PRIMES OF THE FORM $[p^c]$ AND RELATED QUESTIONS

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1. Introduction. Much interest has been shown in determining the range of values of c for which the sequence $[n^c]$ contains infinitely many primes. The result is an elementary deduction from the prime number theorem, of course, if $0<c\leq 1$. In 1953, Piatetski-Shapiro [9] showed that

$$\pi_c(x)\sim\frac{x}{c\ln x}\qquad \text{as } x\to+\infty \tag{1}$$

for $1<c<\frac{12}{11}$, where $\pi_c(x)$ stands for the number of primes in the set $\{[n^c]\mid n\leq x\}$.

Various authors have since extended the range of c for which (1) holds up to $\frac{6121}{5302}$ (see [10]). It has also been shown [1] that $\pi_c(x)>\dfrac{x}{4c\ln x}$ for all large x when $1<c\leq\frac{20}{17}$. Leitmann and Wolke [8] have shown that (1) holds for almost all $c\in(1,2)$, and Deshouillers [3] has demonstrated that $\pi_c(x)\to+\infty$ as $x\to+\infty$ for almost all $c>1$. In this paper we shall attempt to establish the analogous results for $\pi_c^*(x)$ where $\pi_c^*(x)$ denotes the number of primes in the set $\{[p^c]\mid p\leq x\}$ with p standing for a variable which only takes prime values. Balog [2] has obtained the analogue of Deshouillers' result in this situation. In fact, by making a slight adaptation to Deshouillers' method he proved:

THEOREM 1. *For almost all $c>1$ we have*

$$\limsup_{x\to+\infty}\pi_c^*(x)\frac{c(\log x)^2}{x}\geq 1.$$

REMARK: As with the case $[n^c]$ it is expected that the exceptional set consists solely of the integers exceeding 1.

In this paper we shall establish an asymptotic formula for a quantity related to $\pi_c^*(x)$ and investigate related topics concerning the sequence $[n^c]$. We have little hope at present of deducing that there are infinitely many primes in the sequence for any given $c>1$ (see [5] where the equivalent question of the joint distribution of the fractional and integral parts of p^λ is discussed for $\lambda<1$).

The results we prove are as follows:

THEOREM 2. *For almost all $(a,b)\in[1,2]^2$ we have*

$$S(a,b,x)=\sum_{\substack{[n^a]=p\\ n\leq x\\ q-1<p^{1/b}\leq q}}1\sim\frac{bx}{a^2(\ln x)^2}\qquad \textit{as } x\to+\infty \tag{2}$$

where q and p both denote primes.

REMARKS: The significance of Theorem 2 lies in the fact that $\pi_c^*(x)=S(c,c,x)$. It will become clear that much more can be said about the two dimensional "almost all" set of this result, but sadly we cannot as yet determine whether it includes almost all points of the line $a = b$. Theorem 2 follows directly from Theorem 3 and Theorem 4 below.

THEOREM 3. *Let $\mathcal{A}$ be an infinite set of positive integers and write*

$$\mathcal{A}_c(x) = \#\{n \le x \mid [n^c] \in \mathcal{A}\}, \qquad \mathcal{A}(x) = \mathcal{A}_1(x). \tag{3}$$

Suppose that, for some $\alpha \in (0,1]$ we have $\mathcal{A}(y) \gg y^{\alpha}$. Then, for almost all $c \in \left(1, \frac{2}{2-\alpha}\right)$ we have, for any $\varepsilon > 0$,

$$\mathcal{A}_c(x) = \gamma \sum_{\substack{a \le x^c \\ a \in \mathcal{A}}} a^{-1+\gamma} + O(\Psi(x)^{\frac{1}{2}}) \tag{4}$$

where $\gamma = \frac{1}{c}$ and

$$\Psi(x) = \sum_{\substack{a \le x^c \\ a \in \mathcal{A}}} \mathcal{A}(a)^{\frac{1}{2}} a^{-1+\gamma+\varepsilon}.$$

REMARKS: Suppose that, for any $\eta > 0$, $y^{\alpha-\eta} \ll \mathcal{A}(y) \ll y^{\alpha}$. Then the main term in (4) is

$$\gg x^{(\alpha-\eta)c+1-c}$$

while the error term is

$$\ll x^{\frac{3\alpha c}{4}+\frac{1}{2}-\frac{c}{2}+\varepsilon}$$

for any $\varepsilon > 0$. It follows that the main term is a larger order of magnitude than the error term for $c < \frac{2}{2-\alpha}$ (that is $\alpha > 2(1-\gamma)$). After correcting an oversight, Théorème 8.2 of **[10]** requires $c < \frac{2}{8-7\alpha}$ which is weaker for $\alpha < 1$. When $\alpha = 1$, as will be the case here in proving Theorem 2, then (4) holds for almost all $c \in (1,2)$. In Chapitre 8 of **[10]** it is shown how a result like Theorem 3 can be iterated to obtain arbitrarily long finite sequences of reals $c_k > 1$ such that the equation

$$p = [n_1^{c_1}] = [n_2^{c_2}] = \ldots = [n_k^{c_k}]$$

has the expected asymptotic formula for the number of solutions. The improvements we have made with Theorem 3 lead to a particularly neat consequence of this type, which we state as follows:

COROLLARY 1. *Let $g_1, \ldots, g_k$ be a finite sequence of reals with $g_j \in [\frac{1}{2}, 1)$ and*

$$\sum_{j=1}^{t} g_j + g_t > t$$

for $t = 1, \ldots, k$.

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Then, for almost all $(c_1, \ldots, c_k)$ *with* $1 < c_j < g_j^{-1}$, *we have that the number of solutions to*

$$p = [n_1^{c_1}] = [n_2^{c_2}] = \ldots = [n_k^{c_k}], \qquad p \le x$$

is

$$\gamma_1 \ldots \gamma_k \sum_{p \le x} p^{\gamma_1 + \ldots + \gamma_k - k}(1 + O(x^{-\delta}))$$

for some $\delta = \delta(c_1, \ldots, c_k) > 0$. *We remark that* δ *could be stated explictly.*

THEOREM 4. *Let* $\mathcal{P}$ *be the set of positive prime numbers and write*

$$\mathcal{B}(y) = \mathcal{P} \cap \bigcup_{p \in \mathcal{P}} ((p-1)^y, p^y)$$

and

$$\mathcal{B}(y, x) = \sum_{\substack{p \in \mathcal{B}(y) \\ p \le x^y}} 1.$$

Then, for almost all $y > 1$ *we have*

$$\mathcal{B}(y, x) \sim \frac{x^y}{y(\ln x)^2} \qquad \text{as } x \to +\infty. \tag{5}$$

REMARKS: The formula (5) is actually true for *all* $y > \frac{6}{5}$, as the reader should observe from our proof. Also, assuming the Riemann Hypothesis, (5) is valid for *all* $y > 1$. We obtain Theorem 2 by letting $\mathcal{A}$ in Theorem 3 equal $\mathcal{B}(c_2)$ from Theorem 4. We fail to obtain our hoped for analogue of **[8]** because, in general $\mathcal{A}$ in Theorem 3 must not vary with c. Indeed, it is not hard to show that the result fails for certain $\mathcal{A}$ varying with c. In our situation the precise structure of $\mathcal{A}$ would need to be taken into account when analysing various error terms, if we were to obtain our desired conclusion.

Before proving our results we mention some related questions. Leitmann and Wolke also established that the sequence $[\alpha n]$ includes infinitely many primes for *every* positive α. The corresponding problem for $[\alpha p]$ encompasses such notoriously difficult questions as "is $2p + 1$ prime infinitely often?" (take $\alpha = \frac{1}{2}$). It follows from **[4]** that $[\alpha p]$ is prime infinitely often for almost all α. It is a consequence of **[6]** that $[\alpha p^c]$ is prime infinitely often once c is fixed greater than one for almost all α.

2. Proof of Theorem 3.

Let

$$g(\gamma, a) = \begin{cases} 1 & \text{if } a = [n^c] \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Then

$$\begin{aligned} g(\gamma, a) &= (a+1)^\gamma - a^\gamma + (\{-(a+1)^\gamma\} - \{-a^\gamma\}) \\ &= \gamma a^{\gamma-1} + O(a^{\gamma-2}) + (\{-(a+1)^\gamma\} - \{-a^\gamma\}) \\ &= \gamma a^{\gamma-1} + O(a^{\gamma-2}) - \sum_{0<|h|\le H} \frac{1}{2i\pi h}(e(ha^\gamma) - e(h(a+1)^\gamma)) \\ &\quad + O\left(\min\left(1, \frac{1}{H\,\|a^\gamma\|}\right) + \min\left(1, \frac{1}{H\,\|(a+1)^\gamma\|}\right)\right) \end{aligned}$$

for any H, using the familiar truncated Fourier series for $\{\theta\}$,

$$\begin{aligned} &= \gamma a^{\gamma-1} + O(a^{\gamma-2}) - \int_0^1 \sum_{0<|h|\le H} e(h(a+t)^\gamma)(a+t)^{\gamma-1}\gamma\, dt + E_2(a,\gamma), \quad \text{say} \\ &= M(\gamma, a) + O(a^{\gamma-2}) - E_1(a,\gamma) + E_2(a,\gamma), \quad \text{say.} \end{aligned} \tag{7}$$

To prove Theorem 3 we first restrict our attention to an interval $\left[\frac{1}{v}, \frac{1}{u}\right] \subset (1,2)$ where $v = u + \frac{\varepsilon}{4}$. We may then obtain our result by considering almost all $\gamma \in [u, v]$. At this point we choose

$$H = H(a) = [\mathscr{A}(a)^{1/4} a^\beta] + 2, \quad \text{where} \quad \beta = \frac{1-v}{2}.$$

It will become apparent later that this is the optimum choice for H to balance the two errors $E_1(a,\gamma)$ and $E_2(a,\gamma)$. Clearly

$$\sum_{\substack{a \in \mathscr{A} \\ a \le x^c}} M(\gamma, a)$$

is our main term, while

$$\sum_{a \le x^c} O(a^{\gamma-2}) = O(1).$$

Henceforth the variables a and b will both tacitly be assumed to belong to $\mathscr{A}$.
We will show that, for any $s > r \ge 1$,

$$\int_u^v \sum_{r \le a < s} |E_2(a,\gamma)|\, d\gamma \ll \sum_{r \le a < s} H(a)^{-1} \ln(2a) = \Phi_1(r,s), \quad \text{say,} \tag{8}$$

and

$$\int_u^v \left| \sum_{r \le a < s} E_1(a,\gamma) \right|^2 d\gamma \ll \sum_{r \le a < s} H(a)^2 (\ln(H(a)))^3 a^{-4\beta} = \Phi_2(r,s), \quad \text{say.} \tag{9}$$

By the argument used in **[11]** (or a slight addition to the method of **[12]**, Lemma 10) it follows from (7), (8), (9) that, for almost all $\gamma \in [u, v]$, we have

$$\sum_{a \le y} g(\gamma, a) = \gamma \sum_{a \le y} a^{\gamma-1} + O(\Phi_1(1, y+1)^{1+\frac{\varepsilon}{4}} + \Phi_2(1, y+1)^{\frac{1}{2}+\frac{\varepsilon}{4}}). \tag{10}$$

By Cauchy's inequality

$$\Phi_1(1, y+1)^2 \le \Big(\sum_{1\le a\le y} H(a)^2(\ln a)^4 a^{-4\beta}\Big)\Big(\sum_{1\le a\le y} \frac{H(a)^{-4}a^{4\beta}}{(\ln a)^2}\Big)$$
$$\ll \Big(\sum_{1\le a\le y} \mathscr{A}(a)^{\frac{1}{2}}(\ln a)^4 a^{-2\beta}\Big)\Big(\sum_{1\le a\le y} \frac{\mathscr{A}(a)^{-1}}{(\ln a)^2}\Big). \tag{11}$$

Since the second sum in (11) converges, the error in (10) is

$$O\Big(\Big(\sum_{1\le a\le y} \mathscr{A}(a)^{\frac{1}{2}}(\ln a)^4 a^{-2\beta}\Big)^{\frac{1}{2}+\frac{\varepsilon}{4}}\Big) = O\Big(\Big(\sum_{1\le a\le y} \mathscr{A}(a)^{\frac{1}{2}} a^{\gamma-1+\varepsilon}\Big)^{\frac{1}{2}}\Big),$$

as required to complete the proof of Theorem 3.

To obtain (8) we note that if

$$A \le n^\gamma \le A+\delta$$

then γ is restricted to an interval of length

$$\frac{\ln\left(1+\frac{\delta}{A}\right)}{\ln n} < \frac{\delta}{A\ln n}.$$

Thus

$$\int_u^v \min\Big(H, \frac{1}{\|n^\gamma\|}\Big)\,d\gamma \ll \sum_{n^u-1\le m\le n^v+1} \frac{1}{m\ln n} \sum_{k=1}^{1+[\log_2(H)]} 2^{-k}\min(H, 2^k)$$
$$\ll \ln(2H)$$
$$\ll \ln(2n)$$

and this gives (8).

We begin our estimation of the left-hand side of (9) by writing it as

$$\int_u^v \Big|\sum_{k=0}^{\infty}\sum_{|h|\sim 2^k}\sum_{\substack{r\le a<s\\ H(a)\ge|h|}} \int_0^1 e(h(a+t)^\gamma)(a+t)^{\gamma-1}\gamma\,dt\Big|^2 d\gamma$$
$$\le \frac{\pi^2}{6}\sum_{k=0}^{\infty}(k+1)^2 2^k \int_u^v\int_0^1 \sum_{|h|\sim 2^k}\Big|\sum_{\substack{r\le a<s\\ H(a)\ge|h|}} e(h(a+t)^\gamma)(a+t)^{\gamma-1}\Big|^2 \gamma^2\,dt\,d\gamma \tag{12}$$

by three applications of the Cauchy–Schwarz inequality. Here $d\sim D$ means $D\le d<2D$.

Now

$$\int_u^v \Big|\sum_{r\le a<s} e(h(a+t)^\gamma)(a+t)^{\gamma-1}\Big|^2 d\gamma$$
$$= \sum_{r\le a,b<s}\int_u^v (a+t)^{\gamma-1}(b+t)^{\gamma-1} e(h((a+t)^\gamma-(b+t)^\gamma))\,d\gamma$$

(we tacitly assume the restriction $H(a), H(b) \geq |h|$ in the above and following)

$$\ll \sum_{r \leq a,b < s} (a+t)^{\nu-1}(b+t)^{\nu-1} \min\left(1, \frac{1}{|h|\,|(a+t)^{\nu}\ln(a+t) - (b+t)^{\nu}\ln(b+t)|}\right)$$

(by **[13]**, Lemma 4.3)

$$\ll \sum_{r \leq b \leq a < s} a^{-2\beta} b^{-2\beta} \min\left(1, \frac{a^{2\beta}}{|h|\,(a-b)\ln a}\right) \leq \sum_{r \leq a < s} a^{-4\beta} + \frac{1}{|h|} \sum_{r \leq b < a < s} \frac{b^{-2\beta}}{(a-b)\ln a}$$

$$\ll \sum_{r \leq a < s} a^{-4\beta} + \frac{1}{|h|} \sum_{r \leq a < s} a^{-2\beta}. \tag{13}$$

The right-hand side of (12) is thus

$$\ll \sum_{k=0}^{\infty} (k+1)^2 2^k \sum_{|h| \sim 2^k} \left(\sum_{r \leq a < s} a^{-4\beta} + \frac{1}{|h|} \sum_{r \leq a < s} a^{-2\beta} \right)$$

$$\ll \sum_{r \leq a < s} (a^{-4\beta} H(a)^2 (\ln H(a))^2 + a^{-2\beta} H(a)(\ln H(a))^3)$$

(recalling that a was restricted by $H(a) \geq |h|$)

$$\ll \sum_{r \leq a < s} a^{-4\beta} H(a)^2 (\ln H(a))^3 \tag{14}$$

as desired. To obtain (14) we noted that $H(a)a^{-2\beta} \geq 1$ follows from $c \leq \dfrac{2}{2-\alpha}$. We have thus established (9) and completed the proof of Theorem 3. We note that it was the diagonal terms ($a = b$) which gave the most significant contribution to (13). If $\mathscr{A}$ were to depend on c in the way necessary to prove an analogue of **[8]** the bound (8) may be obtained as above, and the diagonal terms in (13) cause no problem. It is the non-diagonal terms which become very difficult to treat in this new situation.

3. Proof of Theorem 4. We write $\zeta(s)$ for the Riemann zeta-function and let $\rho = \beta + i\gamma$ denote a zero of $\zeta(s)$ $\left(\gamma \text{ no longer appears as } \frac{1}{c}\right)$. We put

$$N(\sigma, T) = \sum_{\substack{\rho \\ \beta \geq \sigma \\ |\gamma| \leq T}} 1.$$

In the following Lemma we assemble the information we require on $N(\sigma, T)$.

LEMMA 1. *If T is sufficiently large we have*

$$N(\sigma, T) = 0, \qquad \textit{for } \sigma > 1 - (\ln T)^{-\frac{7}{10}} \tag{15}$$

$$N(\sigma, T) \ll T^{(1-\sigma)(2-\frac{1}{25})}, \qquad \textit{for } \sigma \geq \tfrac{4}{5} \tag{16}$$

$$N(\sigma, T) \ll \begin{cases} T^{\frac{12}{5}(1-\sigma)}(\ln T)^{44}, & \textit{for } \frac{3}{4} \leq \sigma \leq \frac{4}{5} \\ T^{3(1-\sigma)/(2-\sigma)}(\ln T)^5, & \textit{for } \frac{1}{2} \leq \sigma \leq \frac{3}{4}. \end{cases} \tag{17}$$

Proof. See Chapters 6 and 9 of [**13**], in particular the end of the chapter notes.

Proof of Theorem 4. Put

$$X_r = \exp((\ln r)^4), \qquad r = 1, 2, 3, \ldots.$$

We then note that

$$\frac{\ln X_r}{\ln X_{r-1}} = 1 + O(r^{-1}) = 1 + O(\exp(-(\ln X_r)^{\frac{1}{4}})) \tag{18}$$

and

$$\max\left(\frac{X_r}{X_{r-1}} - 1, 1 - \frac{X_{r-1}}{X_r}\right) \ll \frac{(\ln r)^3}{r} \ll \exp(-\ln X_r)^{\frac{1}{5}}). \tag{19}$$

Write

$$S(r) = \mathscr{B}(y, X_r) - \mathscr{B}(y, X_{r-1}),\ \mathscr{I}_r = (X_{r-1}, X_r]$$

$$d_r = X_r - X_{r-1}, \qquad \text{and, for } j = 0, 1,\ \alpha_j = 1 - \frac{1}{X_{r-j}},$$

$$S_j(r) = \sum_{n \in \mathscr{I}_r} \Lambda(n) \sum_{n^y \alpha_j^y < m \le n^y} \Lambda(m) \tag{20}$$

$$M_j(r) = (1 - \alpha_j^y) \sum_{n \in \mathscr{I}_r} \Lambda(n) n^y.$$

Then

$$\frac{S_0(r)}{y(\ln X_r)^2} + O(X_r^{y-\frac{1}{2}}) \le S(r) \le \frac{S_1(r)}{y(\ln X_{r-1})^2}\left(1 + O\left(\frac{1}{X_{r-1}}\right)\right) \tag{21}$$

where the $O(X_r^{y-\frac{1}{2}})$ term in (21) arises from the contribution of prime powers to (20).

We will show that

$$S_j(r) = M_j(r)(1 + E_1(r) + E_2(r)) \tag{22}$$

where $E_1(r) = O(\exp(-(\ln X_r)^{\frac{1}{5}}))$ for all y, and $|E_2(r)| < \exp(-(\ln X_r)^{\frac{1}{5}})$ for almost all $y \in (1, 2)$, for all large r. We now show how Theorem 4 follows from (22). By (19) and the prime number theorem we obtain

$$M_j(r) = (X_r^y - X_{r-1}^y)(1 + O(\exp(-(\ln X_r)^{\frac{1}{5}}))).$$

Now suppose $X_{r-1} < x < X_r$. The formula (18) gives a bound on $\dfrac{\ln x}{\ln X_{r-1}}$ and $\dfrac{\ln X_r}{\ln x}$, while

$$\mathscr{B}(y, x) - \mathscr{B}(y, X_{r-1}) \ll (x - X_{r-1})x^{y-1} \ll x^y \exp(-(\ln x)^{\frac{1}{5}})$$

and

$$\frac{x^y}{y\,(\ln x)^2} - \frac{X_{r-1}^y}{y(\ln X_{r-1})^2} \ll x^y \exp(-(\ln x)^{\frac{1}{5}}).$$

It follows that

$$\mathscr{B}(y, x) = \frac{x^y}{y(\ln x)^2} + O(x^y \exp(-(\ln x)^{\frac{1}{5}}))$$

for all y for which we obtain a suitable bound on $|E_2(r)|$.

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We commence our proof of (22) by appealing to Landau's formula [7]. We henceforth drop the subscript j on α for clarity. We obtain

$$S_j(r) = M(r) - \sum_{\substack{\rho \\ |\gamma|\le T}} \sum_{n\in\mathcal{I}_r} \Lambda(n) n^{y\rho} \frac{(1-\alpha^{y\rho})}{\rho} + O\left(\frac{d_r X_r^y (\ln X_r)^3}{T} + \sum_{n\in\mathcal{I}_r} \Lambda(n)\ln n\right). \tag{23}$$

We pick $T = X_r \exp((\ln X_r)^{\frac{1}{4}})$ $(= rX_r)$ and so obtain a satisfactory estimate for the $O(\cdot)$ term in (23).

We write $\theta = 1 + (\ln X_r)^{-1}$. Then Lemma 3.19 of [13] gives

$$\sum_{n\in\mathcal{I}_r} \Lambda(n) n^{y\rho} = \frac{-1}{2\pi i} \int_{\theta+y\beta-4iT}^{\theta+y\beta+4iT} \frac{\zeta'}{\zeta}(\omega - y\rho) \frac{X_r^\omega - X_{r-1}^\omega}{\omega}\, d\omega + O(X_r^{y\beta}(\ln X_r)). \tag{24}$$

Now

$$\begin{aligned}\sum_{\substack{\rho \\ |\gamma|\le T}} \left|\frac{(1-\alpha^{y\rho})}{\rho}\right| X_r^{y\beta}(\ln X_r) &\le \frac{\ln X_r}{X_{r-1}} \sum_{\substack{\rho \\ |\gamma|\le T}} X_r^{y\beta} \\ &< (\ln X_r)^2 \max_{\beta\ge\frac{1}{2}} X_r^{y\beta-1} N(\beta, T) \\ &< X_r^{\frac{y}{2}+\frac{1}{4}}\end{aligned}$$

using (17). The error term in (24) is therefore of a suitable size.

We next shift the integral in (24) back to $\mathrm{Re}(\omega) = -\infty$, taking care to avoid zeros of $\zeta(s)$ on the contour (which arise from $\omega - y\rho = \rho'$). We thereby find that the integral in (24) is

$$\frac{X_r^{1+y\rho} - X_{r-1}^{1+y\rho}}{1+y\rho} - \sum_{\substack{\rho' \\ |\gamma'+y\gamma|\le 4T}} \frac{X_r^{\rho'+y\rho} - X_{r-1}^{\rho'+y\rho}}{\rho' + y\rho} + O((\ln X_r) X_r^{y\beta}). \tag{25}$$

The error term in (25) may be dealt with as the error term in (24). We have

$$\begin{aligned}\sum_{\substack{\rho \\ |\gamma|\le T}} \frac{X_r^{1+y\beta}}{1+y|\gamma|} \left|\frac{(1-\alpha^{y\rho})}{\rho}\right| &< \frac{2X_r}{X_{r-1}} \sum_{\rho} \frac{X_r^{y\beta}}{1+y|\gamma|} \\ &\ll X_r^y \exp(-(\ln X_r)^{\frac{3}{10}})(\ln X_r)^2 \\ &\ll (X_r^y - X_{r-1}^y)\exp(-(\ln X_r)^{\frac{1}{4}})\end{aligned}$$

using (15).

It remains therefore to consider

$$\left|\sum_{\substack{\rho \\ |\gamma|\le T}} \sum_{\substack{\rho' \\ |\gamma'+y\gamma|\le 4T}} \frac{X_r^{\rho'+y\rho} - X_{r-1}^{\rho'+y\rho}}{\rho'+y\rho} \frac{1-\alpha^{y\rho}}{\rho}\right| \le \sum_{\substack{\rho \\ |\gamma|\le T}} \sum_{\substack{\rho' \\ |\gamma'+y\gamma|\le 4T}} \frac{2X_r^{\beta'+y\beta-1}}{y\beta + \beta' + |\gamma' + y\gamma|}. \tag{26}$$

It should become apparent that, as usual, the terms with $\beta \ge \frac{1}{2}$ are of the greatest significance in (26), so we henceforth tacitly assume $\beta \ge \frac{1}{2}$. We then split the summation ranges for ρ and ρ' to get sums $S(b, B)$ where $b \le \beta < b + (\ln X_r)^{-1}$, and $B \le \beta' < B + (\ln X_r)^{-1}$ in each sum. First suppose that one of $b, B \ge \frac{4}{5}$. We suppose $b \ge B$, the case

$B > b$ follows similarly. Then we tackle (26) by summing first over ρ' then over ρ. This gives

$$S(b,B) \ll (\ln X_r)^2 N(b,T) X_r^{B+yb-1}$$
$$\ll (\ln X_r)^2 T^{(2-\frac{1}{25})(1-b)} X_r^{b+yb-1}. \tag{27}$$

We note that (27) is an increasing function of b since $1+y > 2-\frac{1}{25}$. Hence, using (15)

$$S(b,B) \ll X_r^y \exp(-(\ln X_r)^{\frac{1}{4}})(\ln X_r)^{-2}$$

which we have already seen is a suitable error.

We now suppose that $b, B \le \frac{4}{5}$. We split up the ranges of summation over γ and γ' so that in $S(b,B,A,G)$ we have $|\gamma| \sim G$ and $A-1 \le |\gamma' + y\gamma| < 2A$, where $A, G \in \{2^n \mid n \ge 0\}$. Thus

$$S(b,B,A,G) \ll \sum_{\substack{\rho,\rho' \\ A-1\le|\gamma'+y\gamma|<2A \\ |\gamma|\sim G}} \frac{X_r^{B+yb-1}}{A}.$$

Thus if we write

$$I = \int_{1+\varepsilon}^{2} S(b,B,A,G)^2 X_r^{-2y}\,dy$$

(we deal here only with the "hardest" region: $y \in [1+\varepsilon, 2]$), we obtain

$$I \ll \sum_{\rho_1,\rho_2} \frac{X_r^{2B+2b(1+\varepsilon)-4-2\varepsilon}}{A^2} \int_{1+\varepsilon}^{2} (*) \sum_{\rho_3,\rho_4} 1\,dy. \tag{28}$$

Here $(*)$ indicates that y is constrained by $A-1 \le |\gamma_1 + y\gamma_2| \le 2A-1$. Once γ_1, γ_2 are fixed the integration is thus over a range of length at most $\frac{2A}{G}$. Also, for each fixed y,

$$\sum_{\rho_3,\rho_4} 1 \ll A \ln 2X_r \min(N(B,G), N(b,G))$$
$$\le A(\ln 2X_r)(N(B,G)N(b,G))^{\frac{1}{2}}. \tag{29}$$

To obtain (29) note that once y and γ_3 are fixed then γ_4 is restricted to a range of length $\frac{A}{y}$ and $N(\sigma, R+A) - N(\sigma, R) \ll A \ln 2X_r$ for $R \le X_r$. A similar argument holds if y and γ_4 are fixed. Hence upon writing $\eta = \frac{2\varepsilon}{5}$, and using $b \le \frac{4}{5}$ we obtain

$$I \ll \sum_{\rho_1,\rho_2} \frac{X_r^{2b+2B-4-\eta}}{G} (\ln X_r)(N(B,G)N(b,G))^{\frac{1}{2}}$$
$$\ll \frac{X_r^{2b+2B-4-\eta}}{G} (\ln X_r)(N(B,G)N(b,G))^{\frac{3}{2}}$$
$$\ll \max_{a=b,B} (\ln X_r) \frac{N^3(a,G)}{G} X_r^{4a-4-\eta}. \tag{30}$$

After substituting the bound (17) for $N(a,G)$ in the above, (30) becomes an increasing function of G, so we may put $G = T$. Since

$$\frac{36(1-a)}{5} + 4a - 5$$

is a decreasing function of a, the maximum of (30) for $a \geq \frac{3}{4}$ is attained at $a = \frac{3}{4}$, and this value is

$$\ll (\ln X_r)^{45} \exp((\ln X_r)^{\frac{1}{4}}) X_r^{-\frac{1}{5}}. \tag{31}$$

For $\frac{1}{2} \leq a \leq \frac{3}{4}$ we note that

$$\frac{9(1-a)}{2-a} + 4a - 5 = -\frac{(1-2a)^2}{2-a}.$$

Thus for $a < \frac{3}{4}$ the maximum value for (30) is

$$\ll (\ln X_r)^6 \exp(2(\ln X_r)^{\frac{1}{4}}) X_r^{-\eta} \ll X_r^{-\frac{\eta}{2}}. \tag{32}$$

From (31) and (32) we obtain $I \ll X_r^{-\frac{\eta}{2}}$. We have thus established (22) with an $E_2(r)$ such that, for any $\varepsilon > 0$,

$$\int_{1+\varepsilon}^{2} E_2(r)^2 \, dy \ll X_r^{-\frac{\eta}{3}}.$$

Hence the measure of the set on which $E_2(r) \geq \exp(-(\ln X_r)^{\frac{1}{5}})$ is $\ll r^{-2}$. Since

$$\sum_r r^{-2}$$

converges, we deduce, by the Borel–Cantelli Lemma, that, for almost all $y \in [1+\varepsilon, 2]$, $E_2(r) < \exp(-(\ln X_r)^{\frac{1}{5}})$ for all large r.

This completes the proof.

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