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A tauberian theorem related to the modified Hankel transform

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The modified Hankel transform arises naturally in connection with certain semigroup operations on measures in probability theory. We give a tauberian theorem for this transform when certain higher moments exist. The probabilistic significance of our result is that it translates a regularity condition on the transform into a direct condition on the measure. This complements earlier results by Pitman and Bingham for the trigonometric and the modified Hankel transform respectively.

1. Introduction

Let F be a probability measure on $[0, \infty)$ and let

$$\Phi_\nu(x) = \Gamma(\nu+1) \int_0^\infty (xt/2)^{-\nu} J_\nu(xt) dF(t) \ , \quad \nu \geq -1/2 \ . \tag{1.1}$$

Recently, Bingham [4] gave some abelian and tauberian results for the transform defined by (1.1). He proved that if $L(t)$ is a slowly varying function in the sense of Bojanic and Karamata [5] as $t \to \infty$ and $0 < \alpha < 2$, then

$$1 - F(t) \sim t^{-\alpha} L(t) \ , \quad t \to \infty \ , \tag{1.2}$$

if and only if

$$1 - \Phi_\nu(x) \sim x^\alpha L(1/x) 2^{-\alpha} \frac{\Gamma(1+\nu)\Gamma(1-\alpha/2)}{\Gamma(1+\nu+\alpha/2)} \ , \quad x \to 0 \ . \tag{1.3}$$

Bingham's results are based on those given earlier by Pitman [8] for the

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cosine transform, $\nu = -1/2$. Bingham and Pitman discuss these implications at the boundary points, $\alpha = 0$ and $\alpha = 2$, also. However, for $\alpha > 2$, they give only the abelian implication. Our object in this paper is to give the related tauberian result.

2. Statement of the main result

If we integrate (1.1) by parts and use the relation

$$\frac{d}{dt}\left[t^{-\nu}J_\nu(t)\right] = -t^{-\nu}J_{\nu+1}(t) , \tag{2.1}$$

we obtain

$$G(x) = cx\int_0^\infty (xt)^{-\nu}J_{\nu+1}(xt)g(t)dt , \quad \nu \ge -1/2 , \tag{2.2}$$

where

$$G(x) = 1 - \Phi_\nu(x) , \tag{2.3}$$

$$g(t) = 1 - F(t) , \tag{2.4}$$

and

$$c = 2^\nu\Gamma(\nu+1) . \tag{2.5}$$

For $\alpha > 2$, the Pitman-Bingham Theorem can be stated as follows.

THEOREM A. *If* $n \ge 1$, $2n < \alpha \le 2n+2$, *and*

$$\mu_{2n} = -\int_0^\infty t^{2n}dg(t) < \infty , \tag{2.6}$$

then

$$g(t) \sim t^{-\alpha}L(t) , \quad t \to \infty , \tag{2.7}$$

implies

$$(2.8)\quad G(x) - \sum_{r=1}^{n} \frac{(-1)^{r-1}\Gamma(1+\nu)\mu_{2r}}{2^{2r}\Gamma(1+r)\Gamma(1+\nu+r)} x^{2r}$$

$$\sim \begin{cases} \dfrac{\Gamma(1+\nu)\Gamma(1-\alpha/2)}{2^{\alpha}\Gamma(1+\nu+\alpha/2)} x^{\alpha}L(1/x) , & x \to 0 , \quad 2n < \alpha < 2n+2 , \\ \dfrac{(-1)^{n}\Gamma(1+\nu)}{2^{2n+1}\Gamma(n+1)\Gamma(n+\nu+2)} x^{2n+2} \displaystyle\int_0^{1/x} t^{2n+1}g(t)dt , & \\ & x \to 0 , \quad \alpha = 2n+2 . \end{cases}$$

We prove the following converse.

THEOREM B. *Let* $n \geq 1$, $2n < \alpha \leq 2n+2$, *and let* $G(x)$ *be the transform of* $g(t)$ *defined by* (2.2). *If* $g(t)$ *is bounded, decreases to zero, and*

$$(2.9)\qquad \int_0^{\infty} tg(t)dt < \infty ,$$

then, for some constants $c_1, c_2, \ldots, c_n, c_{n+1}$,

$$(2.10)\qquad G(x) - \sum_{r=1}^{n} c_r x^{2r} \sim c_{n+1} x^{\alpha} L(1/x) , \quad x \to 0 , \quad c_{n+1} \neq 0 ,$$

implies

$$(2.11)\qquad g(t) \sim c_{n+1} \frac{2^{\alpha}\Gamma(1+\nu+\alpha/2)}{\Gamma(1+\nu)\Gamma(1-\alpha/2)} t^{-\alpha}L(t) , \quad t \to \infty , \quad 2n < \alpha < 2n+2$$

or

$$(2.12)\qquad \int_0^{t} u^{2n+1}g(u)du \sim c_{n+1} \frac{(-1)^{n}2^{2n+1}\Gamma(n+1)\Gamma(n+\nu+2)}{\Gamma(1+\nu)} L(t) , \quad t \to \infty , \quad \alpha = 2n+2 .$$

Furthermore,

$$(2.13)\qquad c_r = \frac{(-1)^{r-1}\Gamma(1+\nu)}{2^{2r-1}\Gamma(r)\Gamma(1+\nu+r)} \int_0^{\infty} t^{2r-1}g(t)dt , \quad r = 1, 2, \ldots, n .$$

We note that (2.9) holds if and only if μ_2 , defined by (2.6), is finite. In what follows, we assume that the slowly varying function $L(x)$ is positive and measurable in $0 \leq x < \infty$. Furthermore, without loss of generality, we may also assume that both $L(x)$ and $[L(x)]^{-1}$ are locally bounded.

3. Proof of Theorem B

We prove the theorem with the help of some lemmas. Let $g(s)$ and $G(s)$ be the Mellin transforms of $g(t)$ and $G(t)$ respectively, that is,

$$(3.1)\qquad g(s) = \int_0^\infty t^{s-1}g(t)dt \ , \quad s = \sigma + i\tau \ ,$$

and

$$(3.2)\qquad G(s) = \int_0^\infty t^{s-1}G(t)dt \ .$$

The integral (3.1) converges absolutely in $0 < \sigma \leq 2$. Since

$$(3.3)\qquad t^{-\nu}J_{\nu+1}(t) = \begin{cases} O(t) \ , & t \to 0 \ , \\ O(1) \ , & t \to \infty \ , \quad \nu \geq -1/2 \ , \end{cases}$$

the integral (2.2) converges absolutely, and

$$(3.4)\qquad G(t) = \begin{cases} O\left(t^2\right) \ , & t \to 0 \ , \\ O(t) \ , & t \to \infty \ . \end{cases}$$

Hence the integral (3.2) converges absolutely in $-2 < \sigma < -1$.

LEMMA 1. *Under the assumptions of Theorem* B, *we have*

$$(3.5)\qquad g(s) = \frac{2^{s+\nu}\Gamma(1+\nu+s/2)}{c\Gamma(1-s/2)} G(-s) \ , \quad 1 < \sigma < 2 \ ,$$

where c *is defined by* (2.5).

Proof. By the absolute convergence of the double integral in $2 < \sigma < 3$,

$$(3.6)\qquad \int_0^X x^{-s}G(x)dx = c\int_0^\infty g(t)\left\{\int_0^X (xt)^{-\nu}x^{1-s}J_{\nu+1}(xt)dx\right\}dt$$

$$= c\int_0^\infty t^{s-2}g(t)\left\{\int_0^{Xt} u^{1-s-\nu}J_{\nu+1}(u)du\right\}dt \ .$$

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The inner integral converges absolutely and by [7, p. 326, (1)],

(3.7) $$\int_0^\infty u^{1-s-\nu} J_{\nu+1}(u)du = \frac{2^{1-s-\nu}\Gamma(3/2-s/2)}{\Gamma(\nu+s/2+1/2)} .$$

Hence, by the dominated convergence theorem,

$$G(1-s) = \frac{2^{1-s}\Gamma(\nu+1)\Gamma(3/2-s/2)}{\Gamma(\nu+s/2+1/2)} g(s-1) , \quad 2 < \sigma < 3 ,$$

which proves (3.5).

Proof of Theorem B (continued). Now we consider the integral

(3.8) $$I(x) = \int_0^x t^\beta (x-t)^\gamma g(t)dt$$

where

(3.9) $$\beta = 2n + 1 ,$$

(3.10) $$\gamma = 2n + 4 + [\nu] ,$$

$[\nu]$ denotes the greatest integer function.

Since

(3.11) $$\int_0^x t^{\beta-s}(x-t)^\gamma dt = x^{\beta+\gamma+1-s} \frac{\Gamma(\beta+1-s)\Gamma(1+\gamma)}{\Gamma(\beta+\gamma+2-s)} , \quad \beta > \sigma - 1 ,$$

by the Parseval relation for the Mellin transform [13, p. 60],

$$I(x) = (2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} x^{\beta+\gamma+1-s} \frac{\Gamma(\beta+1-s)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2-s)} g(s)ds , \quad 1 < \delta < 2 .$$

By (3.5),

(3.12) $$I(x) = (2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} x^{\beta+\gamma+1-s} \psi(s)G(-s)ds , \quad 1 < \delta < 2 ,$$

where

(3.13) $$\psi(s) = 2^s \frac{\Gamma(\gamma+1)\Gamma(\beta+1-s)\Gamma(1+\nu+s/2)}{\Gamma(\nu+1)\Gamma(\beta+\gamma+2-s)\Gamma(1-s/2)} .$$

The poles of $\Gamma(1+\nu+s/2)$ lie in the half plane $\sigma < 0$. Therefore, $\psi(s)$ is analytic in $\sigma > 0$ except for a finite number of simple poles at $\sigma = 2n+3, 2n+5, \ldots$. By the well known properties of the Γ-function,

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$$(3.14) \qquad \frac{\Gamma(1+\nu+s/2)}{\Gamma(1-s/2)} = \pi^{-1}\Gamma(1+\nu+s/2)\Gamma(s/2)\sin(\pi s/2)$$

$$= \left(|\tau|^{\sigma+\nu}\right) , \quad s = \sigma + i\tau , \quad |\tau| \to \infty .$$

Thus

$$(3.15) \qquad \psi(s) = O\left(|\tau|^{\nu+\sigma-\gamma-1}\right) , \quad |\tau| \to \infty .$$

By (3.12),

$$I(x) = (2\pi i)^{-1} \int_{\delta-i\infty}^{\delta+i\infty} x^{\beta+\gamma+1-s}\psi(s)\left(\int_0^\infty t^{-s-1}G(t)dt\right)ds .$$

By (3.15), the double integral converges absolutely. Hence,

$$(3.16) \qquad I(x) = (2\pi i)^{-1}x^{\beta+\gamma+1} \int_0^\infty G(t)\left(\int_{\delta-i\infty}^{\delta+i\infty} x^{-s}t^{-s-1}\psi(s)ds\right)dt$$

$$= (2\pi i)^{-1}x^{\beta+\gamma+1} \int_0^\infty G(u/x)\left(\int_{\delta-i\infty}^{\delta+i\infty} u^{-s-1}\psi(s)ds\right)du .$$

Let

$$(3.17) \qquad H(x) = G(x) - \sum_{r=1}^{n} c_r x^{2r} .$$

Our next step is to show that $I(x)$ remains unchanged if G is replaced by H in (3.16).

LEMMA 2.

$$(3.18) \qquad \int_0^\infty u^\mu\left(\int_{\delta-i\infty}^{\delta+i\infty} u^{-s-1}\psi(s)ds\right)du = 2\pi i\psi(\mu) , \quad 2 \le \mu \le 2n+2 ,$$

$$\psi(2n+2) = \lim_{s\to 2n+2} \psi(s) .$$

Proof. Let

$$(3.19) \qquad \phi(u) = \int_{\delta-i\infty}^{\delta+i\infty} u^{\mu-s-1}\psi(s)ds .$$

By (3.15),

$$(3.20) \qquad \phi(u) = O\left(u^{\mu-\delta-1}\right) , \quad u \to 0 , \quad 1 < \delta < 2 .$$

Since $\psi(s)$ is analytic in $\delta \le \mathrm{Re}(s) < 2n + 3$, by (3.15) again

$$(3.21)\qquad \phi(u) = \int_{\delta_1-i\infty}^{\delta_1+i\infty} u^{\mu-s-1}\psi(s)ds\ , \quad 2n+2 < \delta_1 < 2n+3\ ,$$

$$= O\left(u^{\mu-\delta_1-1}\right)\ , \quad u \to \infty\ .$$

Thus the repeated integral in (3.18) converges for $\delta < \mu < \delta_1$.

Obviously,

$$(3.22)\qquad \int_0^1 \phi(u)du = \int_{\delta-i\infty}^{\delta+i\infty} (\mu-s)^{-1}\psi(s)ds$$

$$= \int_{\delta_1-i\infty}^{\delta_1+i\infty} (\mu-s)^{-1}\psi(s)ds + 2\pi i\psi(\mu)\ , \quad 2 \le \mu \le 2n+2\ .$$

Also, by shifting the line of integration from $\mathrm{Re}(s) = \delta$ to $\mathrm{Re}(s) = \delta_1$,

$$(3.23)\qquad \int_1^\infty \phi(u)du = \int_1^\infty \left(\int_{\delta_1-i\infty}^{\delta_1+i\infty} u^{\mu-s-1}\psi(s)ds\right)du$$

$$= -\int_{\delta_1-i\infty}^{\delta_1+i\infty} (\mu-s)^{-1}\psi(s)ds\ , \quad 2 \le \mu \le 2n+2\ .$$

Hence, by (3.22) and (3.23),

$$\int_0^\infty \phi(u)du = 2\pi i\psi(\mu)\ , \quad 2 \le \mu \le 2n+2\ ,$$

which proves the lemma.

For later use, we note the following:

$$(3.24)\qquad \psi(2r) = 0\ , \quad r = 1, 2, \ldots, n\ ,$$

and

$$(3.25)\qquad \psi(2n+2) = \lim_{s\to 2n+2} 2^{2n+2}\,\frac{\Gamma(n+\nu+2)\Gamma(2n+2-s)}{\Gamma(\nu+1)\Gamma(1-s/2)}$$

$$= (-1)^n 2^{2n+1}\,\frac{\Gamma(n+1)\Gamma(n+\nu+2)}{\Gamma(\nu+1)}\ .$$

We now return to the proof of Theorem B. By Lemma 2 and (3.24), (3.26) $$\frac{I(x)}{x^{\beta+\gamma+1}} = (2\pi i)^{-1}\int_0^\infty H(u/x)\left[\int_{\delta-i\infty}^{\delta+i\infty} u^{-s-1}\psi(s)ds\right]du ,$$

where H is defined by (3.17). We are interested in the behavior of $I(x)$ as $x \to \infty$. By (2.10) and (3.4),

(3.27) $$|H(u/x)| \le M(u/x)^\alpha L(x/u) , \quad 2n < \alpha \le 2n+2 ,$$

for some constant M. The dominant behavior of $L(x/u)$, $x \to \infty$, is given by the following lemma. This result is not new and, in a slightly different form, was given by Pitman [8, Lemma 2]. The technique, however, has been used quite often ([3], [5]).

LEMMA 3. *If* $L(t)$ *is slowly varying,* $L(t)$ *and* $\{L(t)\}^{-1}$ *are locally bounded, then, for every* $\eta > 0$,

(3.28) $$\frac{L(xt)}{L(x)} = O\left(t^{-\eta}\right) , \quad x \to \infty ,$$

uniformly in $0 < t \le 1$ *and*

(3.29) $$\frac{L(xt)}{L(x)} = O\left(t^{\eta}\right) , \quad x \to \infty ,$$

uniformly in $1 \le t < \infty$.

Proof. Let

$$L_1(x) = x^{-\eta} \sup_{0<t\le x} \{t^\eta L(t)\}$$

and

$$L_2(x) = x^\eta \sup_{t\ge x} \{t^{-\eta}L(t)\} ;$$

$x^\eta L_1(x)$ is an increasing and $x^{-\eta}L_2(x)$ is a decreasing function of x. Also, it is known ([1], [2], [5]) that $L_j(x) \sim L(x)$, $x \to \infty$, $j = 1, 2$. If $0 < t \le 1$,

$$(xt)^\eta L(xt) \le (xt)^\eta L_1(xt) \le x^\eta L_1(x)$$

and, if $1 \le t < \infty$,

$$(xt)^{-\eta}L(xt) \le (xt)^{-\eta}L_2(xt) \le x^{-\eta}L_2(x) .$$

The relations (3.28) and (3.29) follow immediately from the above inequalities.

We return to the consideration of (3.26). By (3.27) and Lemma 3,

$$\frac{H(u/x)}{x^{-\alpha}L(x)} = O\left(u^{\alpha-\eta}+u^{\alpha+\eta}\right) , \quad x \to \infty ,$$

uniformly in $0 < u < \infty$. Choose η such that $\delta < \alpha - \eta < \alpha + \eta < \delta_1$. By (3.20) and (3.21), we can apply the dominated convergence theorem to the integral in (3.26). Since, $H(u/x) \sim c_{n+1}(u/x)^{\alpha}L(x)$ pointwise as $x \to \infty$, we have

$$\frac{I(x)}{x^{\beta+\gamma+1-\alpha}L(x)} \sim \frac{c_{n+1}}{2\pi i}\int_0^\infty u^\alpha\left(\int_{\delta-i\infty}^{\delta+i\infty} u^{-s-1}\psi(s)ds\right)du$$

or, by Lemma 2,

(3.30) $$I(x) \sim c_{n+1}x^{\beta+\gamma+1-\alpha}L(x)\psi(\alpha) , \quad x \to \infty .$$

It is known ([8, Lemma 3]) that if $\xi(t)$ is monotone and $\xi_1(t) = \int_0^t u^p\xi(u)du$ is of index q as $t \to \infty$, $q > 0$, then $t^p\xi(t)$ is of index $q - 1$. Since $I(x)$ is of index $2n + 2 + \gamma - \alpha$, by repeated application of the above result, we see that

(3.32) $$h(x) = \int_0^x t^{2n+1}g(t)dt$$

is of index $2n + 2 - \alpha$ as $x \to \infty$. Let

(3.32) $$h(x) = x^{2n+2-\alpha}L^*(x) , \quad x \geq 1 ,$$

so that $L^*(x)$ is slowly varying as $x \to \infty$. For $0 \leq x < 1$, define $L^*(x)$ to be a locally bounded and integrable function. Then,

$$I(x) = \gamma \int_0^x (x-t)^{\gamma-1} h(t)dt$$

$$= \gamma \int_0^x (x-t)^{\gamma-1} t^{2n+2-\alpha} L^*(t)dt + \phi_1(x)$$

$$= \gamma x^{\gamma+2n+2-\alpha} \int_0^1 (1-u)^{\gamma-1} u^{2n+2-\alpha} L^*(ux)du + \phi_1(x) ,$$

where $\phi_1(x) = O\left(x^{\gamma-1}\right)$ as $x \to \infty$. To obtain the behavior of the above integral as $x \to \infty$, we may use a known result [3], or we may use the "dominated convergence" technique which is justified by Lemma 3. Hence,

$$I(x) \sim x^{\gamma+2n+2-\alpha} L^*(x) \frac{\Gamma(\gamma+1)\Gamma(2n+3-\alpha)}{\Gamma(\gamma+2n+3-\alpha)} , \quad x \to \infty .$$

By (3.30),

$$L^*(x) = c_{n+1} L(x) \frac{\Gamma(\gamma+2n+3-\alpha)}{\Gamma(\gamma+1)\Gamma(2n+3-\alpha)} \psi(\alpha) . \tag{3.33}$$

If $\alpha = 2n + 2$, $L^*(x) = c_{n+1} L(x)\psi(2n+2)$, so that by (3.25),

$$\int_0^x t^{2n+1} g(t)dt \sim c_{n+1}(-1)^n 2^{2n+1} \frac{\Gamma(n+1)\Gamma(n+\nu+2)}{\Gamma(\nu+1)} L(x) , \quad x \to \infty ,$$

which is (2.12). If $2n < \alpha < 2n + 2$, by (3.13),

$$L^*(x) = c_{n+1} 2^{\alpha} \frac{\Gamma(1+\nu+\alpha/2)}{(2n+2-\alpha)\Gamma(\nu+1)\Gamma(1-\alpha/2)} L(x) .$$

Since $h(x)$ is of index $2n + 2 - \alpha > 0$, $t^{2n+1}g(t)$ is of index $2n + 1 - \alpha$ as $x \to \infty$. We now employ reasoning similar to that used earlier to obtain

$$x^{2n+1} g(x) \sim c_{n+1} 2^{\alpha} \frac{\Gamma(1+\nu+\alpha/2)}{\Gamma(\nu+1)\Gamma(1-\alpha/2)} x^{2n+1-\alpha} L(x) , \quad x \to \infty .$$

This proves (2.11). Finally, we want to prove that the coefficients c_r must satisfy (2.13). If $2n < \alpha < 2n + 2$, this follows directly from Theorem A. However, for the case $\alpha = 2n + 2$, Theorem A is not applicable since (2.12) does not imply (2.7) even when $g(u)$ is decreasing. The proof of this assertion depends on some results given in [11]. However, (2.13) follows from the following lemma.

LEMMA 4. *Let* $n \geq 1$. *If* $g(t)$ *is bounded and decreases to zero as* $t \to \infty$, *then* (2.12) *implies* (2.13).

Proof. Since $g(t) \downarrow 0$, (2.12) implies that

$$t^{2n+2}g(t) = o\{L(t)\} , \quad t \to \infty . \tag{3.34}$$

For the sake of convenience, let

$$k(t) = \Gamma(\nu+1)(t/2)^{-\nu}J_{\nu+1}(t)$$

and

$$a_r = \frac{(-1)^{r-1}\Gamma(\nu+1)}{2^{2r-1}\Gamma(r)\Gamma(\nu+r+1)} ,$$

so that

$$k(t) = \sum_{r=1}^{\infty} a_r t^{2r-1} .$$

Now

$$\begin{aligned} G(x) - \sum_{r=1}^{n} a_r x^{2r} \int_0^\infty t^{2r-1}g(t)dt - a_{n+1}x^{2n+2} \int_0^{1/x} t^{2n+1}g(t)dt \\ = x \int_0^\infty \left\{k(xt) - \sum_{r=1}^{n} a_r (xt)^{2r-1}\right\}g(t)dt - a_{n+1}x^{2n+2} \int_0^{1/x} t^{2n+1}g(t)dt \\ = x \int_0^{1/x} \left\{k(xt) - \sum_{r=1}^{n+1} a_r (xt)^{2r-1}\right\}g(t)dt \\ + x \int_{1/x}^\infty \left\{k(xt) - \sum_{r=1}^{n} a_r (xt)^{2r-1}\right\}g(t)dt \\ = I_1 + I_2 . \end{aligned}$$

We shall prove that $I_j = o\{x^{2n+2}L(1/x)\}$, $x \to 0$, $j = 1, 2$. Since $k(u)$ is bounded, by a known result [3],

$$\int_1^\infty \left\{k(u) - \sum_{r=1}^{n} a_r u^{2r-1}\right\}u^{-2n-2}L(u/x)du \\ \sim L(1/x) \int_1^\infty \left\{k(u) - \sum_{r=1}^{n} a_r u^{2r-1}\right\}u^{-2n-2}du , \quad x \to 0 .$$

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Therefore, by (3.34),

$$I_2 = o\left\{\int_1^\infty \left[k(u) - \sum_{r=1}^{n} a_r u^{2r-1}\right](u/x)^{-2n-2}L(u/x)du\right\}$$

$$= o\{x^{2n+2}L(1/x)\}\ , \quad x \to 0\ .$$

Next, let $\varepsilon > 0$. Choose $0 < \delta' < 1$ such that

$$\left|k(u) - \sum_{r=1}^{n+1} a_r u^{2r-1}\right| < \varepsilon u^{2n+1}\ , \quad 0 < u < \delta'\ .$$

Furthermore, let

$$I_1 = x\left\{\int_0^{\delta'/x} + \int_{\delta'/x}^{1/x}\right\}\left\{k(xt) - \sum_{r=1}^{n+1} a_r (xt)^{2r-1}\right\}g(t)dt$$

$$= I_3 + I_4\ .$$

$$|I_3| < \varepsilon x \int_0^{\delta'/x} (xt)^{2n+1}g(t)dt$$

$$< \varepsilon M_1 x^{2n+2}L(\delta'/x)\ , \quad x \to 0\ ,$$

for some constant M_1 . The relation (2.12) indicates that it is no loss of generality to assume that $L(t)$ is nondecreasing. Hence,

$$|I_3| < \varepsilon M_1 x^{2n+2}L(1/x)\ , \quad x \to 0\ .$$

Finally, for some constant M_2 ,

$$|I_4| \le M_2 x \int_{\delta'/x}^{1/x} (xt)^{2n+1}g(t)dt$$

$$= M_2 x^{2n+2} \int_{\delta'/x}^{1/x} t^{2n+1}g(t)dt\ .$$

By (2.12),

$$\int_{\delta'/x}^{1/x} t^{2n+1}g(t)dt = o\{L(1/x)\}\ , \quad x \to 0\ .$$

Hence,

$$I_4 = o\{x^{2n+2}L(1/x)\} \ , \quad x \to 0 \ .$$

This completes the proof of the assertion

$$I_j = o\{x^{2n+2}L(1/x)\} \ , \quad x \to 0 \ , \quad j = 1, 2 \ .$$

It follows that

$$G(x) \sim \sum_{r=1}^{n} a_r x^{2r} \int_0^\infty t^{2r-1} g(t)dt + a_{n+1} x^{2n+2} \int_0^{1/x} t^{2n+1} g(t)dt \ , \quad x \to 0 \ .$$

Comparing this asymptotic relation with (2.10), we obtain (2.13).

REMARK 1. The assumption (2.9) is not necessary. With the help of some known results [9], [10], it can be shown that (2.10) itself implies $g(t) = O\left(t^{-\alpha_1}\right)$ as $t \to \infty$, for some $\alpha_1 > 0$. This is sufficient to justify the Mellin transform technique used.

REMARK 2. The technique is quite general. In particular, it is applicable when $K(s)$, the Mellin transform of the kernel $k(t)$, has no singularities other than poles in the complex s-plane and, for $\sigma_1 \le 0 \le \sigma_2$, $K(s) = O\left(|\tau|^p\right)$, $|\tau| \to \infty$, $s = \sigma + i\tau$.

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