ISOPERIMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

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1. Introduction. We are concerned with establishing sufficiency theorems for minima of simple integrals of the parametric type in a class of curves with variable end points and satisfying isoperimetric side conditions. The results which are obtained involve no explicit assumptions of normality. Such results can be derived by transforming our problem to a problem of Bolza and using the latest developments in the theory of that problem. More recently [**6**] an indirect method of proof has been published. Our object is to present a direct method of proof without transformation of the problem which is based upon a generalization of the classical theory of fields.

We treat first the case of no isoperimetric side conditions. The main theorem to be proved for this problem is Theorem 2.2, appearing at the end of §2. The proof is based upon a theory of fields which is an extension of the theory of fields for fixed end points, and was suggested by a similar treatment for a problem in non-parametric form [**5**].

The isoperimetric problem is formulated in §6, where the main result of the paper, Theorem 6.1, is stated. The proof of this theorem makes use of a family of broken extremals whose properties are described in §7. The results of §7 are extensions of those of Birkhoff and Hestenes [**1**]. The proof of Theorem 6.1 is completed first for the so-called strongly normal case; in §9 it is shown how the normality assumption may be lifted [cf. **2**].The final §10 is devoted to several corollaries of Theorem 6.1.

2. The non-isoperimetric problem. In the present section we formulate precisely the non-isoperimetric problem of the calculus of variations and we shall study first, and state, some standard definitions and properties.[1]

The function to be minimized will be taken to have the form

$$I(C) = g(a) + \int_{t_1}^{t_2} f(a, y, y')dt, \tag{2.1}$$

and is defined over a class of admissible parametric arcs C of the form

$$a_h, y_i(t) \qquad (h = 1,2,\dots,r;\ i = 1,2,\dots,n;\ t_1 \leqslant t \leqslant t_2) \tag{2.2}$$

in ay-space, satisfying a set of end conditions

$$y_i(t_s) = y_{is}(a) \qquad (s = 1,2). \tag{2.3}$$

Received August 29, 1950.

[1]Throughout this paper we shall use many of the standard results found in reference [**3**].

https://doi.org/10.4153/CJM-1952-024-3 Published online by Cambridge University Press

It is understood that the a_h are constants independent of t. We denote the derivative of $y_i(t)$ by $y'_i(t)$. The functions g, f, y_{is} are assumed to be of class C^4 in a region $\mathfrak{R}$ of points (a, y, y'), and f is taken to be positively homogeneous of degree one. This region is assumed to have the property that if the element (a, y, y') is in $\mathfrak{R}$ then[2] $y'_i y'_i \neq 0$ and (a, y, ky') is in $\mathfrak{R}$ for $k > 0$. An element in $\mathfrak{R}$ is called *admissible*. By an *admissible arc* is meant a continuous arc having a representation[3] (2.2) with the following properties:

(1) $t_1 t_2$ can be partitioned into a finite number of (closed) sub-intervals on each of which the functions $y_i(t)$ are of class C';

(2) each element (a, y, y') of the arc is admissible.

An admissible element is *non-singular* in case the matrix

$$||f_{y'_i y'_k}||$$

has rank $n - 1$ at that element (from the homogeneity of f, this is the maximum rank). Equivalently, an element is non-singular in case the determinant

$$\begin{vmatrix} f_{y'_i y'_k} & y'_i \\ y'_k & 0 \end{vmatrix}$$

does not vanish.

An *extremal* E is an admissible arc (2.2) of class C'' which satisfies the Euler equations

$$f_{y_i} - \frac{d}{dt} f_{y'_i} = 0. \tag{2.4}$$

Such an arc is a *non-singular extremal* in case each of its elements $(a, y\ (t), y'\ (t))$ is non-singular. It is well known [**3**, p. 108] that when a non-singular extremal is represented with arc length as parameter, then the functions (2.2) have the same class as f, namely C^4, in the present case.

From standard existence theorems on differential equation one obtains the following result [**3**].

THEOREM 2.1. *Every non-singular extremal E is a member of an $(r + 2n)$-parameter family of extremals*

$$a_h, \quad y_i(t,a,b,c) \qquad (h = 1,2, \ldots, r;\ i = 1,2, \ldots, n) \tag{2.5}$$

for special values $a_h = a_{h0}$, $b_i = b_{i0}$, $c_i = c_{i0}$, $t_1 \leqslant t \leqslant t_2$. The functions y_i, y'_i are defined and of class C'' in a neighbourhood of the values (t, a, b, c) belonging to E, and satisfy the equation

$$y'_i y''_i = 0. \tag{2.6}$$

The determinant

$$\begin{vmatrix} y_{ib_j} & y_{ic_j} \\ y'_{ib_j} & y'_{ic_j} \end{vmatrix} \tag{2.7}$$

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[2]Here and elsewhere a repeated index indicates summation over that index.

[3]For brevity we often refer to the representation (2.2) as the arc itself.

is different from zero along E. The parameters b_i, c_i may be taken to be the values of y_i, y'_i at a fixed value $t = t_0$ on the interval t_1t_2. Furthermore, there exists a neighbourhood $\mathfrak{F}$ of E in ay-space such that an extremal in $\mathfrak{F}$ with end values (a, y_1) and (a, y_2) sufficiently close to the end values of E is an extremal of the family (2.5).

Although the parameters b, c appearing above are not independent we prefer to leave them in the present form.

An extremal E satisfies the *transversality condition* if, along E, the equation

$$dg + [f_{y'_i}dy_{is}]_1^2 + \int_{t_1}^{t_2} f_{a_h}da_h dt = 0 \tag{2.8}$$

is an identity in da_h. The expression in brackets denotes

$$f_{y'_i}(a,y(t_2),y'(t_2))dy_{i2} - f_{y'_i}(a,y,(t_1),y'(t_1))dy_{i1},$$

where summation over the repeated index i is understood as remarked earlier.

The extremal E satisfies the *Weierstrass condition* II_N if, for every element (a, y, y') in a neighbourhood N of those on E,

$$E(a,y,y',Y') \geqslant 0 \tag{2.9}$$

for every $(Y') \neq (ky')$, $k > 0$, such that a, y, Y') is admissible. Here the function E is defined by

$$E = f(a,y,Y') - Y'_i f_{y'_i}(a,y,y').$$

Using the homogeneity of f, this function may be expressed in other forms. One consequence of the homogeneity is that $E = 0$ whenever $(Y') = (ky')$, $k > 0$.

The extremal E satisfies the *Clebsch condition* if, along E,

$$f_{y'_iy'_j}\sigma_i\sigma_j \geqslant 0$$

for all $(\sigma) \neq (ky')$, k arbitrary. The equality holds automatically for $(\sigma) = (ky')$, k arbitrary.

The second variation of I along an extremal E will be taken to be

$$I_2(\alpha,\eta) = b_{hk}\alpha_h\alpha_k + \int_{t_1}^{t_2} 2\omega(t,\alpha,\eta,\eta')dt \tag{2.10}$$

where

$$b_{hk} = g_{hk} + [f_{y'_i}y_{ishk}]_1^2, \tag{2.11a}$$

$$2\omega = f_{y_iy_j}\eta_i\eta_j + 2f_{y_iy'_j}\eta_i\eta'_j + f_{y'_iy'_j}\eta'_i\eta'_j + 2f_{y_ia_h}\eta_i\alpha_h + 2f_{y'_ia_k}\eta'_i\alpha_k + f_{a_ha_k}\alpha_h\alpha_k. \tag{2.11b}$$

In the derivatives of f, g, and y_{is} we understand that the arguments belong to E. Subscripts h, k on the latter two functions indicate differentiation with respect to a_h, a_k. The constants α_h and the functions $\eta_i(t)$ are required to satisfy continuity conditions like those for admissible arcs. The non-parametric arcs thereby defined in $\alpha t\eta$-space are called *admissible variations*. An admissible variation of the form $(0,\eta)$ with $\eta_i(t) = w(t)y'_i(t)$ is a *tangential variation*.

The second variation I_2 will be said to be *positive* along E if $I_2(a,\eta) > 0$ for every non-tangential variation which satisfies along E the end conditions

$$\eta_i(t_s) = y_{ish}a_h \qquad (s = 1,2). \tag{2.12}$$

From the homogeneity of f it may be proved that for a tangential variation satisfying (2.12), i.e., vanishing at the end points, the equality $I_2 = 0$ holds.

An *accessory extremal* is an admissible variation of class C'' for which

$$\omega_{\eta_i} - \frac{d}{dt}\omega_{\eta'_i} = 0. \tag{2.13}$$

The accessory extremal is *special* [**4**] in case $y'_i\eta'_i =$ constant, i.e.,

$$y'_i\eta''_i + y''_i\eta'_i = 0. \tag{2.14}$$

It may be shown that if $w(t)$ is an arbitrary function of class C' then the variation $(0, wy')$ is a solution of (2.13).

Let P_3P_4 be points on the extremal E defined by parametric values $t_3 \neq t_4$. We say that P_4 is *conjugate* to P_3 if there exists a special accessory extremal of the form $(0,\eta)$ which vanishes at t_3 and t_4 but is not identically zero between these values.

Two special accessory extremals $(0,\eta)$ and $0,u)$ are *conjugate* in case

$$\eta_i\omega_{\eta'_i}(0,u,u') = u_i\omega_{\eta'_i}(0,\eta,\eta'). \tag{2.15}$$

A set of special accessory extremals form a *conjugate system* in case every pair of the set is conjugate. Two such extremals are conjugate if and only if (2.15) holds at one point of the interval; this is a consequence of the well-known fact that the two members of the equation always differ by a constant for special accessory extremals.

Our first objective is to establish the following sufficiency theorem.

THEOREM 2.2 *If a non-singular extremal E which does not intersect itself satisfies the end conditions* (2.3), *the transversality condition* (2.8), *the Weierstrass condition* II_N, *and is such that the second variation of I along E is positive, then there exists a neighbourhood $\mathfrak{F}$ of E in ay-space such that $I(C) > I(E)$ for every admissible arc C in $\mathfrak{F}$ satisfying* (2.3) *and not identical with E.*

3. Mayer fields. We present in this section a theory of fields which is a generalization of the theory of fields for the fixed end point case. Our results here will assume that a given extremal E is already imbedded in a field; in the next section we shall show how this imbedding may be carried out.

By a *Mayer field* we shall mean a region $\mathfrak{F}$ in ay-space and a set of slope functions $p_i(a, y)$ $(i = 1, 2, \ldots, n)$ of class C'' on $\mathfrak{F}$ with the following properties. For every (a, y) in $\mathfrak{F}$ the element $(a, y, p\ (a, y))$ is admissible, and the Hilbert integral

$$I^*(C) = g(a) + \int_{t_1}^{t_2} y'_i f_{y'_i}(a,y,p)dt \tag{3.1}$$

is independent of the path in $\mathfrak{F}$ in the sense that I^* has the same value for any

https://doi.org/10.4153/CJM-1952-024-3 Published online by Cambridge University Press

two admissible arcs in $\mathfrak{F}$ with the same end points (a, y_1) and (a, y_2). Notice that if $\mathfrak{F}$ forms a field with $p_i(a, y)$ then this region forms a field with $k(a, y)p_i(a, y)$ for an arbitrary function $k(a, y) > 0$ of class C''.

For an admissible arc C in $\mathfrak{F}$,

$$I(C) - I^*(C) = \int_{t_1}^{t_2} E(a,y,p,y')dt. \tag{3.2}$$

An arc which has a representation (2.2) that satisfies $y'_i = p_i(a, y)$ will be called *an extremal of the field*. The following results are standard. An extremal of a field is an extremal in the sense of satisfying the Euler equations; through each element (a, y) there passes one and only one extremal of a field; for an extremal of a field, $I^*(E) = I(E)$.

THEOREM 3.1. *Let E be an extremal of a field $\mathfrak{F}$ which satisfies the end conditions* (2.3). *Suppose that for each (a, y) in $\mathfrak{F}$,*

$$E(a,y,p(a,y),y') > 0$$

whenever (a, y, y') is admissible and $(y') \neq (kp), k > 0$. Suppose also that $I^*(C) \geqslant I^*(E)$ for every admissible C in $\mathfrak{F}$ satisfying (2.3), *the equality holding if and only if C and E have the same components a_h. Then $I(C) > I(E)$ for every admissible C in $\mathfrak{F}$ satisfying* (2.3) *and not identical with E.*

For, from (3.2), $I(C) \geqslant I^*(C) \geqslant I^*(E) = I(E)$ for C as in the theorem. Suppose $I(C) = I(E)$. Then the right side of (3.2) is zero, and from the assumption on E it follows that $y'_i(t) = k(t)p_i(a, y)$ with $k(t) > 0$. Introducing the parameter

$$\tau = \int_{t_1}^{\tau} k(t)dt,$$

one readily verifies that C is an extremal of the field. From $I^*(C) = I^*(E)$ it follows by assumption that C and E have the same components (a) and hence the same end points. Since a unique extremal of a field passes through a point we conclude that C and E are identical.

The last theorem suggests the problem of minimizing I^*. Our next theorem deals with that problem. But first we compute the second variation $I_2^*(\alpha, \eta)$ of I^* along E. It is

$$I_2^*(\alpha,\eta) = b_{hk}\alpha_h\alpha_k + 2\int_{t_1}^{t_2} [\omega + (\eta'_i - \pi_i)\omega_{\eta'_i}]dt, \tag{3.3}$$

where

$$\pi_i(t,\alpha,\eta) = p_{ia_h}\alpha_h + p_{iy_j}\eta_j,$$

and the remaining symbols are defined by (2.11); the arguments in ω and its derivatives are (t, α, η, π).

THEOREM 3.2. *Let E be an extremal of a field which satisfies the transversality conditions* (2.8) *and end conditions* (2.3). *Suppose $I_2^*(\alpha, \eta) > 0$ for every admissible*

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variation with $(\alpha) \neq (0)$ *which satisfies the end conditions* (2.12). *Then there exists a neighbourhood* $\mathfrak{F}$ *of* E *in ay-space such that for every admissible* C *in* $\mathfrak{F}$ *satisfying* (2.3) *we have* $I^*(C) \geqslant I^*(E)$, *the equality holding if and only if* C *and* E *have the same components* a_h.

For the proof we may assume that $a_h = 0$ for E. Let $\alpha_{hk}, \eta_{ik}(t)$ $(k = 1, 2, \ldots, r)$ be r admissible variations of class C'' satisfying (2.12), $\alpha_{hk} = \delta_{hk}$ (Kronecker delta). Let

$$Y_i(t, a) = y_i(t) + \eta_{ik} a_k,$$

$$h_{is}(a) = \frac{y_{is}(a) - Y_i(t_s, a)}{t_2 - t_1} \qquad (s = 1,2),$$

where $y_i(t)$ belongs to E. Define an r-parameter family of admissible arcs:

$$a_h, y_i(t,a) = Y_i(t,a) + h_{i1}(a)(t_2 - t) + h_{i2}(a)(t - t_1). \tag{3.4}$$

This family satisfies (2.3) and contains E for $(a) = (0)$. Let $I^*(a)$ be the value of I^* along (3.4). By direct calculation and the Euler equations (2.4) we find that at $a = 0$,

$$dI^* = dg + [f_{y'_i} dy_{is}]_1^2 + \int_{t_1}^{t_2} f_{a_h} da_h dt,$$

$$d^2 I^* = I_2^*(\alpha, \eta),$$

where $\alpha_h = da_h$, $\eta_i = \eta_{ik} da_k$. Hence $dI^* = 0$, $d^2 I^* > 0$ for $(da) \neq (0)$, and $I^*(a)$ has a proper relative minimum at $(a) = (0)$. Therefore $I^*(a) > I^*(0) = I^*(E)$ for $(a) \neq (0)$ in a neighbourhood $\mathfrak{A}$ of (0). Take $\mathfrak{A}$ so small that the arcs (3.4) lie in the given field. Define $\mathfrak{F}$ to be all (a, y) of the field whose projections (a) lie in $\mathfrak{A}$. Consider any admissible C in $\mathfrak{F}$ which satisfies (2.3). The components (a) of C determine an arc of the family (3.4) with the same end points as C. From the invariance of I^*, $I^*(C) = I^*(a) \geqslant I^*(E)$, the equality holding in case C and E have the same (a).

The next theorem deals with a Mayer field for the second variation I_2. Since I_2 is non-parametric, to describe a field for this integral requires a slight modification of the definition of a field already given. For the second variation the slope functions π_i of the field are functions of (t, α, η) and the invariant integral has the form (3.3). With this in mind we state the next theorem; we omit the proof which is similar to that found in [**5**, p. 316].

THEOREM 3.3 *Let* E *be an extremal of a field which satisfies the end conditions* (2.3). *Then the set of points* (t, α, η) *with* $t_1 \leqslant t \leqslant t_2$ *and* (α, η) *arbitrary, and the slope functions*

$$\pi_i(t,\alpha,\eta) = p_{ia_h}\alpha_h + p_{iy_j}\eta_j$$

define an accessory Mayer field for the second variation I_2 *of* I *along* E *subject to the end conditions* (2.12). *The Hilbert integral for this accessory field is the integral* I_2^* *given by* (3.3).

4. Construction of a field. The main result of this section is Theorem 4.1 in which we construct a Mayer field containing a given extremal E. We first establish four lemmas. Throughout the section we *assume that E is represented with arc-length as parameter*. Thus $y'_i y'_i \equiv 1$ and y_i is of class C^4.

LEMMA 4.1. *Let E be an extremal such that $I_2 > 0$ for every admissible variation $(0, \eta)$ satisfying (2.12). Then E has on it no point conjugate to the initial point.*

Suppose P_3, corresponding to $t = t_3$, is conjugate to the initial point P_1. Then there exists a special accessory extremal $(0, \eta)$ vanishing at t_1 and t_3 but not identically on t_1t_3. Define $(\bar{\eta})$ as (η) on t_1t_3 and (0) on t_3t_2. Then

$$I_2(0,\eta) = \int_{t_1}^{t_2} [\bar{\eta}_i \omega_{\eta_i} + \bar{\eta}'_i \omega_{\eta'_i}]dt$$

$$= \int_{t_1}^{t_3}\left[\eta_i \frac{d}{dt}\omega_{\eta'_i} + \eta'_i \omega_{\eta'_i}\right]dt = [\eta_i \omega_{\eta'_i}]_{t_1}^{t_3} = 0.$$

Thus $\eta_i = wy'_i$. Multiplying both sides of the last equation by y'_i and summing we obtain $w = \eta_i y'_i$; thus w is of class C' on t_1t_3. Hence we may differentiate with respect to t in the preceding equation for (η). Doing this, multiplying by y'_i, and summing, we find $w' = k =$ constant, by (2.14). Thus $w = kt + l$. Hence, since w vanishes at t_1 and t_3, it vanishes identically on t_1t_3. The same then holds for (η), contrary to an earlier statement in the proof.

LEMMA 4.2. *Let E be a non-singular extremal which satisfies the end conditions (2.3). Suppose E has on it no point conjugate to the initial point. Then there exist functions $b_i(a), c_i(a)$ defined and of class C'' in a neighbourhood of the value $(a) = (a_0)$ belonging to E such that when b_i, c_i are replaced by $b_i(a), c_i(a)$ in (2.5) the resulting family of extremals*

$$a_h, \ y_i(t,a) = y_i(t, a, b(a), c(a)) \tag{4.1}$$

satisfies (2.3). Also $b_i(a_0), c_i(a_0)$ are the values (b_0, c_0) belonging to E.

To establish this result we show first that the determinant

$$\begin{vmatrix} y_{ib_j}(t_1) & y_{ic_j}(t_1) \\ y_{ib_j}(t_2) & y_{ic_j}(t_2) \end{vmatrix}$$

is different from zero (the arguments not displayed belong to E). Assume the contrary. Then for some constant $(r, s) \neq (0,0)$, the admissible variation

$$a_h = 0, \quad \eta_i = r_j y_{ib_j} + s_j y_{ic_j}$$

would vanish at t_1 and t_2. Furthermore this variation would be a special accessory extremal as one can verify by substituting (2.5) into (2.4) and (2.6) and differentiating. Since P_2 is not conjugate to P_1 we would have $\eta_i \equiv 0$, contrary to the fact that the determinant (2.7) is different from zero along E. Consider now the equations

$$y_{i1}(a) = y_i(t_1, a, b, c),$$

$$y_{i2}(a) = y_i(t_2, a, b, c).$$

They have initial solutions $(a, b, c) = (a_0, b_0, c_0)$, and the functional determinant with respect to (b, c) does not vanish there. By the implicit function theorem the solutions $b_i = b_i(a)$, $c_i = c_i(a)$ of these equations will then satisfy the conclusion of the theorem.

LEMMA 4.3. *Let E satisfy the hypotheses of Lemma 4.2. Then the admissible variations*

$$a_{hk} = \delta_{hk}, \quad \eta_{ik}(t) = y_{ia_k}(t,a_0) \qquad (k = 1,2,\ldots,r),$$

derived from (4.1), *form a set of r special accessory extremals which satisfy* (2.12).

For the proof we need only substitute the family (4.1) into equations (2.3), (2.4), and (2.6) and differentiate.

LEMMA 4.4. *Let E satisfy the hypotheses of Lemma 4.2. Then there exists a conjugate system*

$$a_{hp} = 0, \quad u_{ip}(t) \qquad (p = 1,2,\ldots,n-1)$$

of special accessory extremals which satisfy on t_1t_2 the conditions

$$|u_{ip}(t)\ y'_i(t)| \neq 0, \quad y'_i(t)u_{ip}(t) = 0. \tag{4.2}$$

For the proof let P_1 and P_2 be the left and right end points of E. By standard procedure we can show that there is a point P_3 on the leftward extension of E which has no conjugate point [**3**, p. 123] between t_1 and t_2. Choose constants e_{ip} such that $e_{ip}y'_i(t_3) = 0$ and $|e_{ip}, y'_i(t_3)| \neq 0$. We can choose $n-1$ special accessory extremals $(0, u_p)$ with initial values $u_{ip}(t_3) = 0$, $u'_{ip}(t_3) = e_{ip}$, and the zeros $t \neq t_3$ of the determinant in (4.2) will yield the points on E_{32} conjugate [**4**] to P_3. Thus the first relation of (4.2) holds. The second relation of (4.2) holds because the left side is constant, by the definition of special accessory extremal, and this constant is zero by the choice of e_{ip}.

We now proceed to the construction of a Mayer field containing E.

THEOREM 4.1. *Let E be a non-singular extremal which does not intersect itself and which satisfies the end conditions* (2.3). *Suppose E has on it no point conjugate to its initial point. Then there exists an $(r + n - 1)$-parameter family of extremals.*

$$a_h, y_i(t,a,e) \qquad (h = 1,2,\ldots,r;\ i = 1,2,\ldots,n) \tag{4.3}$$

which satisfies the equation $y'_iy''_i = 0$ and which contains the family (4.1) *for values $e_p = 0$ $(p = 1, 2, \ldots, n-1)$. The functions y_i, y'_i are defined and of class C'' in a neighbourhood of the values (t, a, e) belonging to E. Also, along E we have*

$$y_{ia_k} = \eta_{ih}, \quad y_{ie_p} = u_{ip} \tag{4.4}$$

for the variations of Lemmas 4.3 *and* 4.4. *The extremal E is an extremal of a Mayer field $\mathfrak{F}$ with slope functions*

$$p_i(a,y) = y'_i[t(a,y), a, e(a,y)], \tag{4.5}$$

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where $t(a, y), e(a, y)$ *are the unique solutions of class* C'' *of the equations*

$$y_i = y_i(t,a,e), \tag{4.6}$$

for (a, y) *in* $\mathfrak{F}$.

In the proof we shall employ the functions and notations of Theorem 2.1 and the previous lemmas. Let

$$B_i(a,e) = b_i(a) + u_{ip}(t_0)e_p,$$

$$v_{ip}(t) = \omega_{\eta'_i}(0,u_p) \equiv f_{y'_iy_j}u_{jp} + f_{y'_iy'_j}u'_{jp}.$$

Consider the equations

$$\begin{gathered} f_{y'_i}[a,B(a,c),C]u_{ip}(t_0) = f_{y'_i}[a,b(a),c(a)]u_{ip}(t_0) + v_{iq}(t_0)u_{ip}(t_0)e_q, \\ C_iC_i = c_i(a)c_i(a). \end{gathered} \tag{4.7}$$

They have initial solutions $(e, a, C) = (0, a, c(a))$ for (a) in a neighbourhood of (a_0). The functional determinant with respect to (C) at $(e, a, C) = (0, a_0, c_0)$ equals

$$2|A_{kj}| \qquad \begin{cases} A_{pj} = f_{y'_iy'_j}u_{ip}(t_0) & (p = 1,2,\ldots,n-1), \\ A_{nj} = y'_j(t_0) & (j,k = 1,2,\ldots,n). \end{cases} \tag{4.8}$$

Suppose this determinant were zero. Then there would exist constants m_p, m not all zero such that

$$m_pf_{y'_iy'_j}u_{ip} + m_0y'_j = 0.$$

Multiplying by y'_j and summing, $m_0 = 0$. From non-singularity, $m_pu_{ip} = wy'_i$ for some number w. From Lemma 4.4, $m_p = w = 0$; contradiction. By continuity the functional determinant is different from zero for $(e, a, C) = (0, a, c(a))$ with (a) in a neighbourhood of (a_0). We can then solve equations (4.7) for $C_i = C_i(a, e)$, where these functions are defined and of class C'' near $(a_0, 0)$ and have the value $c_i(a)$ for $(e) = (0)$.

The family extremals (4.3) will be shown to be

$$a_h, \quad y_i(t,a,e) = y_i[t, a, B(a,e), C(a,e)].$$

The first of equations (4.4) follows from Lemma 4.3 and the fact that for $(e) = (0)$ this family is (4.1). To prove the second of equations (4.4), we note first by Theorem 2.1 that

$$\begin{gathered} y_i(t_0,a,e) = B_i(a,e) = b(a) + u_{ip}(t_0)e_p, \\ y'_i(t_0,a,e) = C_i(a,e). \end{gathered} \tag{4.9}$$

Setting $C_i = C_i(a, e)$ in equations (4.7), differentiating with respect to e_q, and setting $(a, e) = (a_0, 0)$ we obtain

$$(f_{y'_iy_j}B_{je_q} + f_{y'_iy'_j}C_{je_q})u_{ip} = v_{iq}u_{ip}, \qquad C_jC_{je_q} = 0.$$

With the help of (4.9) and the definition of v_{iq} these equations become

$$f_{y'_iy'_j}u_{ip} \quad (C_{je_p} - u'_{jq}) = 0, y'_jC_{je_q} = 0.$$

The last equation and (4.2) imply that

$$y'_i(C_{je_q} - u'_{jq}) = 0.$$

But the determinant (4.8) does not vanish. Hence

$$C_{je_q} - u'_{jq} = 0.$$

The last equation and differentiation of (4.9) yield

$$y_{ie_p}(t_0,a_0,0) = u_{ip}(t_0),$$

$$y'_{ie_p}(t_0,a_0,0) = C_{ie_p}(a_0,0) = u'_{ip}(t_0).$$

By substituting (4.3) into (2.4) and (2.6) and differentiating with respect to e_p we find that y_{ie_p} $(t, a_0, 0)$ is a special accessory extremal with $(\alpha) = (0)$. But such an accessory extremal is uniquely determined by the values (η, η') at a single point. This establishes the second of equations (4.4).

Next consider the equations (4.6). They have solutions (y, t, a, e) for the values belonging to E. Furthermore, no two such distinct points have the same projection (y, a), and the functional determinant along E with respect to (t, e) is the determinant (4.2). Hence solutions $t = t(a, y)$, $e_p = e_p(a, y)$ exist, defined and of class C'' in a neighbourhood $\mathfrak{F}$ of the values (a, y) belonging to E. Therefore the slope functions (4.5) are well defined and E is an extremal of the field $\mathfrak{F}$. It remains to show the invariance of the Hilbert integral. On the hypersurface $a_h =$ constant, $y_i = y_i(t_0, a, e)$ this integral becomes

$$\begin{aligned}
&\int f_{y'_i}[a, (t_0,a,e), y'(t_0,a,e)]y_{ie_p}(t_0,a,e)de_p \\
&\qquad = \int f_{y'_i}[a, B(a,e), C(a,e)]u_{ip}(t_0)de_p \\
&\qquad = \int \{f_{y'_i}[a, b(a), c(a)]u_{ip}(t_0) + v_{iq}(t_0)u_{ip}(t_0)e_q\}de_p \\
&\qquad = \int d[f_{y'}u_{ip}e_p + \tfrac{1}{2}v_{iq}u_{ip}e_qe_p],
\end{aligned}$$

by use of (4.7) and (2.15). From the invariance on the hypersurface follows [**3**, p. 126] the invariance in $\mathfrak{F}$.

THEOREM 4.2. *Let E be a non-singular extremal which does not intersect itself and which satisfies the end conditions* (2.3). *Suppose that the second variation of I along E is positive. Let*

$$\pi_i(t,\alpha,\eta) = \eta'_{ik}\alpha_k + u'_{ip}\epsilon_p + y''_i\tau, \tag{4.10}$$

where $\epsilon_p = \epsilon_p(t,\alpha,\eta)$, $\tau = \tau(t,\alpha,\eta)$ are the unique solutions of the equations

$$\eta_i = \eta_{ik}\alpha_k + u_{ip}\epsilon_p + y'_i\tau. \tag{4.11}$$

Then the functions π_i are the slope functions of the accessory Mayer field associated with the field $\mathfrak{F}$ described in the preceding theorem. Furthermore, the invariant integral I_2^* of the accessory field satisfies the condition $I_2^*(\alpha, \eta) > 0$ for every admissible variation with $(\alpha) \neq (0)$ satisfying (2.12).

Using Lemma 4.1 and Theorems 4.1 and 3.3 we obtain (4.10) as the slope functions of the accessory field, where

$$\epsilon_p = e_{pa_k}\alpha_k + e_{py_j}\eta_j, \qquad \tau = t_{a_k}\alpha_k + t_{y_j}\eta_j.$$

In the identity $y_i = y_i[t(a, y), a, e(a, y)]$ replace (a, y) by $(a, y) + b(\alpha,\eta)$, differentiate with respect to b, and set $b = 0$. The result is (4.11). To prove the last part of the theorem let (α, η) be any admissible variation as described in the theorem. Then $\alpha, \bar{\eta} = a_k\eta_k$ has the same end points; from equations (4.11) and (4.10) this variation is an extremal of the accessory field. Therefore

$$I_2^*(\alpha,\eta) = I_2^*(\alpha,\bar{\eta}) = I_2(\alpha,\bar{\eta}) > 0.$$

5. Proof of Theorem 2.2. We are now in a position to prove Theorem 2.2. Imbed E in the field described in Theorem 4.1. From Theorem 4.2 the hypotheses of Theorem 3.2 are satisfied. Hence there exists a neighbourhood $\mathfrak{F}$ of E with the properties described there. Pick a neighbourhood N of non-singular elements (a, y, y') for which the Weierstrass condition holds. By a theorem of Hestenes and Reid [7] the strict inequality holds in the Weierstrass condition. Decrease the field if necessary so that the elements $a, y, p(a, y)$ all lie in N. Then the hypotheses of Theorem 3.1 are satisfied and our proof is complete.

For use in the study of the isoperimetric problem we shall need the following two theorems.

THEOREM 5.1. *Let E be a non-singular extremal which does not intersect itself and satisfies the end conditions* (2.3). *Suppose that E satisfies the Weierstrass condition II_N and has on it no point conjugate to its initial point. Then there exists a field $\mathfrak{F}$ containing E as an extremal of the field such that the E-function is positive in $\mathfrak{F}$. Furthermore, if (a) is sufficiently close to the value (a_0) belonging to E then there exists a unique extremal in $\mathfrak{F}$ with components a_h which satisfies* (2.3). *This extremal is an extremal of the field.*

As in the preceding proof we select a neighbourhood in which the strengthened Weierstrass condition holds. From Theorem 4.1 we obtain a field $\mathfrak{F}$ containing E with all its elements $[a, y, p(a, y)]$ in N. This proves the first part of the theorem. To show the existence of an extremal for every value (a) we exhibit $(a, y\ (t, a,0))$ of (4.3). Furthermore, this extremal is an extremal of the field described in Theorem 4.1. The uniqueness follows from the last statement of Theorem 2.1 and the proof of Lemma 4.2.

The proof of the next theorem is like that of Theorem 3.1.

THEOREM 5.2. *Let E be an extremal of a Mayer field $\mathfrak{F}$ at each point of which $E[a, y, p(a, y), y'] > 0$ for $(y') \neq (kp)$, $k > 0$, with (a, y, y') admissible. Then*

$I(C) \geqslant I(E)$ for every admissible arc C in $\mathfrak{F}$ with the same end values (a, y_s) as E, the equality holding only in case C is identical with E.

6. The isoperimetric problem. We turn now to the problem of minimizing the function I of (2.1) in the class of admissible arcs (2.2) which satisfy conditions

$$y_i(t_s) = y_{is}(a) \qquad (s = 1,2), \tag{6.1a}$$

$$I_p(C) = g_p(a) + \int_{t_1}^{t_2} f_p(a,y,y')dt = 0 \qquad (p = 1,2,\ldots,m). \tag{6.1b}$$

This problem differs from the earlier one in the adjunction of the isoperimetric side conditions (6.1b). We shall assume that the functions f, f_p are positively homogeneous of degree one in the variables y'_i and that the functions appearing above have the same continuity properties that were assumed earlier.

Associated with this problem is an integral,

$$J(C,l) = G(a,l) + \int_{t_1}^{t_2} F(a,l,y,y')dt, \tag{6.2}$$

where $G(a, l) = g(a) + l_p g_p(a)$, $F(a, l, y, y') = f(a, y, y') + l_p f_p(a, y, y')$ and the l's are constants. We shall employ the function (6.2) to relate the theory of the previous pages to the present case.

By an *isoperimetric extremal* E will be meant an admissible arc (2.2) of class C'' together with a set of constant multipliers l_p $(p = 1, 2, \ldots, m)$ which satisfies the Euler-Lagrange equations

$$F_{y_i} - \frac{d}{dt} F_{y'_i} = 0. \tag{6.3}$$

The conditions of non-singularity and transversality, and the Weierstrass and Clebsch conditions are, with a minor exception, identical with those of § 2 provided that in the earlier definitions the functions f, g are replaced by F, G. The exception concerns the Weierstrass condition II_N. This should be slightly reworded as follows: for every element (a, l, y, y') in a neighbourhood N of those belonging to the arc E, $E(a, l, y, y', Y') \geqslant 0$ for $(Y') \neq (ky')$, $k > 0$, with (a, y, Y') admissible.

The second variation I_2 of I along E is given by (2.10) where the left sides of equations (2.11) are understood now to be defined by the right sides with f, g replaced by F, G. We retain the same notations for the left sides. We shall say that I_2 is *positive* in case $I_2(\alpha, \eta) > 0$ for every non-tangential admissible variation which satisfies along E the conditions

$$\eta_i(t_s) = y_{ish}\alpha_h \qquad (s = 1,2), \tag{6.4a}$$

$$I_{p1}(\alpha,\eta) = g_{ph}\alpha_h + \int_{t_1}^{t_2} \omega_p(t,\alpha,\eta,\eta')dt \qquad (p = 1,2,\ldots,m), \tag{6.4b}$$

where

$$\omega_p = f_{pa_h}\alpha_h + f_{py_i}\eta_i + f_{py'_i}\eta'_i.$$

By an *isoperimetric accessory extremal* will be meant an admissible variation (α, η) of class C'' together with constant multipliers λ_p which satisfies the accessory differential equations

$$\Omega_{\eta_i} - \frac{d}{dt}\Omega_{\eta'_i} = 0, \tag{6.5}$$

where $\Omega(t, \alpha, \lambda, \eta, \eta') = \omega + \lambda_p\omega_p$. An isoperimetric accessory extremal which satisfies (2.14) will be called *special.*

Consider an isoperimetric extremal E with multipliers l_p. We shall say that t_4 defines a point P_4 *conjugate* to a point P_3 on E relative to the function $J(C, l)$ of (6.2) in case there exists a special isoperimetric accessory extremal with $(\alpha) = 0, (\lambda) = 0$ which vanishes at t_3 and t_4 but is not identically zero on t_3t_4.

Our main objective is the sufficiency theorem below. Notice that there are no normality restrictions in this result. Our method of proof will consist of proving the theorem under the assumption of "strong normality" and then showing how this restriction may be dropped.

THEOREM 6.1. *If a non-singular isoperimetric extremal E_0 which does not intersect itself satisfies the conditions* (6.1), *the transversality condition, the Weierstrass condition* II_N, *and is such that the second variation of I along E_0 is positive, then there exists a neighbourhood $\mathfrak{F}$ of E in ay-space such that $I(C) > I(E_0)$ for every admissible arc C in $\mathfrak{F}$ satisfying* (6.1) *but not identical with E_0.*

7. A generalization of the Hahn lemma. We first establish the usual type of Hahn lemma.

THEOREM 7.1. *Let E_0: $(a_0, y_0(t))$ be a non-singular isoperimetric extremal with multipliers (l_0) which does not intersect itself. Suppose E_0 satisfies the Weierstrass condition II_N and E_0 has on it no point conjugate to its initial point relative to $J(C, l_0)$. Then there exist neighbourhoods $\mathfrak{F}$ of E_0 in ay-space and $\mathfrak{L}$ of the multipliers (l_0) such that for every pair of points in $\mathfrak{F}$ sufficiently close to the initial and terminal points respectively of E_0, and every set of multipliers in $\mathfrak{L}$ there is a unique isoperimetric extremal E_l in $\mathfrak{F}$ with these end points and multipliers. Furthermore, $J(C, l) > J(E_l, l)$ for every admissible arc C in $\mathfrak{F}$ joining the end points of E_l but not identical with it. Also, for every sub-arc $\bar{E}_l$ of E_l we have $J(C, l) > J(\bar{E}_l, l)$ for every admissible arc C in $\mathfrak{F}$ joining the end points of $\bar{E}_l$ but not identical with it.*

To make the proof consider the problem of minimizing the function (6.2) in the class of admissible arcs

$$a_h, l_p, y_{i1}, y_{i2}, y_i(t)$$

which satisfy end conditions of the form

$$y_i(t_s) = y_{is}.$$

This is a non-isoperimetric problem with the constants a_h replaced by a_h, l_p, y_{is}. For this problem the arc defined by the values belonging to E_0 satisfies the hypotheses of Theorem 5.1. The conclusions of Theorems 5.1 and 5.2 appropriately interpreted yield the desired result.

Consider now an arc E_0 satisfying the hypotheses of Theorem 7.1 with the exception of the assumption on conjugate points. We proceed to make the following geometric construction for such an arc. It is a standard result in the calculus of variations that there exists a positive constant d so small that no sub-arc of E_0 of length not exceeding d has on it a pair of conjugate points relative to $J(C, l_0)$. Let $Q_0, Q_1, \ldots, Q_{q+1}$ be successive points on E_0 such that the arcs Q_jQ_{j+3} do not exceed d in length. We may suppose that the points Q_0 and Q_{q+1} lie respectively on the leftward and rightward extensions of the arc E_0, and that the initial point P_1 of E_0 lies between Q_0 and Q_1 while the terminal point P_2 lies between Q_q and Q_{q+1}. Through the points Q_j pass hyperplanes π^j cutting E_0 orthogonally. By Theorem 7.1 we can select a neighbourhood $\mathfrak{F}''$ of E_0 and $\mathfrak{L}'$ of (l_0) such that, for every pair of points R_j, R_{j+3} with the same components (a) on π^j, π^{j+3} sufficiently close to Q_j, Q_{j+3} respectively and every (l) in $\mathfrak{L}'$, there is a unique isoperimetric extremal E in $\mathfrak{F}''$ with multipliers (l) and end points R_j, R_{j+3} which affords $J(C, l)$ a proper minimum relative to admissible arcs C in $\mathfrak{F}''$ joining the points R_j, R_{j+3} and not crossing the manifolds π^j, π^{j+3}. Let $\mathfrak{F}'$ be a neighbourhood of E_0 contained in $\mathfrak{F}''$, and $\mathfrak{L}$ a neighbourhood of (l_0) contained in $\mathfrak{L}'$ such that every pair of points R_j, R_{j+1} in $\mathfrak{F}'$ with the same components (a) and lying on π^j, π^{j+1} respectively determines together with a set (l) in $\mathfrak{L}$ an isoperimetric extremal E in $\mathfrak{F}''$ with end points on π^{j-1}, π^{j+2} and multipliers (l) such that E has the following property: the arc E intersects each of π^j, π^{j+1} exactly once, at the points R_j, R_{j+1}. Thus the segment E_j of E between these points does not cross π^j, π^{j+1}. Also, by Theorem 7.1, E_j will afford $J(C, l)$ a proper minimum relative to admissible arcs C in $\mathfrak{F}''$ joining the points R_j, R_{j+1} and not crossing π^{j-1}, π^{j+2}. Let π_0 be the end manifold in ay-space determined by $a_h = a_h$, $y_i = y_{is}(a)$ for $s = 1$ and π_{q+1} the manifold for $s = 2$. Let π_j denote π^j $(j = 1, 2, \ldots, q)$. Then we may require that the neighbourhoods $\mathfrak{F}'$ and $\mathfrak{L}$ also satisfy the following condition: for every pair of points R_0, R_1 in $\mathfrak{F}'$ with the same components (a) and lying on π_0, π_1 respectively and every (l) in $\mathfrak{L}$ there is an isoperimetric extremal in $\mathfrak{F}''$ with multipliers (l) and end points R_0, R_1 which does not cross π_1 and affords $J(C, l)$ a proper minimum relative to admissible arcs C in $\mathfrak{F}''$ joining R_0, R_1 but not crossing π_1. A similar result holds for points R_q, R_{q+1} in $\mathfrak{F}'$ on π_q, π_{q+1}. Finally, we may restrict $\mathfrak{F}'$ to include no points of $\mathfrak{F}''$ to the left of π^0 or to the right of π^{q+1}. We are now in a position to state the following important result.

THEOREM 7.2. *Let E_0 be a non-singular isoperimetric extremal which does not intersect itself and satisfies the Weierstrass condition II_N. Let the neighbourhoods $\mathfrak{F}''$, $\mathfrak{F}'$, $\mathfrak{L}$ and the manifolds π_j be defined as in the above paragraph. Then every pair of points R_j, R_{j+1} in $\mathfrak{F}'$ with the same components a_h and lying on the manifolds π_j, π_{j+1} respectively determines together with a set of multipliers l_p in $\mathfrak{L}$ a unique isoperimetric extremal E_j in $\mathfrak{F}''$ with these end points and multipliers such* *that $J(C, l) > J(E_j, l)$ for every admissible arc C in $\mathfrak{F}'$ joining the end points of E_j and not identical with E_j.*

It remains only to prove the asserted inequality. Let C be an admissible arc in $\mathfrak{F}'$ joining the end points of E_j. If C does not cross the hyperplanes π^j, π^{j+1} then, by our earlier discussion, $J(C, l) > J(E_j, l)$ unless $C \equiv E_j$. Suppose C crosses π^j. Let π^{k-1} be the first hyperplane on the left which C does not cross. As the point P moves along C from R_j to R_{j+1} it will intersect π^k at a first point R_k and will subsequently reach π^{k+1} at a first point R_{k+1}. Let C' be the segment of C between R_k and R_{k+1} and let E be the isoperimetric extremal between π^k and π^{k+1} determined by R_k, R_{k+1} and the multipliers l_p. Then $J(C', l) > J(E, l)$, the strict inequality holding because the arc C' is not identical with E (since C' actually crosses π^k). We replace the sub-arc C' of C by E to obtain a new arc C_1 joining the end points of C for which $J(C, l) > J(C_1, l)$. The arc C_1 may be in $\mathfrak{F}''$ but it crosses the hyperplanes at points in $\mathfrak{F}'$. If the new arc C_1 still crosses the hyperplane π^k we apply our lopping-off process to it. From the finite length of the arc C in a finite number of steps we can replace the original arc C by an arc C_k which does not cross π^k and for which $J(C, l) > J(C_k, l)$. In a similar fashion we obtain an arc C_{k+1} which does not cross π^{k+1} and, finally, an arc C_2 which does not cross π^j and satisfies the inequality $J(C, l) > J(C_2, l)$. Proceeding in an analogous manner to the right of the hyperplane π^{j+1} we eventually obtain an admissable arc C_3 in $\mathfrak{F}''$ which joins the end points of E_j and does not cross π^j, π^{j+1}. Hence

$$J(C,l) > J(C_3,l) \geqslant J(E_j,l)$$

and the proof is complete.

By an argument like that above we can establish the following extension of Theorem 7.1.

THEOREM 7.3. *Let E_0 satisfy the hypotheses of Theorem 7.2 with multipliers (l_0). Suppose $\bar{E}_0$ is a sub-arc of E_0 which has on it no pairs of conjugate points. Then there exists a neighbourhood $\mathfrak{F}$ of E_0 in ay-space and a neighbourhood $\mathfrak{L}$ of (l_0) such that every pair of points (a, y_1) and (a, y_2) sufficiently close to the initial and terminal points respectively of $\bar{E}_0$ and every set l_p in $\mathfrak{L}$ determine a unique isoperimetric extremal with these end points and multipliers which affords the function $J(C, l)$ a proper minimum relative to admissible arcs joining its end points and lying in $\mathfrak{F}$.*

8. Proof of Theorem 6.1 in the strongly normal case. We introduce at this point the notion of normality. We shall say that an isoperimetric extremal E is *normal* relative to the isoperimetric conditions (6.1b) if there do not exist constants c_p not all zero such that the following equations hold along E.

$$c_p\left(f_{py_i} - \frac{d}{dt}f_{py'_i}\right) = 0, \tag{8.1}$$

$$c_p\left(dg_p + [f_{py'_i}dy_{is}]_1^2 + \int_{t_1}^{t_2} f_{a_h}da_h dt\right) = 0.$$

https://doi.org/10.4153/CJM-1952-024-3 Published online by Cambridge University Press

In other words, E is normal in case E is not an extremal satisfying the transversality condition for an integral of the form

$$c_p I_p = c_p g_p(a) + \int_{t_1}^{t_2} c_p f_p dt. \tag{8.2}$$

We shall say that E is *strongly normal* relative to the isoperimetric conditions (6.1b) if there do not exist constants c_p not all zero such that the first of equations (8.1) holds or, equivalently, the arc E is not an extremal for a function (8.2). Obviously strong normality implies normality. Equations (8.1) are equivalent to the condition that the first variation

$$c_p I_{p1}(\alpha,\eta) \tag{8.3}$$

of (8.2) along E vanishes for all admissible variations satisfying (6.4a). Similarly, the first equation (8.1) is equivalent to the vanishing of (8.3) for all such variations with $(\alpha) = (0)$.

In the proof of Theorem 6.1 we shall make use of the following result.

THEOREM 8.1. *Let E_0 be an arc, satisfying the hypotheses of Theorem* 6.1, *which is strongly normal with respect to the isoperimetric conditions* (6.1b). *Let the neighbourhoods $\mathfrak{F}'$ and $\mathfrak{L}$ and the manifolds π_j be defined as in Theorem* 7.2. *Then there exists a neighbourhood $\mathfrak{F}$ of E_0 contained in $\mathfrak{F}'$ such that for every succession of points $R_0, R_1, \ldots, R_q, R_{q+1}$ in $\mathfrak{F}$ with the same components a_h and lying on successive manifolds $\pi_0, \pi_1, \ldots, \pi_q, \pi_{q+1}$ there is a set of multipliers*

$$l_p = l_p(R_0, \ldots, R_{q+1})$$

in $\mathfrak{L}$ such that the broken isoperimetric extremal E determined by the points R_j and the multipliers l_p by means of Theorem 7.2 *satisfies conditions* (6.1) *and the inequality $I(E) \geqslant I(E_0)$, the equality holding only in case E is identical with E_0.*

If one accepts for the moment the truth of this theorem then the proof of Theorem 6.1 under the assumption of strong normality may be made as follows. Let $\mathfrak{F}$ be the neighbourhood given in Theorem 8.1. Consider any admissible arc C in $\mathfrak{F}$ satisfying the conditions (6.1). Let R_0, R_{q+1} be the initial and terminal points of C and let R_j $(j = 1, 2, \ldots, q)$ be the last point at which the point P crosses the hyperplane π_j as P moves along C from its initial to its terminal point. Let E be the broken isoperimetric extremal of Theorem 6.1 determined by the points $R_0, \ldots, R_{q+1}$ and the multipliers $l_p = l_p\ (R_0, \ldots, R_{q+1})$. Denote by C_j the segment of C between R_j and R_{j+1}, and by E_j the segments of E between R_j and R_{j+1}. Then by Theorems 7.2 and 8.1, and equations (6.1b) we obtain

$$\begin{aligned} 0 \leqslant \sum_j [J(C_j,l) - J(E_j,l)] &= J(C,l) - J(E,l) \\ &= I(C) - I(E) \leqslant I(C) - I(E_0), \end{aligned}$$

the equality holding only in case C is identical with E_0.

Let us turn now to the proof of Theorem 8.1. Let

$$t_1 = s_0 < s_1 < \ldots < s_q < s_{q+1} = t_2$$

be the parametric values which determine the points of intersection of E_0 with the manifolds $\pi_0, \pi_1, \ldots, \pi_{q+1}$. Let the equations of the hyperplanes π_j $(j = 1, 2, \ldots, q)$ be

(8.4) $$a_h = a_h, \quad y_i = b_{ij}(e_{1,j}, e_{2,j}, \ldots, e_{n-1,j}) \quad (h = 1,2,\ldots,r;\ i = 1,2,\ldots,n)$$

(j not summed). Along E_0

(8.5) $$y_{i0}(s_0) = y_{i1}(a_0), \quad y_{ip}(s_j) = b_{ij}(e_{j0}), \quad y_{i0}(s_{q+1}) = y_{12}(a_0).$$

Also, we have the orthogonality conditions

(8.6) $$y'_{i0}(s_j)\frac{\partial b_{ij}}{\partial e_{kj}} = 0 \qquad (k = 1,2,\ldots,n-1;\quad j \text{ not summed}).$$

Moreover, for each j, the matrix

$$\left\| \frac{\partial b_{ij}}{\partial e_{kj}} \right\|$$

has rank $n - 1$ at $(e_j) = (e_{j0})$. By means of Theorem 7.1 and its proof we can obtain the existence of an $(h + nq + m)$-parameter family of broken isoperimetric extremals

(8.7) $$a_h, \quad y_i = Y_i(t, a_1, \ldots, a_h, b_{11}, \ldots, b_{nq}, l_1, \ldots, l_m)$$

with multipliers (l) which satisfies the following conditions. It contains E_0 for values (a_0, b_0, l_0), and $t_1 \leqslant t \leqslant t_2$. Except possibly at the corner points $t = s_j$ $(j = 1, 2, \ldots, q)$ the functions Y_i, Y_{it} are of class C'' in a neighbourhood of the values (t, a, b, l) belonging to E_0. For fixed values (a, b, l) the corresponding arc of the family satisfies the end conditions (6.1a) and passes through the point (a, b_j), for $t = s_j$ $(j = 1, 2, \ldots, q)$. Finally, the identity

(8.8) $$Y_{it}Y_{itt} = 0$$

holds. Replacing the arguments in (8.7) by the functions of (8.4) we obtain a family

(8.9) $$a_h, \quad y_i = y_i(t,a,e,l)$$

of broken isoperimetric extremals with multipliers (l). This is the family determined by a sequence of points $R_0, \ldots, R_{q+1}$ on the manifolds $\pi_0, \ldots, \pi_{p+1}$. Substitute (8.9) into (6.1b) to obtain

(8.10) $$I_p(a,e,l) = 0.$$

Equations (8.10) have an initial solution (a_0, e_0, l_0) and the functions on the left are of class C''. Assume for the moment that the functional determinant

(8.11) $$\left\| \frac{\partial I_p}{\partial l_k} \right\| \qquad (p,k = 1,2,\ldots,m)$$

is different from zero at the initial solution. Then (8.10) has unique solutions $l_p = l_p(a, e)$ of class C'' near (a, e_0) with $l_p(a_0, e_0) = l_{p0}$. Let (a, e) define the sequence of points $R_0, \ldots, R_{q+1}$. We shall show that the functions $l_p(R)$ of the theorem are the functions $l_p(a, e)$.

The family

$$(8.12) \qquad a_h, \; y_i(t, a, e) = y_i[t, q, e, l(a,e)]$$

with the indicated multipliers satisfy (6.1). It remains only to show that E_0 affords I a proper minimum relative to arcs of (8.12). Consider the variation $(da, \delta y(t))$ where

$$\delta y_i(t) = y_{ia_h}(t,a_0,e_0)da_h + \frac{\partial y_i(t,a_0,e_0)}{\partial e_{kj}} de_{kj}.$$

From the equations

$$y_i(s_0,a,e) = y_{i1}(a), \quad y_i(s_j,a,e) = b_{ij}(e_{kj}), \quad y_i(s_{q+1},a,e,) = y_{i2}(a)$$

(j not summed) we obtain

$$(8.13) \qquad \delta y_i(s_0) = y_{i1h}da_h, \quad \delta y_i(s_j) = \frac{\partial b_{ij}}{\partial e_{kj}} de_{kj}, \quad \delta y_i(s_{q+1}) = y_{i2h}da_h.$$

These relations together with differentiation of the identity

$$I_p(a, e) = I_p[a, e, l(a,e)] = 0$$

yield that $(da, \delta y)$ satisfies (6.4) and has $(\delta y) \equiv 0$ only if $de_{kj} = 0$ for all k, j. Suppose this variation is of the tangential form $(0, wy'_0)$. Then, by an argument like that used in the proof of Lemma 4.1, $w(t)$ is linear on each sub-interval $s_j s_{j+1}$. But multiplying $\delta y_i(s_j) = w(s_j)y'_{i0}(s_j)$ by $y'_{i0}(s_j)$, summing, and employing (8.6) and (8.13) we find

$$w(s_j) = 0 \qquad (j = 0,1, \ldots ,q + 1).$$

Hence $w(t) \equiv 0$, $(\delta y) \equiv 0$, and finally $(de) = 0$. Therefore for $(da, de) \neq (0,0)$ the variation $(da, \delta y)$ is not tangential and hence $I_2(da, \delta y) > 0$. Consider now the function $J(a, e, l_0)$ obtained by evaluating the integral $J(C, ,l_0)$ along (8.12). By computation we find that along E_0 the relations

$$dJ = dG + [F_{y'_i}dy_{is}]_1^2 + \int_{t_1}^{t_2} F_{a_h}da_h dt = 0,$$

$$d^2J = I_2(da,\delta y) > 0,$$

for all $(da, de) \neq (0,0)$. Hence for (a, e) near, but distinct from, (a_0, e_0)

$$0 < J(a,e,l) - J(a_0,e_0,l_0) = I(a,e) - I(E_0)$$

since $I_p(a, e) = 0$.

It remains to establish the non-vanishing of (8.11). Let

$$\bar{\delta} y(t) = y_{l_k}(t,a_0,e_0,l_0)dl_k.$$

By differentiating the functions $I_p(a, e, l)$ we find

$$\frac{\partial I_p}{\partial l_k} dl_k = I_{p1}(0,\bar{\delta}y). \tag{8.14}$$

Differentiation of the equations

$$y_i(s_0,a,e,l) = y_{i1}(a), \quad y_i(e_j,a,e,l) = b_{ij}(e_j), \quad y_i(s_{q+1},a,e,l) = y_{i2}(a)$$

yields $(\bar{\delta}y(s_j)) = (0)$ $(j = 0, 1, \ldots, q+1)$. Substituting (8.9) into the Euler-Lagrange equations (6.3), differentiating with respect to l_k, multiplying by dl_k and summing we see that

$$\omega_{\eta_i}(0,\bar{\delta}y) - \frac{d}{dt}\omega_{\eta'_i}(0,\bar{\delta}y) + dl_k\left(f_{ky_i} - \frac{d}{dt}f_{ky'_i}\right) = 0. \tag{8.15}$$

Suppose now that (8.11) is zero. Then there exist constants $(dl) \neq 0$ such that the left member of (8.14) vanishes. Hence $(0, \bar{\delta}y)$ satisfies (6.4). This variation is not tangential; otherwise it would satisfy (2.13), which is impossible by (8.15) and our assumption of strong normality. Hence $I_2(0, \bar{\delta}y) > 0$. But by (8.15)

$$\begin{aligned} 0 &= \int_{t_1}^{t_2}\left[\bar{\delta}y_i\left\{\omega_{\eta_i}(0,\bar{\delta}y) - \frac{d}{dt}\omega_{\eta'_i}(0,\bar{\delta}y)\right\} + \bar{\delta}y_i\left\{dl_k\left(f_{ky_i} - \frac{d}{dt}f_{ky'_i}\right)\right\}\right]dt \\ &= \int_{t_1}^{t_2}\left[(\bar{\delta}y_i\omega_{\eta_i} + \bar{\delta}y'_i\omega_{\eta'_i}) + dl_k(f_{ky_i}\bar{\delta}y_i + f_{ky'_i}\bar{\delta}y'_i)\right. \\ &\qquad\qquad \left. - \frac{d}{dt}(\omega_{\eta'_i}\bar{\delta}y_i + dl_k f_{ky'_i}\bar{\delta}y_i)\right]dt \\ &= I_2(0,\bar{\delta}y) + dl_k I_{k1}(0,\bar{\delta}y) - \sum_j \int_{s_j}^{s_{j+1}} \frac{d}{dt}(\omega_{\eta'_i}\bar{\delta}y_i + dl_k f_{ky'_i}\bar{\delta}y_i)dt \\ &= I_2(0,\bar{\delta}y). \end{aligned}$$

This contradiction completes the proof.

9. The general case. We shall establish the following result.

THEOREM 9.1. *Let E_0 be an arc satisfying the hypotheses of Theorem* 6.1 *for the function* (2.1) *with the side conditions* (6.1). *Then there exists a function*

$$\bar{I}(C) = \bar{g}(a) + \int_{t_1}^{t_2}\bar{f}(a,y,y')dt \tag{9.1}$$

and a set of isoperimetric conditions

$$\bar{I}_\tau(C) = \bar{g}_\tau(a) + \int_{t_1}^{t_2}\bar{f}_\tau(a,y,y')dt \qquad (\tau = 1,2,\ldots,m_1 \leqslant m) \tag{9.2}$$

such that E_0 is strongly normal relative to the conditions (9.2) *and satisfies the hypotheses of Theorem* 6.1 *for the function* (9.1) *with end conditions* (6.1a) *and isoperimetric conditions* (9.2). *Furthermore, if $I(E_0)$ is a proper strong relative minimum for the modified problem, $I(E_0)$ is a similar minimum for the original problem.*

In view of the proof of Theorem 6.1 made in the last section under the assumption of strong normality, the above theorem establishes Theorem 6.1 in its original form.

To prove the theorem let $l_{p\sigma}$ $(\sigma = 1, 2, \ldots, q \leqslant m)$ be a maximal set of linearly independent multipliers such that E_0 is an extremal for each of the functions $l_{p\sigma}I_p(C)$. Choose $l_{p\tau}$ so that the determinant $|l_{p\sigma}, l_{p\tau}|$ does not vanish. Renumber the subscripts so that $\tau = 1, 2, \ldots, m_1$ and $\sigma = m_1 + 1, \ldots, m$. Let

$$\bar{I}_p = I_\beta l_{\beta p} \qquad (\beta, p = 1,2, \ldots, m).$$

Then in our proof (6.1b) may be replaced by the equivalent

$$\bar{I}_p(C) = 0 \qquad (p = 1,2, \ldots, m). \tag{9.3}$$

(We represent the functions on the left in the form (6.1b) with bars over g and f.) Also, we may take the multipliers (l_0) for E_0 to be (0), by transforming from I to $I + l_{p0}I_p$. The conditions (9.3) fall into two sets,

$$\begin{aligned} \bar{I}_\tau &= 0 \qquad (\tau = 1,2, \ldots, m_1), \\ \bar{I}_\sigma &= 0 \qquad (\sigma = m_1 + 1, \ldots, m). \end{aligned} \tag{9.4}$$

The arc E_0 is strongly normal for the first set while it is an extremal for each function in the second set. For an admissible variation satisfying (6.4a),

$$\begin{aligned} \bar{I}_{\sigma 1}(a,\eta) &= \bar{g}_{\sigma h}a_h + \int_{t_1}^{t_2} (\bar{f}_{\sigma a_h}a_h + \bar{f}_{\sigma y_i}\eta_i + \bar{f}_{\sigma y'_i}\eta'_i)dt \\ &= \bar{g}_{\sigma h}a_h + \left(\int_{t_1}^{t_2} \bar{f}_{\sigma a_h}dt\right)a_h + [\bar{f}_{\sigma y'_i}\eta_i]_{t_1}^{t_2} \\ &= L_\sigma(a), \end{aligned}$$

where L_σ is a linear form in (a) with constant coefficients.

With the aid of the following lemma we will be able to complete the proof of the theorem.

LEMMA 9.1. *There exists a positive constant c such that*

$$I_2(a,\eta) + cL_\sigma(a)\bar{I}_{\sigma 1}(a,\eta) = I_2 + cL_\sigma L_\sigma \tag{9.5}$$

is positive for every admissible variation with $(a) \neq (0)$ *which satisfies* (6.4a) *and*

$$I_{\tau 1}(a,\eta) = 0 \qquad (\tau = 1,2, \ldots, m_1). \tag{9.6}$$

Granting this lemma, we see that

$$\bar{I}(C) = I(C) + cL_\sigma(a - a_0)I_\sigma(C)$$

satisfies Theorem 9.1 with the first set of conditions in (9.4). In particular, the second variation of $\bar{I}$ is (9.5), and to show its positiveness it is sufficient by Lemma 9.1 to consider only non-tangential admissible variations with $(a) = (0)$ which satisfy (6.4a) and (9.6). For such a variation,

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(9.7) $$\bar{I}_{\sigma 1}(a,\eta) = L_\sigma(a) = 0.$$

Thus, from the positiveness of I_2 relative to (6.4a), (9.6), and (9.7) we deduce that (9.5) is positive. This completes the proof of Theorem 9.1.

We turn to the proof of the lemma. For this we consider for a moment the linear differential equations

(9.8)
$$\Omega_{\eta_i} - \frac{d}{dt}\,\Omega_{\eta'_i} + \mu y'_{i0} = 0,$$
$$y'_{i0}\eta''_i + y''_{i0}\eta'_i = 0,$$

where $\Omega = \omega + \lambda_\tau\omega_\tau$, and

$$\omega_\tau = \bar{f}_{\tau a_h} a_h + \bar{f}_{\tau y_i}\eta_i + \bar{f}_{\tau y'_i}\eta'_i.$$

From the non-singularity of E_0 we can solve (9.8) for η''_i, μ. The solutions have the form

(9.9) $$\eta''_i = A_{ih}(t)a_h + B_{i\tau}(t)\lambda_\tau + C_{ij}(t)\eta_j + D_{ij}(t)\eta'_j$$

and $\mu = \mu(a, \lambda, \eta, \eta')$. By multiplying the first equation (9.8) by y'_{i0}, summing, and using the homogeneity properties of $f, \bar{f}_\tau$ and the fact that E_0 is an extremal for the function I we obtain $\mu = 0$. It follows that the solutions a_h, λ_τ, $\eta_i(t)$ of the differential equations (9.9) are precisely the speical accessory (isoperimetric) extremals. From well-known theorems on linear differential equations we obtain the existence of $2n + m_1 + r$ linearly independent solutions

$$a_{hj},\ \lambda_{\tau j},\ \eta_{ij}(t) \qquad (j = 1,2,\ldots,2n + m_1 + r;\quad t_1 \leqslant t \leqslant t_2)$$

of (9.9). Substitute an arbitrary linear combination of these solutions into (9.6) to obtain m_1 linear homogeneous equations in the coefficients c_j. There will be at least $2n + r$ linearly independent solutions c_j of these equations and the correspondingly linearly independent solutions

(9.10) $$a_{hq},\ \lambda_{\tau q},\ \eta_{iq}(t) \qquad (q = 1,2,\ldots,2n + r)$$

of (9.8) will satisfy (9.6). We show that the determinant

(9.11) $$\begin{vmatrix} a_{hq} \\ \eta_{iq}(t_1) \\ \eta_{iq}(t_2) \end{vmatrix}$$

does not vanish. Notice that the first equation (9.8) with $\mu = 0$ may be written

(9.12) $$\omega_{\eta_i} - \frac{d}{dt}\omega_{\eta'_i} + \lambda_\tau\Big(\bar{f}_{\tau y_i} - \frac{d}{dt}\bar{f}_{\tau y'_i}\Big) = 0.$$

Suppose now that the determinant is zero. There exist constants c_q, not all zero, such that the special accessory extremal $(0, \eta) = (c_q a_q, c_q\eta_q)$ with multi-

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pliers $(\lambda) = (c_q \lambda_q)$ satisfies (9.6) and vanishes at the end points. This variation also satisfies (9.7). Since

$$
\begin{aligned}
I_2(0,\eta) &= \int_{t_1}^{t_2} (\omega_{\eta_i}\eta_i + \omega_{\eta'_i}\eta'_i)dt \\
&= \int_{t_1}^{t_2} (\omega_{\eta_i}\eta_i + \omega_{\eta'_i}\eta'_i)dt + \lambda_\tau I_{\tau 1}(0,\eta) \\
&= \int_{t_1}^{t_2} (\Omega_{\eta_i}\eta_i + \Omega_{\eta'_i}\eta'_i)dt = [\Omega_{\eta'_i}\eta']_{t_1}^{t_2} = 0,
\end{aligned}
$$

it follows that $(0, \eta)$ is a tangential variation. By an argument used earlier in the proof of Lemma 4.1 we find $(\eta) \equiv (0)$. But the variation satisfies (9.12). Therefore by the condition of strong normality for the first set of conditions (9.4), $(\lambda) = 0$. This contradicts the independence of the solutions (9.10) and establishes the non-vanishing of (9.11). By taking appropriate linear combinations of the columns of the (9.11) we can obtain r linearly independent special accessory extremals

$$
a_{hk}, \quad \eta_{ik}(t) \qquad (k = 1,2, \ldots ,r) \tag{9.13}
$$

with $a_{hk} = \delta_{hk}, \eta_{ik}(t_s) = y_{isk}$, and multipliers $\lambda_{\tau k}$. Clearly these variations satisfy (6.4a) and (9.6).

Returning now to (9.5) we shall show that it is sufficient to prove the existence of a constant c such that (9.5) is positive for all linear combinations of (9.13) with $(a) \neq (0)$. In fact we shall prove that for every variation satisfying (6.4a) and (9.6) there is a linear combination of (9.13) with the same components (a) which gives I_2 a value not greater than that given by the original variation. It is convenient to introduce the following notation at this point. For any two admissible variations (a, η), $(\bar{a}, \bar{\eta})$ let

$$
I_2(a,\eta;\bar{a},\bar{\eta}) = g_{hk}a_h\bar{a}_k + \int_{t_1}^{t_2} (\omega_{a_h}\bar{a}_h + \omega_{\eta_i}\bar{\eta}_i + \omega_{\eta'_i}\bar{\eta}'_i)dt,
$$

where the arguments in the derivatives of ω are (a, η, η'). It is easy to verify that $I_2(a, \eta; a, \eta) = I_2(a, \eta)$, $I_2(a, \eta; \bar{a}, \bar{\eta}) = I_2(\bar{a}, \bar{\eta}; a, \eta)$, and I_2 is linear in each of its arguments (a, η) and $(\bar{a}, \bar{\eta})$. With this in mind let $(\bar{a}, \bar{\eta}) \neq (0, \bar{\eta})$ be an admissible variation satisfying (9.6) and (6.4a). Let (a, η) be the linear combination $\bar{a}_k a_{hk}$, $\bar{a}_k \eta_{ik}$ of the variations (9.13) and $\lambda_\tau = a_k \lambda_{\tau k}$. The variation $(\bar{a} - a, \bar{\eta} - \eta)$ then has the form $(0, \eta^*)$ with $\eta_i^*(t_1) = \eta_i^*(t_2) = 0$, and satisfies both (9.6) and (9.7). Hence $I_2(0, \eta^*) \geqslant 0$ (see remark following (2.12)). We observe that

$$
\begin{aligned}
I_2(0, \eta^*) &= I_2(\bar{a} - a, \bar{\eta} - \eta; \bar{a} - a, \bar{\eta} - \eta) \\
&= I_2(\bar{a}, \bar{\eta}) - 2I_2(\bar{a}, \bar{\eta}; a, \eta) + I_2(a, \eta) \\
I_2(0, \eta^*; a, \eta) &= I_2(\bar{a} - a, \bar{\eta} - \eta; a, \eta) \\
&= I_2(\bar{a}, \bar{\eta}; a, \eta) - I_2(a, \eta).
\end{aligned} \tag{9.14}
$$

But

$$
\begin{aligned}
I_2(0, \eta^*; a, \eta) &= \int_{t_1}^{t_2} (\omega_{\eta_i}\eta_i^* + \omega_{\eta_i'}\eta_i^{*\prime})dt \\
&= \int_{t_1}^{t_2} (\omega_{\eta_i}\eta_i^* + \omega_{\eta'_i}\eta_i^{*\prime})dt + \lambda_\tau I_{\tau 1}(0, \eta^*) \\
&= \int_{t_1}^{t_2} (\Omega_{\eta_i}\eta_i^* + \Omega_{\eta'_i}\eta_i^{*\prime})dt = [\Omega_{\eta'_i}\eta_i^*]_{t_1}^{t_2} = 0.
\end{aligned}
$$

Whence (9.14) yields $I_2(a, \eta) = I_2(\bar{a}, \bar{\eta}; a, \eta)$, and $I_2(0, \eta^*) = I_2(\bar{a}, \bar{\eta}) - I_2(a, \eta)$. Therefore $0 \leqslant I_2(\bar{a}, \bar{\eta}) - I_2(a, \eta)$, as desired.

Along the family of variations spanned by (9.13) the function I_2 is a quadratic form $Q(a)$ in the finite set of variables a_h. The function $Q(a)$ is positive for all $(a) \neq (0)$ such that $L_\sigma(a) = 0$. Our proof will be complete if we can show the existence of a constant c such that

$$
Q(a) + cL_\sigma(a)_\sigma L(a) \tag{9.15}
$$

is positive for all $(a) \neq (0)$. It is sufficient to restrict ourselves to the unit sphere

$$
\sum_h a_h^2 = 1;
$$

we are to understand that all point sets are taken relative to this space. The points satisfying $L_\sigma = 0$ form a closed set on which Q is positive. Hence there exists an open neighbourhood of this closed set on which Q is positive. On the complement of the neighbourhood $L_\sigma L_\sigma$ attains a positive minimum m, and $|Q|$ attains a maximum M. Let c be any number with $c > M/m$. Then c is the desired constant.

10. Further sufficient conditions for a relative minimum. For the statement of the next two theorems we shall need the following definition. A *normed element* (a, y, y') is one for which $y'_i y'_i = 1$. By a *neighbourhood* of the elements (a, y, y') on an admissible arc E we will now understand a set of elements (a, y, y') whose normed elements lie in a neighbourhood of the normed elements belonging to E. We shall say that an isoperimetric extremal E satisfies the Weierstrass condition II if it satisfies the Weierstrass condition II_N as defined in § 6 with the modification that the elements (a, l, y, y') are restricted to belong to E. The next result deals with sufficient conditions for a weak relative minimum.

THEOREM 10.1. *If a non-singular isoperimetric extremal E_0 which does not intersect itself satisfies the conditions* (6.1), *the transversality condition, the Clebsch condition, and is such that the second variation of I along E_0 is positive, then there exists a neighbourhood $\mathfrak{R}_0$ in ayy'-space of the elements (a, y, y') belonging to E_0 such that $I(C) > I(E_0)$ for every admissible arc C in R_0 satisfying* (6.1) *but not identical with E_0.*

To prove the theorem we employ a standard result in the calculus of variations [**3**, pp. 110-111], which states that for a non-singular isoperimetric extremal E_0 satisfying the Clebsch condition there exists a neighbourhood $\mathfrak{R}_0$ and a neighbourhood $\mathfrak{L}$ such that $E[a, ly, y', Y'] \geqslant 0$ for (a, y, y') and (a, y, Y') in $\mathfrak{R}_0$, and (l) in $\mathfrak{L}$. Then for this region $\mathfrak{R}_0$ the arc E_0 satisfies the hypotheses of Theorem 6.1 and the theorem follows.

THEOREM 10.2 *Let the region $\mathfrak{R}$ of admissible elements consist of all sets (a, y, y') with (a, y) in an open set of ay-space and $y'_i y'_i \neq 0$. Then if a non-singular isoperimetric extremal E_0 which does not intersect itself satisfies the conditions* (6.1), *the transversality condition, the Weierstrass condition* II, *and is such that the second variation of I along E_0 is positive, then there exists a neighbourhood $\mathfrak{F}$ of E_0 in ay-space such that $I(C) > I(E_0)$ for every admissible arc C in $\mathfrak{F}$ satisfying* (6.1) *but not identical with E_0.*

The Weierstrass condition II and non-singularity imply that the Weierstrass and Clebsch conditions hold alone E_0 with the strict inequality sign [**7**]. These conditions in turn imply, for a region of the type described in the theorem, that the Weierstrass condition II_N holds [**3**, pp. 130-131]. Our theorem is then a consequence of Theorem 6.1.

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