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FINITE SUBSCHEMES OF GROUP SCHEMES

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If G is an ordinary group and H is a non-empty subset of G, then there are two elementary criteria for H to be a subgroup of G. The first and more general is that the mapping $H \times H \to G \times G \to G$, *via* $\langle x, y\rangle \mapsto xy^{-1}$ factor through H. The second is that H be *finite* and closed under multiplication.

In the category of group schemes, if one writes down the hypotheses for the first criterion in diagram form, one can supply the proof by a suitable translation of the classical arguments. The only point that causes any difficulty whatsoever is that one must assume that the structure morphism $\pi_H\colon H \to S$ (S is the base scheme) is an epimorphism in order to factor the identity section $\epsilon_G\colon S \to G$ through H. The second criterion is also true for group schemes under a mild finite presentation hypothesis. It is our aim to provide a simple proof for the following theorem.

THEOREM. *Let G be a group scheme over a scheme S, and let H be a closed subscheme of G, finite over S and locally finitely presented as $\mathcal{O}_S$-module. Denote by λ the composed morphism*

$$H \underset{S}{\times} H \to G \underset{S}{\times} G \underset{\mu_G}{\to} G$$

and assume that $\pi_H\colon H \to S$ is an epimorphism and that λ factors through H. Then H is a subgroup scheme of G.

Proof. Let $i\colon H \to G$ be the closed immersion, and let j_H, j_G be the morphisms

$$j_H\colon\ H \underset{S}{\times} H \xrightarrow{1 \times \Delta} H \underset{S}{\times} H \underset{S}{\times} H \xrightarrow[\mu_H \times 1]{} H \underset{S}{\times} H,$$

$$j_G\colon\ G \underset{S}{\times} G \xrightarrow{1 \times \Delta} G \underset{S}{\times} G \underset{S}{\times} G \xrightarrow[\mu_G \times 1]{} G \underset{S}{\times} G.$$

Here, μ_H is the map guaranteed to exist by hypothesis, namely λ factored through $i\colon H \to G$. The commutative diagram

$$\begin{array}{ccc} H \underset{S}{\times} H & \xrightarrow[j_H]{} & H \underset{S}{\times} H \\ \Big\downarrow {\scriptstyle i \times i} & & \Big\downarrow {\scriptstyle i \times i} \\ G \underset{S}{\times} G & \xrightarrow[j_G]{} & G \underset{S}{\times} G \end{array}$$

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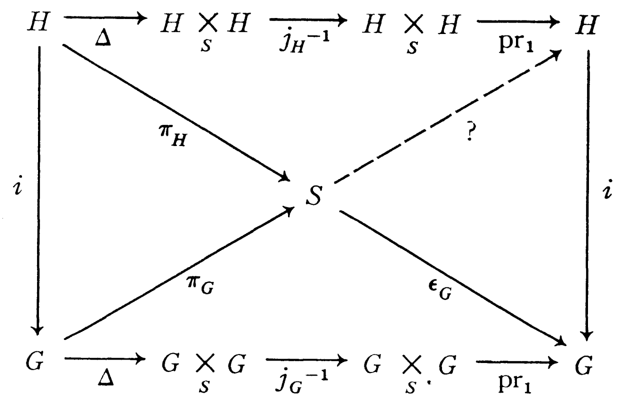
and the fact that j_G is an isomorphism (since G is a group scheme, it acts on itself as principal homogeneous space) show that j_H is a closed immersion. But this implies that j_H is an isomorphism. To see this, observe that the problem is local on S; hence, we may assume that S is affine and H is finitely presented. Then we may apply [**1**, *proposition* 8.9.3] which yields in the present case the fact that j_H is an isomorphism. (Recall that the cited proposition states that a surjective endomorphism of a finitely presented module over a commutative ring is always an isomorphism.)

Now consider the commutative diagram:



in which ϵ_G is the identity section for G. Since the lower horizontal map factors through ϵ_G, and since π_H is an epimorphism, we see easily that the upper horizontal map factors through a morphism $S \to H$ (shown above as a dotted arrow). It follows that this morphism, ϵ_H, is an identity section for H and that it is consistent with ϵ_G. Since G is a group scheme, we verify immediately that the composed morphism

$$G \xrightarrow{\simeq} S \times_S G \xrightarrow{\epsilon_G \times 1} G \times_S G \xrightarrow{j_G^{-1}} G \times_S G \xrightarrow{\text{pr}_1} G$$

is the inverse mapping, inv_G. But then a similar composed map with H replacing G everywhere in the above defines the morphism inv_H which is consistent with inv_G and which satisfies all the axioms for an inverse map. This proves our theorem.

Remarks and counter-examples.

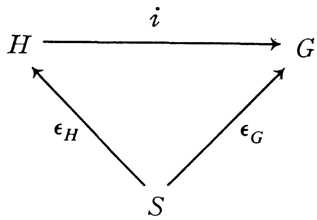
(1) Of course, the finiteness hypothesis is essential as the standard example of the constant group scheme $\mathbf{Z}$ and the closed subscheme consisting of the "positive" elements shows.

(2) The hypothesis, "π_H: $H \to S$ is an epimorphism", cannot be discarded, even if one assumes that H is flat over S. To see this, let k be a field, and let $S = \text{Spec}\, k \amalg \text{Spec}\, k = \text{Spec}(k \oplus k)$. We set G equal to μ_2 over S, and

$H = \text{Spec}(k[X]/(X^2 - 1)) = \mu_2$ over k. The mapping

$$(k \oplus k)[X]/(X^2 - 1) \rightarrow k[X]/(X^2 - 1)$$

via $\langle a, b \rangle \mapsto a$, $X \mapsto X$ defines a closed immersion $H \rightarrow G$. H is closed under multiplication, and H is flat over S. However, H is *not* a subgroup scheme of G for there simply is no morphism $\epsilon_H \colon S \rightarrow H$ which will make the diagram



commute. The problem evidently arises because H is not faithfully flat over S.

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