BOUNDARY REGULARITY IN THE SOBOLEV IMBEDDING THEOREMS

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1. In **(6)** (see also **7**), Sobolev introduced a class of function spaces $W^{m,p}(\Omega)$ (m a non-negative integer, $1 < p < \infty$) defined on open subsets Ω of Euclidean space E^n, which have important applications in partial differential equations. They are defined as follows. For each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers let

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad D^\alpha = \prod_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^{\alpha_i}.$$

Then $W^{m,p}(\Omega)$ is the space of (classes of) functions $u \in L^p(\Omega)$ such that $D^\alpha u$ is also in $L^p(\Omega)$, where the derivatives are taken in the sense of distributions. $W^{m,p}(\Omega)$ is a Banach space under the norm

$$(1) \qquad ||u||_{W^{m,p}(\Omega)} = \left\{ \sum_{|\alpha| \leqslant m} ||D^\alpha u||^p_{L^p(\Omega)} \right\}^{1/p},$$

where $||f||_{L^p(\Omega)}$ is the usual L^p norm on Ω (with Lebesgue n-dimensional measure).

The results that are essential in the applications are the imbedding and compactness theorems of Sobolev and Kondrashov **(7)**, which may be stated as follows:

THEOREM 1. *Let Ω be an open set in E^n with sufficiently regular boundary, and let Ω' be the intersection of Ω with a hyperplane of dimension $s \leqslant n$. Suppose that $u \in W^{l,p}(\Omega)$. Then*

(a) *if $n \geqslant lp$ and $s > n - lp$, then $u \in L^q(\Omega')$ for any q such that*

$$p \leqslant q < sp/(n - lp);$$

also there is a constant C, independent of u, such that

$$(2) \qquad ||u||_{L^q(\Omega')} \leqslant C||u||_{W^{l,p}(\Omega)}$$

(*if Ω is bounded, the condition $p \leqslant q$ is trivially unnecessary*).

(b) *If $n < lp$, then u coincides* a.e. *with a continuous function $\tilde{u}$ on Ω and there is a constant C, independent of u, such that*

$$\max_{\Omega} |\tilde{u}| \leqslant C||u||_{W^{l,p}(\Omega)}.$$

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The content of Theorem 1 is that, in the cases (a) and (b), $W^{l,p}(\Omega)$ can be imbedded continuously in $L^q(\Omega')$ and $C(\Omega)$ respectively.

THEOREM 2. *Let Ω be bounded. Then the injections of $W^{l,p}(\Omega)$ into $L^q(\Omega')$ and $C(\Omega)$, corresponding to the cases* (a) *and* (b) *of Theorem* 1 *respectively, are both compact mappings.*

New proofs of these results, together with some improvements, were given by Gagliardo **(1)**. More general results corresponding to case (b) may be found in **(4)**.

The main concern of this note is to improve the assumptions on the boundary of Ω under which the (a) parts of both theorems hold. Sobolev established his results under the assumption that Ω is a finite union of domains that are star-shaped with respect to some fixed sphere; i.e., such that every point of the domain is the vertex of a cone over the sphere; see §2. The results of Gagliardo were obtained for open sets satisfying the cone property, defined as follows:

DEFINITION 1. *The open set Ω has the cone property if for every $y \in \Omega$ there exists a finite right spherical cone $K(y)$ with vertex at y lying in Ω, and whose altitude and angle are independent of y.*

Domains having the cone property are already more general than finite unions of star-shaped domains (e.g., an open n-ball B with a countable number of points, which have a limit point in B, deleted). However, the cone property still does not allow domains that, for example, have cusps at the boundary. In this note we show how the proof of Sobolev can be simply modified to establish the parts (a) of Theorems 1 and 2 for domains satisfying a generalized L^p-type of cone condition. As a consequence, assumptions on boundary regularity can be weakened in several existence theorems for weak solutions of partial differential equations.

The proof will utilize a result of Kantorovich **(2)** that was presented originally as a generalization of Sobolev's approach to the imbedding theorem. It concerns integral operators of the form

$$Au(y) = \int_\Omega k(x, y)u(x)\, dx, \tag{3}$$

where $k(x, y)$ is a measurable function of $x \in \Omega \subset E^n$ (Lebesgue n-dimensional measure) for a.e. $y \in \Omega' \subset E^s$ (Lebesgue s-dimensional measure). We present here a proof of Kantorovich's theorem (slightly modified), since it is very simple.

THEOREM 3 (Kantorovich). *Let $k_1(x, y)$ and $k_2(x, y)$ be non-negative measurable functions on $\Omega \times \Omega'$ (product measure) such that*

$$|k(x, y)| \leqslant k_1(x, y)k_2(x, y)$$

holds a.e. *in* $\Omega \times \Omega'$. *Suppose that*

(i) *for* a.e. $y \in \Omega'$, $k_1(x, y) \in L^{p'}(\Omega)$ *as a function of* $x \in \Omega$;

(ii) *with* $C_1(y) = ||k_1(\cdot, y)||_{L^{p'}(\Omega)}$, *we have* $C_1(y)k_2(x, y) \in L^q(\Omega')$ *as a function of* $y \in \Omega'$ *for* a.e. $x \in \Omega$, *with* $||C_1(\cdot)k_2(x, \cdot)||_{L^q(\Omega')} \leqslant C_2$ (*a constant*).

Then if $p \leqslant q$, *the integral operator* A *defined by* (3) *is a continuous linear mapping from* $L^p(\Omega)$ *to* $L^q(\Omega')$ *with* $||A|| \leqslant C_2$.

Proof. We have

$$|Au(y)| \leqslant \int_\Omega k_1(x, y)k_2(x, y)u(x)\, dx$$
$$\leqslant \int_\Omega |u(x)|^{p/q}k_2(x, y)k_1(x, y)|u(x)|^{1-p/q}\, dx,$$

and, using Hölder's inequality,

$$|Au(y)| \leqslant \left\{ \int_\Omega |u(x)|^p[k_2(x, y)]^q\, dx\right\}^{1/q} \left\{ \int_\Omega [k_1(x, y)]^{p'}\, dx\right\}^{1/p'}$$
$$\times \left\{ \int_\Omega |u(x)|^p\, dx\right\}^{1/p-1/q} \leqslant ||u||^{1-p/q}_{L^p(\Omega)} C_1(y) \left\{ \int_\Omega |u(x)|^p[k_2(x, y)]^q\, dx\right\}^{1/q}.$$

Thus

$$\left\{ \int_{\Omega'} |Au(y)|^q dy\right\}^{1/q} \leqslant ||u||^{1-p/q}_{L^p(\Omega)} \left\{ \int_\Omega [C_1(y)]^q \left[\int_\Omega |u(x)|^p[k_2(x, y)]^q\, dx\right] dy\right\}^{1/q},$$

and, using Tonelli's theorem,

$$\left\{ \int_{\Omega'} |Au(y)|^q dy\right\}^{1/q} \leqslant ||u||^{1-p/q}_{L^p(\Omega)} \left\{ \int_\Omega |u(x)|^p \left[\int_{\Omega'} [C_1(y)k_2(x, y)]^q\, dy\right] dx\right\}^{1/q}$$
$$\leqslant C_2||u||_{L^p(\Omega)}.$$

2. We now present the geometric assumptions under which the imbedding theorems will be proved. Let y be any point in E^s and let $S = \{x: |x - x_0| < r\}$ be any open sphere not containing y; here $x = (x_1, \ldots, x_n)$ and $|x|$ is the usual Euclidean distance. Then the open cone with vertex y over S is defined to be

$$K_S(y) = \{z \in E^n: z = y + \tau(x - y), x \in S, 0 < \tau < 1\}.$$

Now let D be any bounded open set in E^n. For each $y \in D$ there is at least one sphere $S \subset D$ not containing y such that the cone $K_S(y)$ lies entirely in D. Let $\mathfrak{S}_D(y)$ be the set of all such spheres, and define

$$R_D(y) = \frac{1}{2} \sup_{S \in \mathfrak{S}_{D(y)}} [\text{radius of } S]. \tag{4}$$

Clearly $0 < R_D(y) \leqslant \frac{1}{4}$ diam D for $y \in D$. Also it is easy to see from geometric considerations that $R_D(y)$ is lower semi-continuous and hence measurable.

The regularity of the boundary will be stated in terms of the local behaviour of functions of the form $1/R_D(y)$ near the boundary. In the following section we shall prove the following theorem.

THEOREM 4. *Suppose it is possible to cover Ω, up to a set of measure zero, by the union of disjoint uniformly bounded open sets X_k $(k = 1, 2, \ldots)$ such that, with $D_k = \Omega \cap X_k$, $D'_k = \Omega' \cap X_k$, the functions $1/R_{D_k}(y)$ are in $L^r(D'_k)$, where $r = \max[nq, (n-1)q\beta]$ and $\beta > 1$, with norms bounded independently of k (the condition is interpreted as vacuously satisfied if D'_k is empty). Then the* (a) *parts of Theorems* 1 *and* 2 *for $l = 1$ remain valid if the condition*

$$q < sp/(n-p)$$

is replaced by $q\beta' < sp/(n-p)$, where $1/\beta + 1/\beta' = 1$.

A corresponding result for arbitrary l can be obtained by an inductive argument or by using Sobolev's representation formula in the general case; see §3.

Theorem 4 immediately yields parts (a) of Theorems 1 and 2 if Ω has the cone property, for in that case the functions $1/R_{D_k}(y)$ are uniformly bounded, and we may take $\beta' = 1$. If we insist that q be chosen as close as possible to the optimum value of Theorem 1, then Ω must have the cone property. However, if a lower value of q is adequate, then the boundary regularity may be relaxed. An important special case is obtained when $q = p$, in which case the imbedding theorems hold if $\beta > s/(s + p - n)$.

3. Before proceeding with the proof of Theorem 4 we note several standard simplifications:

(i) In order to prove the (a) parts of Theorems 1 and 2 for a certain class of domains, it suffices to prove Theorem 1(a) for that class of domains. For Gagliardo **(1)** gives a very simple proof of Theorem 2(a) based only on part (a) of Theorem 1 (and the regularity of Ω there required) and a general compactness result due to M. Riesz **(5)**, which is independent of the regularity of Ω.

(ii) The (a) part of Theorem 1 need only be established for

$$u \in W^{l,p}(\Omega) \cap C^\infty(\Omega).$$

For if the theorem were established for all such u, then the imbedding of $W^{l,p}(\Omega) \cap C^\infty(\Omega)$ into $L^q(\Omega')$ would be continuous and hence the completion of $W^{l,p}(\Omega) \cap C^\infty(\Omega)$ would also lie in $L^q(\Omega')$. But by **(3)** this completion is exactly $W^{l,p}(\Omega)$ (the result of **(3)** is independent of the regularity of Ω).

(iii) To show (2) it is sufficient to prove that there is a constant C such that for any open set X in the covering of Theorem 4 we have

$$||u||_{L^q(\Omega' \cap X)} \leqslant C||u||_{W^{l,p}(\Omega \cap X)},$$

where C is independent of the particular open set X. For then

$$\int_{\Omega'} |u|^q = \sum_{k=1}^{\infty} \int_{\Omega' \cap X_k} |u|^q \leqslant C \sum_{k=1}^{\infty} \Big\{ \sum_{|\alpha| \leqslant m} \int_{\Omega \cap X_k} |D^\alpha u|^p \Big\}^{q/p}$$
$$\leqslant C \Big\{ \sum_{k=1}^{\infty} \sum_{|\alpha| \leqslant m} \int_{\Omega \cap X_k} |D^\alpha u|^p \Big\}^{q/p}$$

since $q/p \geqslant 1$, and so

$$\int_{\Omega'} |u|^q \leqslant C \Big\{ \sum_{|\alpha| \leqslant m} \int_{\Omega} |D^\alpha u|^p \Big\}^{q/p}$$

as desired.

Now let X be any open set in the covering of Theorem 4. Put $D = \Omega \cap X$, $D' = \Omega' \cap X$, and assume D' non-empty. For each $y \in D$ there is at least one cone $K_{S(y)}(y)$ over a sphere $S(y)$ with radius $R_D(y)$ which lies entirely in D. Make a definite assignment of spheres $S(y)$ with centres $x_0(y)$ and associated cones $K_{S(y)}(y)$ to each $y \in D$. Because of the arbitrariness of the choice of $S(y)$, the function $x_0(y)$ could be very badly behaved; but at least

$$|x_0(y) - y| \leqslant C$$

where here, and in the following, C generically denotes any constant not depending on X (or later, on u). In fact we may take C to be an upper bound for the diameters of the X_k.

Consider the function

$$(5) \quad v(x, y) = \begin{cases} \kappa(y) \exp\left(\dfrac{|x - x_0(y)|^2}{|x - x_0(y)|^2 - [R_D(y)]^2}\right) & \text{if } |x - x_0(y)| < R_D(y), \\ 0 & \text{if } |x - x_0(y)| \geqslant R_D(y), \end{cases}$$

where $\kappa(y)$ is chosen so that

$$\int_{S(y)} v(x, y)\, dx = 1.$$

As a function of x, $v(x, y) \in C^\infty$ and has support in $S(y)$. Notice again that $v(x, y)$ may be badly behaved as a function of y, but at least $|v(x, y)| \leqslant \kappa(y)$. To obtain an estimate for $\kappa(y)$ notice that

$$1 = \int_{S(y)} v(x, y)\, dx = \kappa(y) \int_{S_{n-1}} d\sigma \int_0^{R_D(y)} \exp\left(\frac{r^2}{r^2 - [R_D(y)]^2}\right) r^{n-1}\, dr,$$

where S_{n-1} is the unit $n - 1$ sphere, and so

$$1 \geqslant C\kappa(y) \int^{R_D(y)\sqrt{[k/k+1]}} r^{n-1}\, dr = C\kappa(y)[R_D(y)]^n,$$

where $k = -\log 1/2$, the last inequality resulting from the observation that

$$\exp\left(\frac{r^2}{r^2 - [R_D(y)]^2}\right) \geqslant 1/2$$

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when

$$r \leqslant R_D(y) \sqrt{\frac{k}{k+1}}\,.$$

Thus, with $f(y) = [R_D(y)]^{-n}$,

$$\kappa(y) \leqslant Cf(y). \tag{6}$$

Next we define the function $\chi(x, y)$ for $x \neq y$ by

$$\chi(x, y) = \chi(y + \sigma r, y) = -\int_r^\infty v(y + \sigma t, y)t^{n-1}\,dt,$$

where $r = |x - y|$ and $\sigma = x - y/|x - y| \in S_{n-1}$, and put $\chi(y, y) = 0$. Then, for any x, y, it follows from the mean-value theorem that

$$\begin{aligned} |\chi(x, y)| &\leqslant 2|\chi(x_0(y), y)| \\ &\leqslant 2\kappa(y)\int_{|x_0(y)-y|}^{|x_0(y)-y|+R_D(y)} t^{n-1}\,dt \leqslant C\kappa(y)R_D(y) \leqslant Cg(y), \end{aligned} \tag{7}$$

where $g(y) = f(y)R_D(y)$.

Using the functions $v(x, y)$ and $\chi(x, y)$ we have the following representation, due in essence to Sobolev (compare with (**7**, formula 7.12) for the case $l = 1$), for $u \in C^\infty(\Omega)$ in terms of $\partial u/\partial x_i$ $(i = 1, \ldots, n)$, holding in D:

$$u(y) = \int_D v(x, y)u(x)\,dx + \sum_{i=1}^n \int_D \frac{\chi(x, y)}{[r(x, y)]^{n-1}} \frac{\partial x_i}{\partial r} \frac{\partial u}{\partial x_i}\,dx.$$

From (6) and (7) and the fact that $|\partial x_i/\partial r| \leqslant 1$, there results

$$|u(y)| \leqslant Cf(y)\int_D |u(x)|dx + C\sum_{i=1}^n \int \frac{g(y)}{[r(x, y)]^{n-1}} \left|\frac{\partial u}{\partial x_i}\right| dx. \tag{8}$$

The inequality (8) will now be used to show that for all $u \in W^{1,p}(\Omega) \cap C^\infty(\Omega)$, $u \in L^q(D')$, and

$$||u||_{L^q(D')} \leqslant C||u||_{W^{1,p}(D)}. \tag{9}$$

Consider the first term on the right of (8) as a function of y. We have

$$||f(\cdot)\int_D |u(x)|\,dx||_{L^q(D')} \leqslant C||f||_{L^q(D')}\int_{D'} |u(x)|\,dx.$$

Now $f(y) \in L^q(D')$ with norm independent of X since $r \geqslant nq$. Thus

$$||f(\cdot)\int_D u(x)\,dx||_{L^q(D')} \leqslant C\int_D |u(x)|\,dx \leqslant C||u||_{L^p(D)},$$

by Hölder's inequality. Consider next a representative term in the sum on the right of (8). Apply Theorem 3 with

$$k_1(x, y) = r^{-(n-1)\alpha}, \qquad k_2(x, y) = g(y)r^{-(n-1)(1-\alpha)},$$

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where $0 \leqslant \alpha \leqslant 1$ is to be chosen. By the elementary convergence properties of integrals of potential type in E^n, $k_1(x, y) \in L^{p'}(D)$ with norm bounded independently of $y \in D'$ if

$$(n-1)\alpha p' < n. \tag{11}$$

Also

$$||k_2(x, \cdot)||^q_{L^q(D')} \leqslant ||g^q||_{L^\beta(D')}||r^{-(n-1)(1-\alpha)q}||_{L^{\beta'}(D')}$$

by Hölder's inequality. The first factor on the right is finite since $r \geqslant (n-1)q\beta$. Also the second factor on the right is bounded independently of $x \in D$ if $(n-1)(1-\alpha)q\beta' < s$. This can be simplified, using (11), to

$$q\beta' < sp/(n-p).$$

Thus, under the conditions of Theorem 4, the right-hand side of (8), considered as a function of y, is in $L^q(D')$ and, from Theorem 3, it has a norm bounded above by $C||u||_{W^{1,p}(D)}$. The same is then true of u and Theorem 4 is established.

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