Wolstenholme's inequality and its relation to the Barrow and Garfunkel-Bankoff inequalities

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Wolstenholme's inequality

Many readers are familiar with the celebrated Finsler-Hadwiger and Weitzenböck's inequality in triangle geometry. On the other hand, Wolstenholme's inequality is not widely known, but is equally important and in an indirect way can be used to derive these two. We will introduce it and then show its strength.

Let x, y, z be real numbers and ABC a triangle with sides a, b, c, semiperimeter s, circumradius R, inradius r and area Δ. Then the inequality

$$x^2 + y^2 + z^2 \geqslant 2yz \cos A + 2zx \cos B + 2xy \cos C \qquad (1)$$

holds. This sublime but underused inequality is named after Joseph Wolstenholme (1829-1891), an English mathematician who published the result in [1]. For its proof consider the incentre I of the triangle ABC and the feet of its perpendiculars X, Y, Z to the sides, see Figure 1. Since $|\overrightarrow{IX}| = |\overrightarrow{IY}| = |\overrightarrow{IZ}| = r$, expanding the obvious inequality for the squared dot product $\left(x\overrightarrow{IX} + y\overrightarrow{IY} + z\overrightarrow{IZ}\right)^2 \geqslant 0$ yields

$$r^2\left(x^2 + y^2 + z^2\right) + 2xy\overrightarrow{IX}\cdot\overrightarrow{IY} + 2yz\overrightarrow{IY}\cdot\overrightarrow{IZ} + 2zx\overrightarrow{IZ}\cdot\overrightarrow{IX} \geqslant 0.$$

In the quadrilateral $XCYI$ we have $\angle XIY = \pi - C$. Hence

$$\overrightarrow{IX}\cdot\overrightarrow{IY} = r^2 \cos(\pi - C) = -r^2 \cos C$$

and similarly for the other two dot products. Now (1) follows readily.

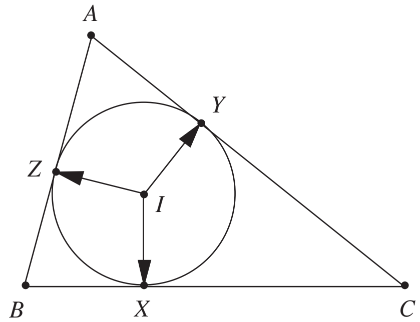

FIGURE 1

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The Barrow and Garfunkel-Bankoff inequalities

Let P be an interior point of a triangle ABC, let the feet of its perpendiculars to the sides BC, CA, AB be denoted by X, Y, Z and let $2\alpha = \angle BPC$, $2\beta = \angle CPA$, $2\gamma = \angle APB$. We denote by PD, PE, PF the angle bisectors of $\angle BPC$, $\angle CPA$, $\angle APC$ in triangles BPC, CPA, APC respectively, see Figure 2. Barrow's inequality

$$PA + PB + PC \geqslant 2(PD + PE + PF) \qquad (2)$$

is an obvious sharpening of the celebrated Erdős-Mordell inequality [2]

$$PA + PB + PC \geqslant 2(PX + PY + PZ).$$

For seven proofs of Erdős-Mordell inequality, with one of them proving (2), see [3]. Our first application of Wolstenholme's inequality is to give another proof of Barrow's inequality (2).

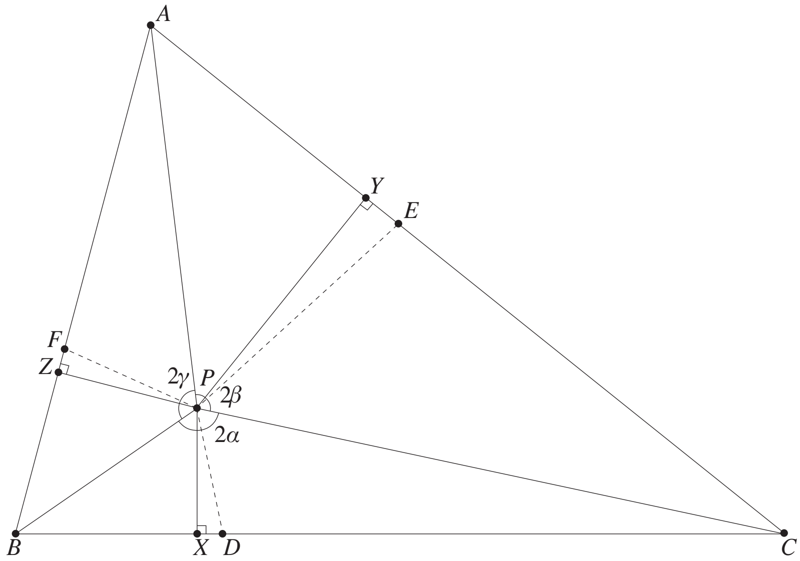

FIGURE 2

The formula for the length of the angle bisector AM in triangle ABC is $AM = \dfrac{2bc}{b + c}\cos\frac{1}{2}A$ and it follows from considering the area of ABC as sum of the areas of triangles ABM and AMC. Hence for the length of the angle bisector PD in triangle BCP we have

$$PD = \frac{2PB \cdot PC}{PB + PC}\cos\alpha$$

and similarly for PE, PF. Since α, β, γ are angles of a triangle, we can apply (1) to $\cos\alpha$, $\cos\beta$, $\cos\gamma$, to get

$$x^2 + y^2 + z^2 \geqslant \frac{PB + PC}{PB \cdot PC}yz\,PD + \frac{PC + PA}{PC \cdot PA}zx\,PE + \frac{PA + PB}{PA \cdot PB}xy\,PF.$$

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Putting $x = \sqrt{PA}$, $y = \sqrt{PB}$, $z = \sqrt{PC}$ in the last inequality, the AM-GM inequality yields

$$PA + PB + PC \geqslant \frac{PB + PC}{\sqrt{PB \cdot PC}} PD + \frac{PC + PA}{\sqrt{PC \cdot PA}} PE + \frac{PA + PB}{\sqrt{PA \cdot PB}} PF$$
$$\geqslant 2(PD + PE + PF),$$

and the proof of Barrow's inequality (2) is completed.

In a second interesting application of Wolstenholme's inequality we use it to prove the Garfunkel-Bankoff inequality [4]

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geqslant 2 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}. \qquad (3)$$

We put

$$x = \tan \frac{A}{2}, \qquad y = \tan \frac{B}{2}, \qquad z = \tan \frac{C}{2}$$

in Wolstenholme's inequality (1) and obtain (with cyclic sum)

$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} \geqslant 2 \sum \tan \frac{B}{2} \tan \frac{C}{2} \cos A. \qquad (4)$$

To calculate the right-hand side of this inequality, we use the well-known trigonometric identity

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1, \qquad (5)$$

which follows from Euler's formula $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, and the identity

$$\frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C} + \frac{\sin \frac{1}{2}B}{\cos \frac{1}{2}C \cos \frac{1}{2}A} + \frac{\sin \frac{1}{2}C}{\cos \frac{1}{2}A \cos \frac{1}{2}B} = 2. \qquad (6)$$

For the proof of (6) we invoke the formula for the product of cosines

$$\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{s}{4R},$$

which follows from Euler's half-angle formula $\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$, Heron's formula for the area $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ and $abc = 4R\Delta$. Hence

$$\sum \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}B \cos \frac{1}{2}C} = \frac{4R}{s} \sum \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$$
$$= \frac{2R}{s} \sum \sin A = 2,$$

where, in the last equation, the sine rule $\sin A = \frac{a}{2R}$ is used. From (5), (6)

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and $1 - \cos A = 2\sin^2 \frac{1}{2}A$, we get

$$\sum \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C \cos A = 1 + \sum \tan\tfrac{1}{2}B \tan\tfrac{1}{2}C \left(\cos A - 1\right)$$

$$= 1 - 2\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C \sum \frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}B \cos\frac{1}{2}C}$$

$$= 1 - 4\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C,$$

which by (4) proves the Garfunkel-Bankoff inequality (3).

One can have fun putting whatever one wants into (1) to obtain other interesting inequalities. As an instance of a third application, Schur's inequality of second degree for the sides of a triangle

$$a^2(a-b)(a-c) + b^2(b-a)(b-c) + c^2(c-a)(c-b) \geqslant 0,$$

is easily obtained by taking $x = a^2$, $y = b^2$, $z = c^2$ and rearranging with the cosine formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$. We remark that the general Schur's inequality

$$x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) \geqslant 0,$$

holds for any three positive numbers x, y and z and all $k \geqslant 0$.

Relations in a sextet of triangle inequalities

We finish this Article by considering the relations between six well-known triangle inequalities. The first Garfunkel-Bankoff inequality (3) is equivalent to the familiar Kooi's inequality [4, 5]

$$s^2 \leqslant \frac{R(4R+r)^2}{2(2R-r)}. \tag{7}$$

Actually in [4] Bankoff used Kooi's inequality to solve the problem posed by Garfunkel, and showed that the two inequalities can be transformed into each other. The two inequalities are equivalent to the more recent [6]

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3 + \frac{R-2r}{R}} + (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{8}$$

The three equivalent inequalities are equalities only for the equilateral triangle. We shall call (8) the 'strong Finsler-Hadwiger inequality' since by Euler's inequality $R \geqslant 2r$ it is an obvious sharpening of the famous Finsler-Hadwiger inequality [7, 8, 9]

$$a^2 + b^2 + c^2 \geqslant 4\Delta\sqrt{3} + (a-b)^2 + (b-c)^2 + (c-a)^2. \tag{9}$$

The fourth inequality in our sextet is named after the Swiss mathematicians Paul Finsler (1894-1970) and Hugo Hadwiger (1908-1981). Between the strong Finsler-Hadwiger and the Finsler-Hadwiger inequality,

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we encounter still another prominent one, Gerretsen's inequality

$$s^2 \leqslant 4R^2 + 4Rr + 3r^2. \tag{10}$$

Gerretsen's inequality implies the Finsler-Hadwiger inequality (9), see [10], and is implied by Kooi's inequality [6]. It has many applications in triangle geometry, see for example [11] where it is used in connection with the inarc centres [12]. For another application, see [13] where Gerretsen's inequality in its equivalent form $a^2 + b^2 + c^2 \leqslant 8R^2 + 4r^2$ is used to prove the inequality $OH \geqslant OI$, where O, H and I are the circumcentre, orthocentre and incentre of the triangle. The last result for the distances between triangle centres plays a central role in the proof of a conjectured inequality for the altitudes of the excentral triangle, see [14].

Finsler-Hadwiger seems to be stronger, but surprisingly is actually equivalent to the last, sixth, Weitzenböck's inequality [15]

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta, \tag{11}$$

named after the Austrian mathematician Roland Weitzenböck (1885-1955). In [16, 17] the excentral and circummidarc triangle [12] are employed to prove the logical equivalence of the two. So, when Weitzenböck's inequality is applied to the sides of these triangles, the Finsler-Hadwiger inequality pops up. Weitzenböck's inequality itself is given eleven proofs in [18], but, for the convenience of the reader, to the multiple proofs we add yet another, very simple, one-line proof:

Twelfth proof. It uses nothing more than the well-known $s \geqslant 3\sqrt{3}r$ and the Cauchy-Schwarz inequality

$$4\sqrt{3}\Delta \leqslant \frac{4s^2}{3} = \frac{(a + b + c)^2}{3} \leqslant a^2 + b^2 + c^2.$$

Equality holds if, and only if, the triangle is equilateral. The inequality is a perennial mathematics contest problem and had a prominent appearance as question number 2 at the third International Mathematical Olympiad held in 1961 in Hungary. Had the unknown author of the contest problem known about reference [15] As IMO problems are supposed to be original, this does not seem likely.

To summarise, we have the following relationship in the sextet of triangle inequalities:

Garfunkel-Bankoff $\Leftrightarrow$ **Kooi** $\Leftrightarrow$ **strong Finsler-Hadwiger**

$\Downarrow$

Gerretsen $\Rightarrow$ **Finsler-Hadwiger** $\Leftrightarrow$ **Weitzenböck**

That is

$$(3) \Leftrightarrow (7) \Leftrightarrow (8) \Rightarrow (10) \Rightarrow (9) \Leftrightarrow (11).$$

And it all begins with Wolstenholme's inequality (1).

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