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# THE CAUCHY PROBLEM OF LINEAR PARABOLIC EQUATIONS WITH DISCONTINUOUS AND UNBOUNDED COEFFICIENTS

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## § 1. Introduction.

In this article we shall prove the uniqueness and existense of a weak solution for the Cauchy problem of linear parabolic equations with discontinuous and unbounded coefficients

$$
\begin{aligned}
(1.1) \qquad Lu &= u_t - \{\sum_{i,j=1}^{n}(a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{j=1}^{n} b_j(x,t)u_{x_j} + c(x,t)u\} \\
&= \sum_{j=1}^{n}(f_j(x,t))_{x_j} + g(x,t).
\end{aligned}
$$

In the case where the coefficients are bounded for large $|x|$, Aronson [1] proved the uniqueness and existence using the weighted energy type estimates for weak solutions. Bodanko [2] also discussed the questions of a regular solution for the Cauchy problem of linear parabolic equations

$$
\begin{aligned}
(1.2) \qquad Lu &= \sum_{i,j=1}^{n} a_{i,j}(x,t)u_{x_i x_j} + \sum_{j=1}^{n} b_j(x,t)u_{x_j} + c(x,t)u - u_t \\
&= f(x,t)
\end{aligned}
$$

with unbounded coefficients under some assumption.

In § 2 we shall state some notations and definitions. § 3 is devoted to derive the energy estimates for weak solutions of the equation (1.1) and we prove main theorems in § 4.

## § 2. Some notations and definitions.

Let $x = (x_1, \cdots, x_n)$ be a point in the $n$-dimensional Euclidean space $E^n$ and $t$ a point on the real line. Let $T$ be a fixed positive number and set $Q = E^n \times (0, T]$. For some fixed positive number $R_0$ we put $Q_0 = \sum_{R_0} \times (0, T]$,

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where $\sum_{R_0} = \{x \in E^n | \ |x| < R_0\}$. Now let $\omega$ be a domain in $E^n$. The space $L^q[0, T; L^p(\omega)]$ is the set of real functions $w(x, t)$ with the following properties;

(i) $w$ is defined and measurable in $\tilde{\omega} = \omega \times (0, T]$,

(ii) $w(x, t) \in L^p(\omega)$ for almost all $t \in (0, T]$,

(iii) $\|w\|_{L^p(\omega)}(t) \in L^q((0, T))$.

The space $L^q[0, T; L^p(\omega)]$ is denoted by $L^{p,q}(\tilde{\omega})$ for simplicity. For $w \in L^{p,q}(\tilde{\omega})$ with $1 \leqq p, q < \infty$ we define the norm by

$$\|w\|_{p,q,\tilde{\omega}} = \left\{ \int_0^T \left( \int_\omega |w|^p \, dx \right)^{q/p} dt \right\}^{1/q}.$$

In the case either $p$ or $q$ is infinte, $\|w\|_{p,q,\omega}$ is defined in a similar fashion using $L^\infty$-norms rather than integrals.

We shall consider the following Cauchy problem:

$$(2.1) \qquad \begin{cases} Lu = \sum_{j=1}^n (f_j)_{x_j} + g & \text{for} \quad (x, t) \in Q, \\ u(x, 0) = u_0(x) & \text{for} \quad x \in E^n, \end{cases}$$

where the coefficients $a_{ij}$, $b_j$ and $c$ are measurable, real valued functions in $Q$, and $f_j$ and $g$ are given functions in $Q$. We assume the following conditions:

Condition A.

(A.1) For all $\xi \in E^n$ and for almost all $(x, t)$ there exist positive constants $k$ and $K_1$ such that

$$k(1 + |x|^2)^{1-\lambda} |\xi|^2 \leqq \sum a_{ij}(x, t) \xi_i \xi_j \leqq K_1(1 + |x|^2)^{1-\lambda} |\xi|^2$$

where $\lambda$ is any fixed number with $0 \leqq \lambda \leqq 1$.

(A.2) The restriction of every coefficient $b_j$ to $Q_0$ belongs to some space $L^{p_j, q_j}(Q_0)$, where $p_j$ and $q_j$ satisfy

$$(*) \qquad 2 < q_j, p_j \leqq \infty \quad \text{and} \quad \frac{n}{2p_j} + \frac{1}{q_j} < \frac{1}{2},$$

and there exists a non-negative constant $K_2$ such that

$$|b_j(x, t)| \leqq K_2(1 + |x|^2)^{\frac{1}{2}} \quad \text{for} \quad (x, t) \in Q - Q_0.$$

(A.3) The restriction of $c$ to $Q_0$ belongs to $L^{p,q}(Q_0)$, where $p$ and $q$

satisfy

$$(**)\qquad 1<p,q\leqq\infty \quad\text{and}\quad \frac{n}{2p}+\frac{1}{q}<1,$$

and $c(x,t)\leq K_3(1+|x|^2)^\lambda$ in $(x,t)\in Q-Q_0$ for a non-negative constant $K_3$.

A function $u(x)$ defined and measurable in $\omega$ is said to belong to $H^{1,p}(\omega)$ if $u$ possesses a distribution derivative $(u_{x_1},\cdots,u_{x_n})$ and $\|u\|_{L^p(\omega)}+\|u_x\|_{L^p(\omega)}<\infty$, where $\|u_x\|^p_{L^p}=\sum_{i=1}^n\|u_{x_i}\|^p_{L^p}$.
The space $H_0^{1,p}(\omega)$ is the completion of the $C_0^\infty(\omega)$ functions in this norm. The space $H^{1,p}(E^n)$ is the completion of the $C_0^\infty(E^n)$ functions in the norm

$$\|\varphi\|_{L^p(E^n)}+\|\varphi_x\|_{L^p(E^n)}.$$

Definition 2.1. A function $u=u(x,t)$ is said to be a weak solution of the problem (2.1) in $Q$ for the initial data $u_0\in L^2_{\mathrm{loc}}(E^n)$ if $u$ belongs to $L^\infty[0,T;L^2(\omega)]\cap L^2[0,T;H^{1,2}(\omega)]$ for any $\omega$ compact in $E^n$, that is, if

$$u\in L^\infty[0,T;L^2_{\mathrm{loc}}(E^n)]\cap L^2[0,T;H^{1,2}_{\mathrm{loc}}(E^n)]$$

and if $u$ satisfies

$$(2.2)\qquad \iint_Q\{-u\varphi_t+\sum a_{ij}u_{x_i}\varphi_{x_j}-\sum b_ju_{x_j}\varphi-cu\varphi+\sum f_j\varphi_{x_j}-g\varphi\}\,dxdt =0$$

for any $\varphi\in C_0^1(Q)$ and further if

$$(2.3)\qquad \lim_{t\to0}\int_{E^n}u(x,t)\psi(x)dx=\int_{E^n}u_0(x)\psi(x)dx\quad\text{for all } \psi\in C_0^1(E^n).$$

At the end of this section we state a lemma which is often used in the following sections.

Lemma 2.1. (*Aronson* [1]). *If* $w\in L^\infty[0,T;L^2(E^n)]\cap L^2[0,T;H_0^{1,2}(E^n)]$, *then* $w\in L^{2p',2q'}(Q)$ *for all values of* $p'$ *and* $q'$ *whose Hölder conjugates* $p$ *and* $q$ *satisfy*

$$\frac{n}{2p}+\frac{1}{q}\leqq1,$$

*where if* $n=2$, *then the strict inequality holds. Moreover*

$$\|w\|^2_{2p',2q',Q}\leqq KT^\theta\{\|w\|^2_{2,\infty,Q}+\|w_x\|^2_{2,2,Q}\},$$

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*where* $|w_x|^2=\sum_{j=1}^n w_{x_j}^2$, $\theta=1-\frac{1}{q}-\frac{n}{2p}$, *and K is a positive constant which depends only on* $n$ *for* $n\neq 2$ *and only on* $p$ *for* $n=2$.

## §3. Energy Estimates.

Let $\Omega$ be any bounded domain in $E^n$ such that $\Omega\subset\sum_{R_0}$, and set $Q_1=\Omega\times(0,T]$. In $Q_1$ we consider the equation

$$Lu=\sum(f_j)_{x_j}+g, \tag{3.1}$$

where $f_j\in L^{2,2}(Q_1)$ and $g\in L^{p,q}(Q_1)$ for any $p$ and $q$ satisfying (**).

LEMMA 3.1. *Let* $u\in L^\infty[0,T;L^2(\Omega)]\cap L^2[0,T;H^{1,2}(\Omega)]$ *be a weak solution of* (3.1) *with the initial data* $u_0$, *that is, let* $u$ *satisfy*

$$\iint_{Q_1}\{-u\varphi_t+\sum a_{ij}u_{x_i}\varphi_{x_j}-\sum b_ju_{x_j}\varphi-cu\varphi+\sum f_j\varphi_{x_j}-g\varphi\}\,dxdt \tag{3.2}$$
$$=0$$

*for any* $\varphi\in C_0^1(Q_1)$ *and* (2.3), *and let* $\zeta=\zeta(x)$ *be a non-negative smooth function such that* $\zeta u\in L^2[0,T;H_0^{1,2}(\Omega)]$. *Then for any positive number* $\mu_0$, *there exist positive constants* $\mathscr{C}$ *and* $\mu$ *such that*

$$\|\zeta e^{-\mu(1+|x|^2)^\lambda}u\|_{2,\infty,Q_1'}^2+\|\zeta e^{-\mu(1+|x|^2)^\lambda}u_x\|_{2,2,Q_1'}^2 \tag{3.3}$$
$$\leqq\mathscr{C}\Big(\int_\Omega\zeta^2e^{-2\mu_0(1+|x|^2)^\lambda}u_0^2dx+\|\zeta_xe^{-\mu_0(1+|x|^2)^\lambda}$$
$$\times(1+|x|^2)^{\frac{1-\lambda}{2}}u\|_{2,2,Q_1'}^2+\|\zeta e^{-\mu_0(1+|x|^2)^\lambda}f_j\|_{2,2,Q'}^2$$
$$+\zeta e^{-\mu_0(1+|x|^2)^\lambda}g\|_{\frac{2p}{p+1},\frac{2q}{q+1},Q_1'}\Big)$$

*where* $Q_1'=\Omega\times(0,T')$ *with* $T'<\frac{\pi}{2}$. *The constants* $T'$, $\mathscr{C}$ *depend only on* $k$, $K_1$, $K_2$, $K_3$, $\lambda$, $\mu_0$, $\|b_j\|_{p_j,q_j,Q_0}$ *and* $\|c\|_{p,q,Q_0}$; *and* $\mu$ *depends on* $\mu_0$ *and* $T'$.

*Proof.* Let $h(x,t)=-\alpha(t)(1+|x|^2)^\lambda$, where $\alpha(t)\in C^1(\tau',\tau'+\sigma)$ for any $\tau'$ and a sufficiently small number $\sigma$ with $0\leqq\tau'<\tau'+\sigma\leqq T$. Then we obtain for any $\tau\in[\tau',\tau'+\sigma]$

$$\frac{1}{2}\int_\Omega\zeta^2e^{2h}u^2dx\,|_{t=\tau}+\int_{\tau'}^\tau\int_\Omega\zeta^2e^{2h}\{\sum a_{ij}u_{x_i}u_{x_j}-\sum b_juu_{x_j} \tag{3.4}$$
$$+(-h_t-c)u^2+\sum f_ju_{x_j}-gu\}\,dxdt+2\int_{\tau'}^\tau\int_\Omega\zeta e^{2h}u(\sum a_{ij}u_{x_i}\zeta_{x_j}$$

$$+ \sum a_{ij} u_{x_i} \zeta h_{x_j}) + \zeta e^{2h} u \sum f_j (\zeta_{x_j} + \zeta h_{x_j})\} \, dx dt$$
$$= \frac{1}{2} \int_\Omega \zeta^2 e^{2h} u^2 dx \,|_{t=\tau'}$$

(See §2 in [1]).

Now we shall estimate each term of (3.4) by using Hölder's inequality and Young's inequality together with Lemma 2.1.

First we see from Condition (A.1) that

$$\iint \zeta^2 e^{2h} \sum a_{ij} u_{x_i} u_{x_j} dx dt \geqq k \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dx dt, \tag{3.5}$$

$$2 \iint \sum a_{ij} u_{x_i} u \zeta \zeta_{x_j} e^{2h} \, dx dt \leqq \frac{k}{10} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dx dt \tag{3.6}$$
$$+ \frac{10 n^2 K_1^2}{k} \iint |\zeta_x|^2 e^{2h} (1 + |x|^2)^{1-\lambda} u^2 dx dt,$$

and

$$2 \iint \sum a_{ij} \zeta^2 e^{2h} u u_{x_i} h_{x_j} dx dt \leqq \frac{k}{10} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} |u_x|^2 dx dt \tag{3.7}$$
$$+ \frac{10 n^2 K_1^2}{k} \iint \zeta^2 e^{2h} (1 + |x|^2)^{1-\lambda} u^2 |h_x|^2 dx dt.$$

Next we have

$$\int_{\tau'}^{\tau} \int_\Omega \sum b_i \zeta^2 e^{2h} u_{x_j} u \, dx dt \leqq \int_{\tau'}^{\tau} \int_{\Sigma_{R_0}} \sum |b_i| \zeta^2 e^{2h} |u_{x_j}| \; |u| \, dx dt$$
$$+ K_2 \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h} (1 + |x|^2)^{\frac{1}{2}} |u| \sum |u_{x_j}| \, dx dt$$
$$\equiv B_1 + B_2, \quad \text{say.}$$

By the inequalities of Hölder and Young it is clear that

$$B_1 \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{5}{k} \sum \|b_j^2\|_{p,q} \cdot \|\zeta e^h u\|_{2p',2q'}^2,$$

where $p$ and $q$ satisfy (**). Thus by Lemma 2.1, we see

$$B_1 \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{5}{k} MK\sigma^\theta \{\|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2\},$$

where we assumed $\sum \|b_j^2\|_{p,q} \leqq M$.

Using Young's inequality again we see

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$$B_2 \leqq \frac{k}{10} \iint \zeta^2 e^{2h}(1 + |x|^2)^{1-\lambda} |u_x|^2 \, dxdt + \frac{5K_2^2 n}{k} \iint \zeta^2 e^{2h}(1 + |x|^2) u^2 dxdt.$$

Thus we obtain

$$\begin{aligned}(3.8)\qquad \sum \iint b_j \zeta^2 e^{2h} u_{x_j} u dxdt &\leqq \frac{k}{5} \iint \zeta^2 e^{2h}(1 + |x|^2)^{1-\lambda} |u_x|^2 dxdt \\ &+ \frac{5}{k} MK\sigma^\theta \{ \|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2 \} \\ &+ \frac{5K_2^2 n}{k} \iint \zeta^2 e^{2h}(1 + |x|^2)^\lambda u^2 dxdt.\end{aligned}$$

Similarly we have

$$\begin{aligned}(3.9)\qquad \iint c\zeta^2 e^{2h} u^2 dxdt &\leqq MK\sigma^\theta \{ \|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2 \} \\ &+ K_3 \iint \zeta^2 e^{2h}(1 + |x|^2)^\lambda u^2 dxdt,\end{aligned}$$

where we assumed $\|c\|_{p,q} \leqq M$.

It is easily seen that

$$\begin{aligned}(3.10)\qquad & 2\iint \sum f_i \zeta e^{2h}(\zeta_{x_j} + \zeta h_{x_j}) u dxdt \leqq \\ & 2\sum \|\zeta e^h f_j\|_{2,2}^2 + \|\zeta_x e^h u\|_{2,2}^2 + \|\zeta e^h u h_x\|_{2,2}^2,\end{aligned}$$

$$(3.11)\qquad \iint \sum f_j \zeta^2 e^{2h} u_{x_j} dxdt \leqq \frac{k}{10} \|\zeta e^h u_x\|_{2,2}^2 + \frac{10}{k} \sum \|\zeta e^h f_j\|_{2,2}^2,$$

and

$$\begin{aligned}(3.12)\qquad \iint \zeta^2 e^{2h} g u dxdt &\leqq \|\zeta e^h g\|_{\frac{2p}{p+1}, \frac{2q}{q+1}} \|\zeta e^h u\|_{2p', 2q'} \\ &\leqq \frac{1}{2M} \|\zeta e^h g\|^2_{\frac{2p}{p+1}, \frac{2q}{q+1}} + \frac{M}{2} K\sigma^\theta \{ \|\zeta e^h u\|_{2,\infty}^2 + \|(\zeta e^h u)_x\|_{2,2}^2 \}.\end{aligned}$$

Finally we note that

$$(3.13)\qquad \|(\zeta e^h u_x)\|_{2,2}^2 \leqq 2\|\zeta e^h u_x\|_{2,2}^2 + 4\|u e^h \zeta_x\|_{2,2}^2 + 4\|\zeta e^h u h_x\|_{2,2}^2.$$

Combining these estimates from (3.5) to (3.13), we have

$$\begin{aligned}(3.14)\qquad & \frac{1}{2} \int_\Omega \zeta^2 e^{2h} u^2 dx |_{t=\tau} + \left\{ \frac{k}{2} - \left( \frac{10}{k} + 3 \right) MK\sigma \right\} \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h} \\ & (1 + |x|^2)^{1-\lambda} |u_x|^2 dxdt + \int_{\tau'}^{\tau} \int_\Omega \zeta^2 e^{2h} u^2 \left\{ - h_t - \left[ \frac{10 n^2 K_1^2}{k} (1 + |x|^2)^{1-\lambda} |h_x|^2 \right.\right.\end{aligned}$$

$$+\left(\frac{20}{k}n^2MK\sigma^\theta+6MK\sigma^\theta+1\right)|h_x|^2+\left(\frac{5K_2^2n}{k}+K_3\right)(1+|x|^2)^\lambda\Big]\Big\}dxdt$$

$$\leqq\frac{1}{2}\left(\frac{10}{k}+3\right)MK\sigma^\theta\|\zeta e^h u\|_{2,\infty}^2+\frac{1}{2}\int_\Omega \zeta^2 e^{2h}u^2dx|_{t=\tau'}$$

$$+C_1\left(\int_{\tau'}^{\tau}\int_\Omega|\zeta_x|^2e^{2h}(1+|x|^2)^{1-\lambda}u^2dxdt+\sum\|\zeta e^h f_j\|_{2,2}^2+\|\zeta e^h g\|^2_{\frac{2p}{p+1},\frac{2q}{q+1}}\right),$$

where we used the fact that $1\leqq(1+|x|^2)^{1-\lambda}$.

Since $h(x,t)=-\alpha(t)(1+|x|^2)^\lambda$, we see

$$-h_t(x,t)=\alpha(t)(1+|x|^2)^\lambda \text{ and } |h_x|^2\leqq\alpha^2(t)4\lambda^2(1+|x|^2)^{2\lambda-1}.$$

Noting that $2\lambda-1\leqq\lambda$, that is, $(1+|x|^2)^{2\lambda-1}\leqq(1+|x|^2)^\lambda$, we obtain

$$-h_t-\Big[\frac{10n^2K_1^2}{k}(1+|x|^2)^{1-\lambda}|h_x|^2+\left(\frac{20}{k}n^2MK\sigma^\theta+6MK\sigma^\theta+1\right)|h_x|^2$$

$$+\left(\frac{5K_2^2n}{k}+K_3\right)(1+|x|^2)^\lambda\Big]\geqq(1+|x|^2)^\lambda\{\alpha'(t)-A\alpha^2(t)-B\},$$

where

$$A=4\left(\frac{10n^2K_1^2+20n^2MK\sigma^\theta}{k}+6MK\sigma+1\right)\lambda^2,\text{ and } B=\frac{5K_2^2n}{k}+K_3.$$

If $\lambda\neq0$ we put $\alpha(t)=\sqrt{\frac{B}{A}}\tan\{\sqrt{AB}\,t+m\}$ for $0\leqq t\leqq\tau'$, where $m(>0)$ is a constant with $\sqrt{\frac{B}{A}}\tan m=\mu_0\left(0<m<\frac{\pi}{2}\right)$. Then

$$\alpha'(t)-A\alpha^2(t)-B=0.$$

Now we put $\sigma$ so small that $\frac{k}{2}-\left(\frac{10}{k}+3\right)MK\sigma>0$, $\frac{1}{2}-\frac{1}{2}\left(\frac{10}{k}+3\right)$. $MK\sigma>0$ and $m+\sqrt{AB}\,\sigma<\min\left(\frac{\pi}{2},\ T\right)$. Then from (3.14) we have

$$\max_{\tau'\leq t\leq\tau'+\sigma}\int_\Omega\zeta^2e^{2h}u^2dx+\int_\tau^{\tau'+\sigma}\int_\Omega\zeta^2e^{2h}|u_x|^2dxdt \tag{3.15}$$

$$\leq\mathscr{C}_2\Big(\int_\Omega\zeta^2e^{2h}u^2dx|_{t=\tau'}+\|\zeta_xe^h(1+|x|^2)^{\frac{1-\lambda}{2}}u\|_{2,2}^2$$

$$+\sum\|\zeta e^hf_j\|_{2,2}^2+\|\zeta e^hg\|^2_{\frac{2p}{p+1},\frac{2q}{q+1}}\Big),$$

where $h=h(x,t)=-\sqrt{\frac{B}{A}}\tan(\sqrt{AB}\,t+m)(1+|x|^2)$.

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Let $X(t)=\int\zeta^2(x)e^{2h(x,t)}u^2(x,t)dx$ and $J=\|\zeta_x e^h(1+|x|^2)^{\frac{1-\lambda}{2}}u\|_{2,2}^2$ $+\sum\|\zeta e^h f_j\|_{2,2}^2+\|\zeta e^h g\|^2_{\frac{2p}{p+1},\ \frac{2q}{q+1}}$.

Then from (3.14) we see

$$X(t)\leqq \mathscr{C}_2\{X(\tau')+J\}\ \text{ for } \tau'<t\leqq\tau'+\sigma.$$

If $(j-1)\sigma\leqq t<j\sigma$, it follows by iteration that

$$X(t)\leqq \mathscr{C}_2^j X(0)+\frac{\mathscr{C}_2(\mathscr{C}_2^j-1)}{\mathscr{C}_2-1}J\ \text{ for } (j-1)\sigma<t\leqq j\sigma. \tag{3.16}$$

On the other hand,

$$\begin{aligned}\int_{(j-1)\sigma}^{j\sigma}\int_\Omega \zeta^2e^{2h}|u_x|^2dxdt &\leqq \mathscr{C}_2\{X((j-1)\sigma)+J\}\\ &\leqq \mathscr{C}_2^j X(0)+\frac{\mathscr{C}_2^j-1}{\mathscr{C}_2-1}J.\end{aligned} \tag{3.17}$$

Suppose that $(l-1)\sigma<T'<l\sigma$ for some integer $l$, where $\sqrt{AB}\,T'+m<\frac{\pi}{2}$. Then summing (3.16) on $j$ from 1 to $l$, we have

$$\|\zeta e^h u\|^2_{2,\infty,Q_1'}\leqq \mathscr{C}_3\{X(0)+J\}.$$

Similarly from (3.16),

$$\|\zeta e^h u_x\|^2_{2,2,Q_1'}\leqq \mathscr{C}_4\{X(0)+J\}.$$

Putting $\sqrt{\dfrac{B}{A}}\tan(\sqrt{AB}\,l+m)=\mu$, and combining these two inequalities we obtain (3.3).

If $\lambda=0$, then we put $\alpha(t)=Bt+\mu_0$ and we have (3.3) in the same manner as above.

## §4. The Cauchy problem.

In this section we consider the Cauchy problem (2.1). A measurable function $u(x,t)$ on $Q$ is said to be in the class $\mathscr{E}_\mu^\lambda(Q)$ if there exist numbers $\lambda\geqq 0$ and $\mu>0$ such that

$$\iint_\Omega e^{-2\mu|x|^{2\lambda}}u^2dxdt<\infty.$$

THEOREM 1. *If there are solutions $u$ of the Cauchy problem* (2.1) *in the class $\mathscr{E}_{\mu_1}^\lambda(Q)$ for some positive constant $\mu_1<\mu_0$, then $u$ is uniquely determined in $Q$.*

*Proof. Let us assume that there are two solutions $u_1$, $u_2$ of the Problem* (2.1) *in* *the class* $\mathscr{E}^{\lambda}_{\mu_1}(Q)$. *Put* $u = u_1 - u_2$. *Then we see that* $u$ *is in* $\mathscr{E}^{\lambda}_{\mu_1}(Q)$ *and* $u$ *is a weaksolution of the problem*

$$Lu = 0 \quad \textit{for} \quad (x, t) \in Q,$$
$$u(x, 0) = 0, \quad \textit{for} \quad x \in E^n.$$

For each $R \geqq R_0$, we define a function $\zeta_R(x)$ in such a way that

(i) $\qquad |\zeta_R(x)| \leqq 1$ in $(-\infty, \infty)$,

(ii) $$\zeta_R(x) = \begin{cases} 1 & |x| \leqq R, \\ 0 & |x| \geqq R+1, \end{cases}$$

(iii) $\quad |\zeta_{R_x}(x)| \leqq C$ in $(-\infty, \infty)$, where $C$ is independent of $R$.

By Lemma 3.1,

$$\text{(4.1)} \qquad \|\zeta_R e^{-\mu(1+|x|^2)^\lambda} u\|^2_{2,\infty}$$
$$\leq \|\zeta_{R_x} e^{-\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{\frac{1-\lambda}{2}} u\|^2_{2,2,Q'}$$
$$\leqq \mathscr{C}_1 \int_\tau^{\tau'} \int_{|x| \geqq R} e^{-2\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{1-\lambda} u^2 dx dt,$$

where $\mathscr{C}_1$ is independent of $R$, and $\mu_1 < \mu_0 < \mu$.
Since

$$e^{-\mu_1(1+|x|^2)^\lambda} u \in L^2[0, T'; L^2(E^n)] \quad \text{for} \quad \mu_0 > \mu_1,$$

we see

$$e^{-\mu_0(1+|x|^2)^\lambda} (1+|x|^2)^{1-\lambda} u \in L^2[0, T'; L^2(E^n)].$$

Therefore the integral on the right in (4.1) tends to zero as $R \to \infty$. Hence

$$\max_{0 \leq t \leq \tau'} \int_{|x| \leq \rho} e^{-2\mu(1+|x|^2)^\lambda} u^2 dx = 0$$

for an arbitrary $\rho > 0$. This means that $u \equiv 0$ in $E^n \times (0, T']$ that is, $u_1 \equiv u_2$ in $E^n \times (0, T']$.

Repeating the same argument on $E^n \times (NT', (N+1)T']$ inductively, we conclude that $u_1 \equiv u_2$ in $E^n \times (0, T]$.

THEOREM 2. *Suppose that* $e^{-\mu_0|x|^{2\lambda}} f_j \in L^{2,2}(Q)$, $e^{-\mu_0|x|^{2\lambda}} g \in L^{p,q}(Q)$ *with* $p$ *and* $q$ *satisfying* (**) *and* $e^{-\mu_0|x|^{2\lambda}} u_0 \in L^2(E^n)$. *Then there exists a weak solution* $u$ *of the Cauchy problem* (2.1) *in* $Q' = E^n \times (0, T']$, *where* $T'$ *depends on the constant in*

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*Condition* (A) *and* $\mu_0$. *Moreover there exists a constant* $\mu$ *depending on* $T'$ *and* $\mu_0$ *such that*

$$\|e^{-\mu(1+|x|^2)^\lambda}u\|^2_{2,\infty,Q'} + \|e^{-\mu(1+|x|^2)^\lambda}u_x\|^2_{2,2,Q'}$$
$$\leqq \mathscr{C}\{\|e^{-\mu_0(1+|x|^2)^\lambda}u_0\|^2_{L^2(E^n)} + \sum \|e^{-\mu_0(1+|x|^2)^\lambda}f_j\|^2_{2,2,Q'}$$
$$+ \|e^{-\mu_0(1+|x|^2)^\lambda}g\|_{p,q,Q'}\}.$$

Since the proof of Theorem 2 is almost parallel to that of [1], we omit it here.

*Remark*. In Lemma 3.1, if $\lambda = 0$, then we put $\alpha(t) = Bt + \mu_0$. Thus Lemma 3 is valid for $Q = E^n \times (0, T]$ if $\lambda = 0$. Therefore if $\lambda = 0$, a weak solution of the problem (2.1) exists in $Q = E^n \times (0, T]$.

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