X_n,\, n \geqslant 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geqslant 1$. Define $S_n = \sum_{i=1}^{n} X_i$ and $s_n^2 = \sum_{i=1}^{n} EX_i^2$. If $s_n^2 \to \infty$ and

$$|X_n| \leqslant k_n \sqrt{\frac{s_n^2}{\log\log s_n^2}} \text{ almost surely for } n \geqslant 1, \tag{1}$$

where $\{k_n\}$ is a sequence of real numbers such that $k_n \to 0$, then by Kolmogorov's law of the iterated logarithm (LIL)

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log\log s_n^2}} = 1 \text{ almost surely.} \tag{2}$$

Furthermore, it is well known that the LIL does not necessarily hold if (1) is replaced by the weaker condition

$$|X_n| \leqslant k \sqrt{\frac{s_n^2}{\log\log s_n^2}} \text{ almost surely for } n \geqslant 1,$$

where k is a positive constant. If $\{X_n,\, n \geqslant 1\}$ is a sequence of independent and identically distributed random variables with $EX_1 = 0$ and $EX_1^2 < \infty$, then by Hartman and Wintner's LIL, (2) holds.

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Now let $\{X_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of independent random variables with $EX_{ni} = 0$ and $EX_{ni}^2 < \infty$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Define $S_n = \sum_{i=1}^{n} X_{ni}$ and $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2$. In the case of independent and identically distributed Bernoulli random variables $\{X_{ni}\}$ with $P(X_{11} = \pm 1) = 1/2$, Hu [1] showed that

(3)
$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log s_n^2}} = 1 \text{ almost surely} \quad \text{and} \quad \liminf_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log s_n^2}} = -1 \text{ almost surely.}$$

Hu and Weber [2] proved the result (3) under the weaker condition that $\{X_{ni}\}$ is an array of independent and identically distributed random variables with $EX_{11} = 0$ and $E|X_{11}|^4 < \infty$. Qi [3] proved that for an array of independent and identically distributed random variables $\{X_{ni}\}$, (3) holds if and only if $EX_{11} = 0$ and $E|X_{11}|^4 \left(\log^+ |X_{11}|\right)^{-2} < \infty$. Note that from (3) it follows that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log\log s_n^2}} = \infty \text{ almost surely,}$$

so Hartman and Wintner's LIL can not hold for arrays. Thus it is natural to ask if Kolmogorov's LIL holds for arrays. In this paper we show that Kolmogorov's LIL does not holds for the arrays and obtain an analogue of Kolmogorov's LIL.

2. Main results

Throughout this section, $\{X_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ will denote an array of rowwise independent random variables with $EX_{ni} = 0$ and $EX_{ni}^2 < \infty$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Define $S_n = \sum_{i=1}^{n} X_{ni}$ and $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2$. Note that rowwise independence plus independent rows is equivalent to independence of the variables of the array.

The following example shows that Kolmogorov's LIL does not hold for arrays.

EXAMPLE 1. Let $\{Y_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of independent Bernoulli random variables with

$$P(Y_{ni} = 1) = \log\log n/n = 1 - P(Y_{ni} = 0) \quad \text{for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1.$$

Let $X_{ni} = Y_{ni} - EY_{ni}$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Then $EX_{ni} = 0$ and $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2 = \log\log n(1 - \log\log n/n) \sim \log\log n$. Since $|X_{ni}| \leqslant 1$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$, there exists a sequence $\{k_n\}$ of real numbers such that $k_n \to 0$ and

$$|X_{ni}| \leqslant k_n \sqrt{\frac{s_n^2}{\log\log s_n^2}} \text{ almost surely for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1.$$

Thus $\{X_{ni}\}$ satisfies Kolmogorov's condition. But, it follows by the Theorem of Rosalsky [4] that

$$\limsup_{n\to\infty} \frac{S_n \log\log n}{\log n} = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} (Y_{ni} - \log\log n/n) \log\log n}{\log n}$$
$$= \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} Y_{ni} \log\log n}{\log n} = 1 \text{ almost surely.}$$

From this result we can easily see that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log\log s_n^2}} = \infty \text{ almost surely.}$$

Now we develop an analogue of Kolmogorov's LIL for arrays. The following lemma, in contrast with those which follow, does not require independent rows.

Lemma 1. *Let $\{X_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of rowwise independent random variables with $EX_{ni} = 0$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Let $\{k_n\}$ be a sequence of positive constants such that $k_n \to 0$ as $n \to \infty$. Suppose that the following conditions hold.*

(i) $s_n^2 \leqslant n$ *for* $n \geqslant 1$.

(ii) $|X_{ni}| \leqslant k_n \sqrt{n}/\sqrt{\log n}$ *almost surely for* $1 \leqslant i \leqslant n$ *and* $n \geqslant 1$.

Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log n}} \leqslant 1 \text{ almost surely.} \tag{4}$$

Proof: Let $b_n = \sqrt{2n\log n}$ for $n \geqslant 1$. Then, by the Borel-Cantelli lemma, it suffices to show that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{S_n}{b_n} > 1 + \varepsilon\right) < \infty. \tag{5}$$

From the inequality $e^x \leqslant 1 + x + \frac{x^2}{2} e^{|x|}$ for all $x \in R$, noting that $|X_{ni}|/b_n \leqslant k_n\sqrt{n}/b_n\sqrt{\log n}$ almost surely, we have for $t > 0$

$$E\Big\{exp\Big(t\frac{X_{ni}}{b_n}\Big)\Big\} \leqslant E\Big\{1 + t\frac{X_{ni}}{b_n} + \frac{t^2}{2b_n^2} X_{ni}^2 exp\Big(t\frac{|X_{ni}|}{b_n}\Big)\Big\}$$
$$\leqslant 1 + \frac{t^2}{2b_n^2} exp\Big(t\frac{k_n\sqrt{n}}{b_n\sqrt{\log n}}\Big) EX_{ni}^2$$
$$\leqslant exp\Big\{\frac{t^2}{2b_n^2} exp\Big(t\frac{k_n\sqrt{n}}{b_n\sqrt{\log n}}\Big) EX_{ni}^2\Big\}.$$

By independence

$$E\left\{exp\left(t\frac{S_n}{b_n}\right)\right\} = \prod_{i=1}^{n} E\left\{exp\left(t\frac{X_{ni}}{b_n}\right)\right\} \leqslant exp\left\{\frac{nt^2}{2b_n^2} exp\left(t\frac{k_n\sqrt{n}}{b_n\sqrt{\log n}}\right)\right\},$$

since $s_n^2 \leqslant n$. Thus, choosing $t = 2(1+\varepsilon)\log n$, we have

$$\begin{aligned} P\left(\frac{S_n}{b_n} > 1+\varepsilon\right) &\leqslant e^{-t(1+\varepsilon)} E\left\{exp\left(t\frac{S_n}{b_n}\right)\right\} \\ &\leqslant exp\left\{-t(1+\varepsilon) + \frac{nt^2}{2b_n^2} exp\left(t\frac{k_n\sqrt{n}}{b_n\sqrt{\log n}}\right)\right\} \\ &= n^{-(1+\varepsilon)^2\left(2-exp\left(\sqrt{2}(1+\varepsilon)k_n\right)\right)}, \end{aligned}$$

which guarantees (5) since $k_n \to 0$ implies that $(1+\varepsilon)^2\left(2 - exp(\sqrt{2}(1+\varepsilon)k_n)\right) > 1$ for n sufficiently large. Thus (4) is proved. ∎

To obtain the opposite inequality of (4), we need the next lemma, which is well known (see, Stout [**5**, Theorem 5.2.2(iii)]).

Lemma 2. *Let $X_1, \cdots, X_n$ be independent random variables with $EX_i = 0$ for $1 \leqslant i \leqslant n$. Set $S_n = \sum_{i=1}^{n} X_i$ and $s_n^2 = \sum_{i=1}^{n} EX_i^2$. Let $|X_i| \leqslant cs_n$ almost surely for $1 \leqslant i \leqslant n$ and $n \geqslant 1$, where $c > 0$ is a constant. Then for any $\gamma > 0$ there exist constants $\varepsilon(\gamma)$ and $\pi(\gamma)$ such that if $\varepsilon \geqslant \varepsilon(\gamma)$ and $\varepsilon c \leqslant \pi(\gamma)$, then*

$$P\left(\frac{S_n}{s_n} > \varepsilon\right) \geqslant exp\left[-\frac{\varepsilon^2}{2}(1+\gamma)\right].$$

The following theorem establishes a LIL type result for arrays of independent, but not necessarily identically distributed, random variables.

Theorem 3. *Let $\{X_{ni}\}$ be as in Lemma 1. If the $\{X_{ni}\}$ are assumed, in addition, to have independent rows and $s_n^2 = n$ for $n \geqslant 1$, then*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} = 1 \textit{ almost surely.}$$

Proof: From Lemma 1, we need only to prove the inequality

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} \geqslant 1 \text{ almost surely.}$$

By the Borel-Cantelli lemma, it is enough to show that for every $\delta > 0$

$$\sum_{n=1}^{\infty} P\left(\frac{S_n}{\sqrt{2n\log n}} > 1-\delta\right) = \infty. \tag{6}$$

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We apply Lemma 2 with $\gamma > 0$ chosen such that $(1-\delta)^2(1+\gamma) \leqslant 1$. Note that

$$(1-\delta)\sqrt{2\log n} \to \infty$$

and

$$(1-\delta)\sqrt{2\log n}\max_{1\leqslant i\leqslant n}|X_{ni}|/\sqrt{n} \leqslant (1-\delta)\sqrt{2}k_n \to 0$$

as $n \to \infty$. Thus, by Lemma 2, for n sufficiently large

$$P\left(\frac{S_n}{\sqrt{n}} > (1-\delta)\sqrt{2\log n}\right) \geqslant exp[-(1-\delta)^2(1+\gamma)\log n]$$
$$= n^{-(1-\delta)^2(1+\gamma)},$$

and so (6) holds. ∎

From Theorem 3, we obtain an analogue of Kolmogorov's LIL for arrays.

COROLLARY 4. *Let $\{X_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of independent random variables with $EX_{ni} = 0$ and $EX_{ni}^2 < \infty$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Suppose that*

$$(7)\qquad |X_{ni}| \leqslant k_n\sqrt{\frac{s_n^2}{\log n}} \text{ almost surely for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1,$$

where $\{k_n\}$ is a sequence of positive constants satisfying $k_n \to 0$. Then

$$(8)\qquad \limsup_{n\to\infty}\frac{S_n}{\sqrt{2s_n^2\log n}} = 1 \text{ almost surely.}$$

PROOF: Let $Y_{ni} = \sqrt{n}X_{ni}/s_n$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Then $\sum_{i=1}^{n} EY_{ni}^2 = n$ and $|Y_{ni}| \leqslant k_n\sqrt{n}/\sqrt{\log n}$ almost surely for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. Thus $\{Y_{ni}\}$ satisfies the conditions of Theorem 3, and so (8) follows from Theorem 3. ∎

REMARK. Even if the condition (7) of Corollary 4 is replaced by

$$|X_{ni}| \leqslant k_n\sqrt{\frac{s_n^2}{\log s_n^2}} \text{ almost surely for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1,$$

the result (8) can not be replaced by

$$\limsup_{n\to\infty}\frac{S_n}{\sqrt{2s_n^2\log s_n^2}} = 1 \text{ almost surely,}$$

as shown in Example 1.

Finally, using an example we show that (8) can fail if condition (7) of Corollary 4 is replaced by the weaker condition $|X_{ni}| \leqslant k\sqrt{s_n^2}/\sqrt{\log n}$ almost surely for $1 \leqslant i \leqslant n$ and $n \geqslant 1$. To present the example, we restate the following lemma due to Teicher [6].

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LEMMA 5. (Teicher [6]). *Let $\{Y_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of independent Bernoulli random variables with*

$$P(Y_{ni} = 1) = p_n = 1 - P(Y_{ni} = 0) \text{ for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1.$$

Let $np_n = \log n/d$. If $1 < d < exp\{e^{-1}\}$, then $\limsup_{n\to\infty} \sum_{i=1}^{n} Y_{ni}/\log n \leqslant 1/\log d$ almost surely, whereas if $d \geqslant exp\{1-e^{-1}\}$, then $\limsup_{n\to\infty} \sum_{i=1}^{n} Y_{ni}/\log n \geqslant 1/\log d$ almost surely.

EXAMPLE 2. Let $\{Y_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of Bernoulli random variables with

$$P(Y_{ni} = 1) = \log n/dn = 1 - P(Y_{ni} = 0) \text{ for } 1 \leqslant i \leqslant n \text{ and } n \geqslant 1,$$

where $d = exp\{1-e^{-1}\}$. Let $X_{ni} = Y_{ni} - EY_{ni}$. Then $EX_{ni} = 0$ and $s_n^2 = \sum_{i=1}^{n} EX_{ni}^2 = (1 - \log n/dn)\log n/d \sim \log n/d$. Since $|X_{ni}| \leqslant 1$ for $1 \leqslant i \leqslant n$ and $n \geqslant 1$, there exists a positive constant k such that $|X_{ni}| \leqslant k\sqrt{s_n^2}/\sqrt{\log n}$. But, it follows from Lemma 5 that

$$\begin{aligned}\limsup_{n\to\infty} \frac{S_n}{\sqrt{2s_n^2 \log n}} &= \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} (Y_{ni} - \log n/dn)}{\sqrt{2s_n^2 \log n}} \\ &= \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} Y_{ni}\sqrt{d}}{\sqrt{2}\log n} - \frac{1}{\sqrt{2d}} \geqslant \frac{\sqrt{d}}{\sqrt{2}\log d} - \frac{1}{\sqrt{2d}} > 1 \text{ almost surely.}\end{aligned}$$

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