DUALITY THEOREMS FOR CONVEX PROGRAMMING
WITHOUT CONSTRAINT QUALIFICATION

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Abstract

Invoking a recent characterization of Optimality for a convex programming problem with finite dimensional range without any constraint qualification given by Borwein and Wolkowicz, we establish duality theorems. These duality theorems subsume numerous earlier duality results with constraint qualifications. We apply our duality theorems in the case of the objective function being the sum of a positively homogeneous, lower-semi-continuous, convex function and a subdifferentiable convex function. We also study specific problems of the above type in this setting.

1. Introduction

We consider the following pair of problems:

Problem (P). Minimize $f(x)$ subject to

$$g_i(x) \leq 0, \qquad i = 1,\ldots,m \quad \text{and} \quad x \in C.$$

Problem (D).   Maximize $f(x) + y'g(x)$ subject to
$$y \geq 0, \qquad x \in C \quad \text{and}$$
(1.1) $$0 \in \partial f(x) + \Sigma_m^1 y_i \, \partial g_i(x) + N(x/C).$$

Here f and g_i are continuous convex function on a real locally convex space X. C is a convex subset of X and ∂ denotes subdifferential. $y = (y_1, \ldots, y_m)$ is a vector in R^m and t denotes transpose. $N(x/C)$ denotes the normal cone to C at x defined by

$$N(x/C) = \{ y \in X^* : \langle y, z - x \rangle \leqslant 0, \text{ for every } z \in C \}$$

where X^* is the topological dual space of X.

Recently, Schechter [19] established a duality theorem between the problems (P) and (D) assuming Slater's Constraint qualification, namely, there exists an $x' \in C$ such that $g_i(x') < 0$, for $i = 1, \ldots, m$. His duality theorem [19, Theorem 2.2] is the following.

THEOREM 1.1. *Suppose that $\bar{x}$ is optimal for problem* (P). *Then there exists $\bar{y}$ such that $(\bar{x}, \bar{y})$ is optimal for problem* (D) *and the two problems have the same extremal values.*

If f and g_i, $i = 1, \ldots, m$, are differentiable, $X = R^n$ and $C = X$, then (1.1) becomes

$$\nabla f(x) + \sum_1^m y_i \nabla g_i(x) = 0, \qquad y \geqslant 0,$$

where ∇ denotes gradient vector and Theorem 1.1 becomes the duality theorem of Wolfe [21], where he has assumed the constraint qualification of Kuhn and Tucker [10, page 483].

Recently, Ben-Israel, Ben-Tal and Zlobec [2, 3] established necessary and sufficient conditions for a vector x_0 to be optimal for problem (P) without the need for a constraint qualification when X is a finite dimensional space. Their results are called 'BBZ' conditions. Using 'BBZ' conditions, Mond and Zlobec established duality theorems in [13]. More recently, the above characterization for optimality was established by Borwein and Wolkowicz [4] in a general locally convex space without assuming any constraint qualification. In this paper we utilize the results of Borwein and Wolkowicz [4] to establish duality results without assuming any constraint qualification. Duality results in [18], including those of Wolfe [21] and Schechter [19], are special cases of our duality results. In Section 3, we also give a converse duality theorem. In Section 4, we apply our duality results to the case where the objective function is the sum of a positively homogeneous, lower-semi-continuous, convex function and a subdifferentiable convex function. Finally, we reformulate many other duality results in the literature so as to eliminate the need for any constraint qualification.

2. Preliminaries

In this paper we consider the convex programming problem:

Problem (P_1).   Minimize $p(x)$ subject to
$$g(x) \in -S \quad \text{and} \quad x \in C$$

where $p: X \to R \cup \{+\infty\}$, $g: X \to Y \cup \{+\infty\}$; X and Y are real locally convex spaces; Y is finite dimensional with an abstract maximal element $+\infty$ (see [14]); $C \subset X$ and $S \subset Y$ are convex and S is also a cone; p is a convex functional and g is S-convex (on C), that is,

$$tg(x_1) + (1-t)g(x_2) - g(tx_1 + (1-t)x_2) \in S,$$

for any $x_1, x_2 \in C$ and $0 \leqslant t \leqslant 1$.

The convex cone S induces a partial ordering '$\geqslant$' on Y given on
$$x_1 \geqslant x_2 \Leftrightarrow x_1 - x_2 \in S.$$
We make the additional routine assumption that the feasible set
$$A = g^{-1}(-S) \cap C$$
is a non-empty subset of dom p, where dom $p = \{x \in X: p(X) < \infty\}$. Let X^* and Y^* denote the continuous dual spaces of X and Y, respectively. We suppose throughout that X^* is endowed with the weak star topology $\sigma(X^*, X)$ (see [15] for details). We write $\langle x^*, x \rangle$ for the value of the continuous linear functional $x^* \in X^*$ at the point $x \in X$.

Given any set K in X, the *dual cone* of K is the set in X^*
$$K^+ = \{x^* \in X^*: \langle x^*, x \rangle \geqslant 0, \text{ for every } x \in K\}.$$
We denote the annihilator of a set K in X by $K^\perp = K^+ \cap (-K^+)$ and the convex cone generated by K by cone K.

A continuous linear operator $T: X \to Y$ is a *subgradient* for g at a if
$$T(x-a) \leqslant g(x) - g(a) \quad \text{for every } x \in X.$$
The set of all such subgradients is denoted by $\partial g(a)$.

The S-convex operator $g: X \to Y$ is said to be *regularly-sub-differentiable* at a if
$$\partial(x^* \circ g)(a) = x^* \circ \partial g(a)$$
for every $x^* \in S^*$. It is easy to see that, if g is continuous and differentiable at a, that is $\partial g(a)$ contains a unique element (for instance if g is Fréchet differentiable), then g is regularly subdifferentiable at a. For more details about regular subdifferentiability, the reader can refer to [1, 22].

K is a face of a convex cone S if K is a convex cone, and s_1, $s_2 \in S$, $s_1 + s_2 \in K \Rightarrow s_1, s_2 \in K$. We denote by S^f, the (unique) smallest face of S which contains $-g(A)$.

Let $a \in A$. We define

$$(2.1) \quad D_g^{\leq}(a) = \{d: \text{there exists } \alpha > 0 \text{ with } g(a + td) \in S^f - S \text{ if } 0 \leq t \leq \alpha\}.$$

If $Y = R^m$ and $S = R_+^m$, then (2.1) corresponds to the intersection of the cones of non-increase of the equality constraints, that is, the constraints which are identically zero on the feasible set [2, 3].

3. Duality and converse duality theorems

The following theorem can be proved using the definition of regular subdifferentiability and Corollaries 4.2 and 4.3 of Borwein and Wolkowicz [4] (see also Zowe [22]).

THEOREM 3.1. *Suppose that $S^+ + (S^f)^\perp = (S^f)^+$ holds. Suppose also that f is a convex functional and g is S-convex on X and that they are continuous at a. Further, we assume that g is regularly subdifferentiable at a. Then $a \in A$ is optimal for (P_1) if and only if there exists $s^+ \in S^+$ such that*

$$0 \in \partial p(a) + s^+ \partial g(a) - \left(D_g^{\leq}(a) \cap \text{Cone}(C - a)\right)^+ \quad \text{and} \quad s^+ g(a) = 0.$$

Based on the above characterization of optimality without any constraint qualification, we can formulate the following dual problem (D_1) to the primal problem (P_1).

Problem (D_1).   Maximize $p(x) + s^+ g(x)$ subject to
$$s^+ \in S^+, x \in A^f \quad \text{and}$$
$$0 \in \partial p(x) + s^+ \partial g(x) - (D_g^{\leq}(x) \cap \text{Cone}(C - x))^+$$

where $A^f = g^{-1}(S^f - S) \cap C$.

Note. $A \subset A^f$. Further $D_g^{\leq}(x) \cap \text{Cone}(C - x)$ is the smallest cone containing A^f (see [4] for details).

Now, we establish the duality theorem between the problems (P_1) and (D_1) without any constraint qualification.

THEOREM 3.2 (*duality*). *Let us assume the hypothesis of Theorem 3.1. If $a_0 \in A$ is optimal for problem* (P_1), *then there exists $s_0^+ \in S^+$ such that (a_0, s_0^+) is optimal for problem* (D_1) *and furthermore the two problems have the same extremal values.*

PROOF. Since $a_0 \in A$ is optimal for (P_1), Theorem 3.1 guarantees the existence of feasible solutions to problem (D_1).

Let (a, s^+) be a feasible solution for problem (D_1). Then $a \in A^f$, $s^+ \in S^+$ and

$$0 \in \partial p(a) + s^+ \partial g(a) - \left(D_g^\leqslant(a) \cap \operatorname{Cone}(C - a) \right)^+ .$$

This implies that there exist $v^* \in \partial p(a)$ and $T \in \partial g(a)$ and $w^* \in (D^\leqslant(a) \cap \operatorname{Cone}(C - a))^+$ such that

(3.1) $0 = v^* + s^+ T - w^*.$

Now, consider

$$
\begin{aligned}
p(a_0) - [p(a) + s^+ g(a)] &= [p(a_0) - p(a)] - s^+ g(a) \\
&\geqslant \langle v^*, a_0 - a \rangle - s^+ g(a), \text{ since } v^* \in \partial p(a) \\
&= \langle -s^+ T + w^*, a_0 - a \rangle - s^+ g(a), \quad \text{from (3.1)} \\
&= \langle -s^+ T, a_0 - a \rangle + \langle w^*, a_0 - a \rangle - s^+ g(a) \\
&\geqslant \langle s^+, g(a) - g(a_0) \rangle + \langle w^*, a_0 - a \rangle - s^+ g(a), \\
&\hspace{6cm} \text{since } T \in \partial g(a) \\
&= -s^+ g(a_0) + \langle w^*, a_0 - a \rangle \\
&\geqslant 0,
\end{aligned}
$$

since $s^+ \in S^+$, $g(a_0) \in -S$ and $w^* \in (D_g^\leqslant(a) \cap \operatorname{Cone}(C - a))^+ = \partial I_{A^f}(A)$ (see [16], Corollary 23.5.4), where I_{A^f} is the indicator function of A^f; that is, $I_{A^f}(x)$ is 0 on A^f and $+\infty$ elsewhere. Thus,

(3.2) $p(a_0) \geqslant p(a) + s^+ g(a)$

for every feasible solution (a, s^+) of (D_1).

Since a_0 is an optimal solution of (P_1), we have from Theorem 3.1, that there exists $s_0^+ \in S^+$ such that $s_0^+ g(a_0) = 0$ and

$$0 \in \partial p(a_0) + s_0^+ \partial g(a_0) - \left(D_g^\leqslant(a_0) \cap \operatorname{Cone}(C - a_0) \right)^+ .$$

In other words, (a_0, s_0^+) is a feasible solution for (D_1) and

(3.3) $p(a_0) = p(a_0) + s_0^+ g(a_0).$

Hence, from (3.2) and (3.3), (a_0, s_0^+) is an optimal solution of (D_1) and that the extremal values of the two problems (P_1) and (D_1) are equal.

REMARKS 3.3. Lemma 3.1(a) in [4] shows that when the convex cone S has non-empty interior, then $S^f = S$ exactly when Slater's constraint qualification holds for (P_1), that is, there exists $x \in C$ with $g(x) \in -\text{int } S$, where int denotes interior. Thus, we can draw the following conclusions.

(a) If Slater's constraint qualification is satisfied for (P_1), then $S^f = S$ and in this case $D_g^\leqslant(a) = X$ and $D_g^\leqslant(a) \cap \text{Cone}(C - a) = \text{Cone}(C - a)$. Hence $(D_g^\leqslant(a) \cap \text{Cone}(C - a))^+ = (\text{Cone}(C - a))^+ = N(a/C)$ (see Holmes [6], page 24).

When $Y = R^m$, $S = R_+^m$, problem (D_1) now reduces to problem (D) and Theorem 3.2 yields Schechter's duality theorem [19], that is, Theorem 1.1 in the introduction.

(b) In addition to the assumptions in (a), if we further assume that $X = R^n$, $C = X$, then $D_g^\leqslant(a) \cap \text{Cone}(C - a))^+ = \{0\}$. Problem (D_1) becomes

Maximize $p(x) + s^t g(x)$ subject to
$$s \geqslant 0 \quad \text{and} \quad 0 \in \partial p(x) + s^t \partial g(x)$$

and Theorem 3.2 becomes Schechter's duality theorem 1 in [18].

(c) In addition to the conditions in (b) if p and g are differentiable, then problem (D_1) becomes

Maximize $p(x) + s^t g(x)$ subject to
$$s \geqslant 0, \qquad \nabla p(x) + s^+ \nabla g(x) = 0$$

and Wolfe's duality theorem 2 in [21] in recovered from Theorem 3.2.

Assuming Slater's constraint qualification a duality theorem in the spirit of Schechter [18, 19] has been proved in a locally convex space with operatorial constraints in [8, 9].

We shall now prove a converse duality theorem between the problems (P_1) and (D_1) without any constraint qualification.

THEOREM 3.3 (*converse duality*). *Let us assume the hypothesis of Theorem* 3.1 *and assume that the problem* (P_1) *has a solution* $\bar{a}$. *If* (a_0, s_0^+) *is an optimal solution of the problem* (D_1), *and if p is strictly convex at* a_0, *then* $a_0 = \bar{a}$. *Hence* a_0 *solves the problem* (P_1). *Furthermore, the extremal values of the two problems are the same.*

PROOF. Supposing $a_0 \neq \bar{a}$. Since $\bar{a}$ is a solution of (P_1), it follows from Theorem 3.1 that there exists $\bar{s}^+ \in S^+$ such that

$$0 \in \partial p(\bar{a}) + \bar{s}^+ \partial g(\bar{a}) - \left(D_g^<(\bar{a}) \cap \mathrm{Cone}(C - \bar{a}) \right)^+$$

and

(3.4) $\bar{s}^+ g(\bar{a}) = 0.$

Let $L(a, s^+) = p(a) + s^+ g(a)$ be the Lagrangian of (P_1). Then

$$L(\bar{a}, \bar{s}^+) = L(a_0, s_0^+) = \max_{(a, s^+) \in N} L(a, s^+)$$

where $N = \{(a, s^+): a \in A^f, s^+ \in S^+ \text{ and } 0 \in \partial p(a) + s^+ \partial g(a) - (D_g^<(a) \cap \mathrm{Cone}(C - a))^+\}$. Note that $(\bar{a}, \bar{s}^+) \in N$.

Since $(a_0, s_0^+) \in N$, we have $0 \in \partial p(a_0) + s_0^+ \partial g(a_0) - (D_g^<(a_0) \cap \mathrm{Cone}(C - a_0))^+$. Hence there exist $u^* \in \partial p(a_0)$, $T \in \partial g(a_0)$ and $w^* \in (D_g^<(a_0) \cap \mathrm{Cone}(C - a_0))^+$ such that

(3.5) $0 = u^* + s_0^+ T - w^*.$

Now, consider

$$
\begin{aligned}
L(\bar{a}, s_0^+) - L(a_0, s_0^+) &= p(\bar{a}) + s_0^+ g(\bar{a}) - p(a_0) - s_0^+ g(a_0) \\
&= [p(\bar{a}) - p(a_0)] + s_0^+ [g(\bar{a}) - g(a_0)] \\
&> \langle u^*, \bar{a} - a_0 \rangle + \langle s_0^+, g(\bar{a}) - g(a_0) \rangle, \\
&\qquad\qquad \text{since } p \text{ is strictly convex at } a_0 \\
&\geqslant \langle u^*, \bar{a} - a_0 \rangle + \langle s_0^+, T(\bar{a}) - T(a_0) \rangle, \\
&\qquad\qquad \text{since } T \in \partial g(a_0) \\
&= \langle u^*, \bar{a} - a_0 \rangle + \langle s_0^+ T, \bar{a} - a_0 \rangle \\
&= \langle w^*, \bar{a} - a_0 \rangle \geqslant 0, \quad \text{by (3.4).}
\end{aligned}
$$

since $w^* \in (D_g^<(a_0) \cap \mathrm{Cone}(C - a_0))^+$ (see the proof of Theorem 3.2). It follows that, $L(\bar{a}, s_0^+) > L(a_0, s_0^+) = L(\bar{a}, \bar{s}^+)$. That is,

(3.6) $p(\bar{a}) + s_0^+ g(\bar{a}) > p(\bar{a}) + \bar{s}^+ g(\bar{a}).$

Hence, (3.6) becomes by (3.4), $s_0^+ g(\bar{a}) > 0$, which is a contradiction to the fact that $s_0^+ \in S^+$, $g(\bar{a}) \in -S$. Hence $\bar{a} = a_0$ and a_0 solves problem (P_1).

Furthermore, we have

$$
\begin{aligned}
p(a_0) = p(\bar{a}) &= p(\bar{a}) + \bar{s}^+ g(\bar{a}) = L(\bar{a}, \bar{s}^+) = L(a_0, s_0^+) \\
&= p(a_0) + s_0^+ g(a_0).
\end{aligned}
$$

Hence the extremal values of the two problems are equal.

EXAMPLE. The following is an example of a minimization problem where the Slater's constraint qualification is not satisfied. We will show that Wolfe's duality theorem is not applicable to this example. However, the duality theorem of this section can be applied.

Problem (P'). Minimize $p(x) = -x$ subject to

$$g(x) \leqslant 0,$$

where $g(x) = x^2$ if $x \geqslant 0$ and $g(x) = 0$ otherwise.

The feasible set F for this problem (P') is

$$F = \{x \in R: g(x) \leqslant 0\} = (-\infty, 0].$$

Clearly $x = 0$ is the optimal solution of problem (P'). The functions p and g are differentiable everywhere. The derivative of g at x is denoted by $g'(x)$, where

$$g'(x) = \begin{cases} 2x & \text{if } x \geqslant 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Corresponding to the primal problem (P'), the Wolfe's [21] dual problem (D') is

Problem (D'). Maximize $-x + \lambda x^2$ subject to

(3.7) $x \in F, \lambda \geqslant 0, \qquad -1 + \lambda g'(x) = 0.$

For any $x \in F, \lambda \geqslant 0$, the condition (3.7) is not satisfied, which implies feasible solutions for problem (D') do not exist and hence there is no optimal solution for problem (D'). Wolfe's duality theorem does not hold in the case.

It is clear that $S^f = \{0\}$ and the condition that $S^+ + (S^f)^\perp = (S^f)^+$ is also satisfied.

By definition

$$\begin{aligned} D_g^\leqslant(x) &= \{d: \exists \alpha > 0 \text{ with } g(x + td) \in S^f - S, 0 \leqslant t \leqslant \alpha\} \\ &= \{d: \exists \alpha > 0 \text{ with } g(x + td) \in (-\infty, 0], 0 \leqslant t \leqslant \alpha\} \\ &= R \quad \text{if } x < 0 \\ &= (-\infty, 0] \quad \text{if } x = 0 \\ &= \text{empty set} \quad \text{if } x > 0. \end{aligned}$$

As per the dual problem (D$_1$) in this section, the dual problem (D'') in this case is:

Problem (D'').   Maximize $-x + \lambda x^2$ subject to
$$x \in F, \lambda \in S^+ = [0, \infty) \quad \text{and}$$
$$-1 + g'(x) \in (D_g^\leqslant(x))^+.$$

It is clear that the set of feasible solutions for problem (D″) is $\{(x, \lambda): x = 0$ and $\lambda \geqslant 0\}$ and any feasible solution is an optimal solution of problem (D″). Further the optimal value zero is same for the two problems (P′) and (D″).

4. Applications

We shall now apply the theorems in Section 3 to the case where the objective function is the sum of a positively homogeneous, lower-semi-continuous, convex function and a continuous convex function. Specific problems of this type are studied in detail in [11, 12] assuming a constraint qualification such as Slater's type. In the next section, we shall examine the forms which the dual problem takes in these special cases without a constraint qualification.

We shall need the following definition and propositions whose proofs can be found in [7, 8].

DEFINITION 4.1. Let V be a non-empty subset of a locally convex space X^*. Then *the support function of V*, denoted by $s(\cdot /V)$ is defined by

$$s(u/V) = \sup\{\langle u^*, u \rangle : u^* \in V\}.$$

Note. Let p be a positively homogeneous, lower-semi-continuous, convex function, defined on a locally convex space X. Then

$$\partial p(0) = \{u^* \in X^*: p(u) \geqslant \langle u^*, u \rangle \text{ for all } u \in X\},$$

since $p(0) = 0$.

PROPOSITION 4.2. *Let p be a positively homogeneous, lower-semi-continuous, convex function defined on a locally convex space X. Then p is the support function $\partial p(0)$.*

REMARK 4.3. If p is a positively homogeneous lower-semi-continuous, convex function, then $\partial p(0)$ is a non-empty, convex, subset of X^* (see [7], page 3). In fact, there is a one to one correspondence between convex subsets of X^* and positively homogeneous, lower-semi-continuous, convex functions on X.

PROPOSITION 4.4 *Let p be a positively homogeneous, lower-semi-continuous convex function defined on a locally convex space X and let $u \neq 0$. Then*

$$\partial p(u) = \{u^* \in \partial p(0): p(u) = \langle u^*, u \rangle\}.$$

Let the objective function $p: X \to R$ be of the form $p = p_1 + p_2$, where p_1 is a continuous convex function and p_2 is a positively homogeneous, lower-semi-continuous, convex function.

The primal problem (P_2) is

Problem (P_2). Minimize $p_1(x) + p_2(x)$ subject to
$$g(x) \in -S \quad \text{and} \quad x \in C.$$

Let (D_2) denote the following dual problem.

Problem (D_2). Maximize $p_1(x) + \langle x^*, x \rangle + s^+ g(x)$ subject to
$s^+ \in S, x \in A^f, x^* \in \partial p_2(0)$ and
$0 \in \partial p_1(x) + x^* + s^* \partial g(x) - (D_g^\leqslant (x) \cap \text{Cone}(C - x))^+$.

We shall now prove a duality theorem between the problems (P_2) and (D_2).

THEOREM 4.5. *Let us assume the hypothesis of Theorem 3.1 If $a_0 \in A$ is optimal for (P_2), then there exist x_0^* and s_0^+ such that (a_0, x_0^*, s_0^+) is optimal for (D_2).*

PROOF. Since $a_0 \in A$ is optimal for (P_2), by Theorem 3.1, there exists $s^+ \in S^+$ such that $s^+ g(a_0) = 0$ and
$$0 \in \partial(p_1 + p_2)(a_0) + s^+ \partial g(a_0) - \left(D_g^\leqslant (a_0) \cap \text{Cone}(C - a_0) \right)^+ .$$
But $\partial(p_1 + p_2)(a_0) = \partial p_1(a_0) + \partial p_2(a_0)$ by the Moreau-Rockafellar Theorem [16]. Also,
$$\partial p_2(a_0) = \{ x^* \in \partial p_2(0) : p_2(a_0) = \langle x^*, a_0 \rangle \}.$$
Therefore, by Proposition 4.4,
$$0 \in \partial p_1(a_0) + \{ x^* \in \partial p_2(0) : p_2(a_0) = \langle x^*, a_0 \rangle \}$$
$$+ s^+ \partial g(a_0) - \left(D_g^\leqslant (a_0) \cap \text{Cone}(C - a_0) \right)^+ .$$
Hence, there is $x^* \in \partial p_2(0)$ satisfying $p_2(a_0) = \langle x^*, a_0 \rangle$ such that
$$0 \in \partial p_1(a_0) + x^* + s^+ \partial g(a_0) - \left(D_g^\leqslant (a_0) \cap \text{Cone}(C - a_0) \right)^+ .$$
Thus, feasible solutions to problem (D_2) exist.

As in the proof of Theorem 3.2, if (a, x^*, s^+) is any feasible solution for (D_2), then we can prove the following inequality by invoking the definition of support

function, the convexity of the functions p_1 and g and using the Propositions 4.2 and 4.4,

$$(4.1) \qquad p_1(a_0) + p_2(a_0) \geqslant p_1(a) + \langle x^*, a \rangle + s^+ g(a)$$

for every feasible solution (a, x^*, s^+) of (D_2). Now, since a_0 is optimal for (P_2), there exists $s_0^+ \in S^+$, $x_0^* \in \partial p_2(0)$ satisfying $p_2(a_0) = \langle x_0^*, a_0 \rangle$ such that

$$0 \in \partial p_1(a_0) + x_0^* + s_0^+ \, \partial g(a_0) - \left(D_g^{\leqslant}(a_0) \cap \mathrm{Cone}(C - a_0) \right)^+$$

and such that $s_0^+ \, g(a_0) = 0$. Therefore

$$p_1(a_0) + p_2(a_0) = p_1(a_0) + p_2(a_0) + s_0^+ \, g(a_0).$$

Hence, by (3.6), $p_1(a_0) + \langle x_0^*, a_0 \rangle + s_0^+ \, g(a_0) = p_1(a_0) + p_2(a_0) + s_0^+ \, g(a_0) \geqslant p_1(a) + \langle x^*, a \rangle + s^+ g(a)$ for every feasible solution (a, x^*, s^+) of (D_2), which implies that (a_0, x_0^*, s_0^+) is optimal for (D_2). Further, it is clear that

$$p_1(a_0) + p_2(a_0) = p_1(a_0) + \langle x_0^*, a_0 \rangle + s_0^+ \, g(a_0),$$

since $p_2(a_0) = \langle x_0^*, a_0 \rangle$ and $s_0^+ \, g(a_0) = 0$. Hence, the extremal values of the two problems are the same.

5. Special cases

We shall now apply the theory developed in Section 4 to special cases of the problems that have been found in the literature.

(1) First we consider a programming problem whose objective function contains a norm. Such types of problems are studied by Watson in [20].

Let X be a real normed linear space. Consider the problem:

Problem (P_2'). Maximize $k(x) + \|x\|$ subject to
$$g_i(x) \leqslant 0, \qquad i, \ldots, m, \, x \in C.$$

Here $k: X \to R$ is a continuous convex functional, $g_i: X \to R$, $i = 1, \ldots, m$, are subdifferentiable convex functionals and C is a convex subset of X. Let $g: X \to R^m$ be defined by $g(x) = (g_1(x), \ldots, g_m(x))$.
Now, let us formulate the dual problem:

Problem (D_2'). Maximize $k(x) + \|x\| + y'g(x)$ subject to
$$y \in R_+^m, \, x \in A^f,$$
$$0 \in \partial k(x) + \partial \|x\| + \Sigma_1^m y_i \, \partial g_i(x) - (D_g^{\leqslant}(x) \cap \mathrm{Cone}(C - x))^+.$$

It is a well known fact that the norm as a function from X into R is a continuous convex functional which is positively homogeneous. Therefore, the norm function can be expressed as a support function of the weak star closed convex set $\partial\|0\|$, by Proposition 4.2. Furthermore, it is proved in ([7], page 46), that

$$\partial\|x\| = \begin{cases} \{x^* \in X^*: \|x^*\| = 1, \langle x^*, x \rangle = \|x\|\}, & \text{if } x \neq 0, \\ \{x^* \in X^*: \|x^*\| \leqslant 1\} = B'(0, 1), & \text{if } x = 0, \end{cases}$$

where X^* is the topological dual of X.

Thus, the subdifferential of the norm, in a normed linear space, at the origin coincides with the closed unit ball of the dual space. Hence, by Proposition 4.2, the norm is the support function of the unit ball of the dual space X^*.

Hence, by the foregoing argument, the dual problem (D$_2'$) becomes

Problem (D$_2'$).   Maximize $k(x) + \langle x^*, x \rangle + y'g(x)$ subject to
$y \in R_+^m$, $x \in A^f$, $x^* \in B'(0, 1)$ and
$0 \in \partial k(x) + x^* + \Sigma_1^m y_i \partial g_i(x) - (D_g^<(x) \cap$
Cone$(C - x))^+$.

If we assume Slater's constraint qualification, then $(D_g^<(x) \cap \text{Cone}(C - x))^+$ $= N(x|C)$ and $A^f = \{0\}$ (see Holmes [6]).

REMARK 5.1. If k is a zero functional, then problem (P$_2'$) becomes a problem in approximation theory, that is, finding a vector of minimum norm from the convex set $K = \{x \in X: g_i(x) \leqslant 0, i = 1,\ldots,m\} \cap C$.

(2) Let us define the function $h: R^n \to R$ by $h(x) = (x'Bx)^{1/2}$, where B is a (symmetric) positive semi-definite $n \times n$ matrix. Then the following lemma is proved in [5].

LEMMA 5.2. *The function h is convex, and $z \in \partial h(a)$ if and only if $z = w'B$, $w'Bw \leqslant 1$, and $h(a) = w'Ba$.*

Now, consider the following dual pair.

Problem (P$_3'$).   Minimize $k(x) + (x'Bx)^{1/2}$ subject to
$g_i(x) \leqslant 0, i = 1,\ldots,m, x \in C$.

Problem (D$_3'$).   Minimize $k(x) + x'Bz + y'g(x)$ subject to
$y \in R_+^m$, $x \in A^J$, $Bz \in \partial h(0)$ and
$0 \in \partial k(x) + Bz + \Sigma_1^m y_i \, \partial g_i(x) - (D_g^{\leqslant}(x) \cap$
Cone$(C - x))^+$.

Here k and the g_i are convex functions on R^n. Hence, they are continuous and subdifferentiable everywhere. It is clear that $(x'Bx)^{1/2} = s(x/\partial h(0))$, where $\partial h(0)$ $= \{By: y'By \leqslant 1\}$.

If we assume Slater's constraint qualification and $C = X$, then (P$_3'$) and (D$_3'$) are the dual pair studied in the differentiable case in [11, 18, 19].

REMARK 5.3. If $A = \{0\}$, then the pair (P$_3'$) and (D$_3'$) becomes the duality pair in Remark 3.3(b).

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