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Raf Bocklandt

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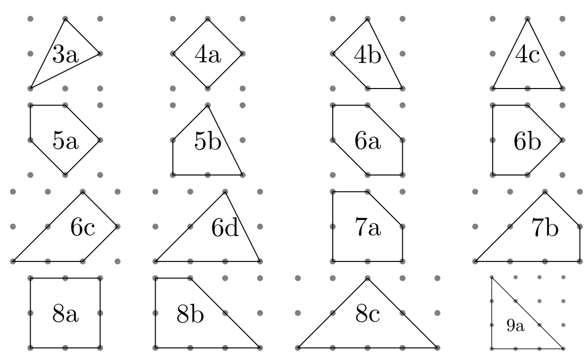
Abstract

In their paper [*Exceptional sequences of invertible sheaves on rational surfaces*, Compositio Math. **147** (2011), 1230–1280], Hille and Perling associate to every cyclic full strongly exceptional sequence of line bundles on a toric weak del Pezzo surface a toric system, which defines a new toric surface. We interpret this construction as an instance of mirror symmetry and extend it to a duality on the set of toric weak del Pezzo surfaces equipped with a cyclic full strongly exceptional sequence.

1. Reflexive polygons and weak del Pezzo surfaces

A convex integral polygon in $\mathbb{Z}^2$ that has exactly one internal lattice point is called a *reflexive polygon*. Up to integral affine transformations, there are precisely 16 reflexive polygons, which are shown below.



Fix a reflexive polygon P, let $(0, 0)$ be the internal lattice point, and let $v_1, \dots, v_k$ be the lattice points on the boundary of the polygon in cyclic order. From this polygon we can construct a toric fan

$$\{0, [v_1], \dots, [v_k], [v_1, v_2], \dots, [v_k, v_1]\}$$

where $[u_1, \dots, u_l]$ is shorthand for $\mathbb{R}^+u_1 + \cdots + \mathbb{R}^+u_l$. This fan defines a projective smooth toric surface X_{P}. This surface is a del Pezzo surface, i.e. with ample anticanonical bundle, if all of the v_i are corners of the polygon, and it is a weak del Pezzo surface otherwise.

To this fan we can associate a sequence of numbers $(a_1, \dots, a_k)$ such that

$$v_{i-1} + a_i v_i + v_{i+1} = 0,$$

and, up to cyclic shifts and inversion of the order, this sequence determines the polygon up to affine transformations and the toric variety up to isomorphism.

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In [HP11a], Hille and Perling studied full cyclic strongly exceptional sequences of line bundles on weak del Pezzo surfaces. These are infinite sequences of line bundles $\dots, \mathscr{L}_i, \mathscr{L}_{i+1}, \dots$ such that:

- $\mathtt{Ext}^r(\mathscr{L}_i, \mathscr{L}_j) = \mathtt{Ext}^r(\mathscr{L}_j, \mathscr{L}_i) = 0$ if $r > 0$ and $i \leqslant j < i + k$;
- $\mathtt{Hom}(\mathscr{L}_i, \mathscr{L}_j) = 0$ if $i > j$;
- $\mathscr{L}_{i+k} = \mathscr{L}_i \otimes \mathscr{K}^{-1}$.

Here $\mathscr{K}$ is the canonical bundle and k is the rank of the Grothendieck group, which is the same as the number of vectors v_i.

Hille and Perling classified these full cyclic strongly exceptional sequences and proved the following strange and remarkable result.

Theorem 1.1 [HP11a, Per10]. *Given a cyclic full strongly exceptional sequence $(\mathscr{L}_i)$ on a toric surface, the sequence of numbers*

$$(b_1, \dots, b_k) := (\dim \mathtt{Hom}(\mathscr{L}_i, \mathscr{L}_{i+1}) - 2, \dots, \dim \mathtt{Hom}(\mathscr{L}_{i+k-1}, \mathscr{L}_{i+k}) - 2)$$

corresponds to the sequence of a new weak del Pezzo surface.

The origin and interpretation of this new surface might seem mysterious at first; however, recent developments in the study of mirror symmetry for punctured Riemann surfaces [AAEKO11] and its relation to dimer models [Boc11] have shed new light on this issue.

To every cyclic full strongly exceptional sequence on a toric weak del Pezzo surface one can associate a consistent dimer model, which is a quiver embedded in a Riemann surface. The dimer contains enough information to recover both the surface and the exceptional collection. More precisely, the (a_i) sequence and the (b_i) sequence can be determined from the dimer model, but not the other way round. A list of all dimers coming from these full cyclic strongly exceptional sequences can be found in [Boc12a], while examples of the quivers without their dimer structure appear in [Per10].

A dimer model can be used to define two categories: a Fukaya category and a category of matrix factorizations. In [Boc11] it is shown that there is a duality on the set of dimer models, such that (under certain consistency conditions) the category of matrix factorizations is $(\mathtt{A}_\infty)$-equivalent to the Fukaya category of the dual dimer. This duality gives a combinatorial description of mirror symmetry for punctured Riemann surfaces.

The main result of this paper is that the aforementioned duality acts as an involution on the set of dimers coming from full strongly exceptional sequences of line bundles on weak toric del Pezzo surfaces and interchanges the (a_i) sequence and the (b_i) sequence. This is an extension of Hille and Perling's result in the following way: the dimer duality not only associates to a cyclic full strongly exceptional sequence of line bundles a new toric weak del Pezzo surface but also equips it with a new full strongly exceptional sequence; moreover, this process is a duality and therefore the toric system of the new exceptional sequence will give us back the original toric del Pezzo surface.

The paper is organized as follows. We start with a brief introduction to dimer models, and then we explain the phenomenon of dimer duality and its relation to mirror symmetry. In § 4 we apply this to the special situation of weak toric del Pezzo surfaces and prove our main result. We end with an illustration of the duality for reflexive polygons with eight lattice points on the boundary.

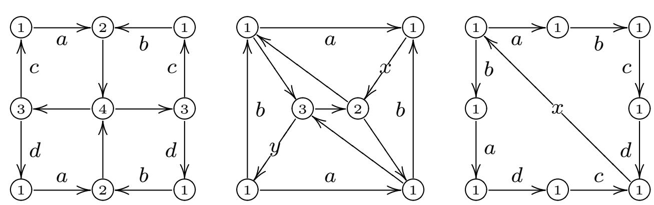
2. Dimer models

A *quiver* Q is an oriented graph. We denote the set of vertices by Q_0 and the set of arrows by Q_1; the maps h and t assign to each arrow its head and tail. Paths are defined in the usual way as sequences of arrows $a_k \dots a_0$ such that $t(a_i) = h(a_{i+1})$. A path is *cyclic* if its head and tail coincide, and a *cycle* is the equivalence class of a cyclic path up to cyclic permutation. A *trivial path* is just a vertex. The *path algebra* $\mathbb{C}Q$ is the vector space spanned by the paths with product given by the concatenation of paths.

Let S be a compact orientable surface without boundary. A quiver is said to be *embedded in* S if the vertices are a subset of S and the arrows can be viewed as smooth curves connecting the heads and tails such that the only intersections occur at endpoints. The surface in which a quiver Q is embedded will often be denoted by $|Q|$.

An embedded quiver is called a *dimer model* if the complement of the arrows is a disjoint union of open discs, each bounded by a cyclic path of length at least 3. We will call the anticlockwise boundary cycles the positive cycles and group them in a set Q_2^+, while the clockwise cycles will be grouped in Q_2^-. Note that the dimer and its surface are completely determined by the sets $Q_2^\pm$. For more information on dimers we refer to [Boc12b, Bro11, Ken04].

Example 2.1. Three dimer models are shown below. The first two are embedded in a torus, the last in a double torus. Arrows and vertices with the same label are identified.



The *Jacobi algebra of a dimer model* is the quotient of the path algebra by the ideal generated by relations of the form $r_a := r_+ - r_-$ where $r_+ a \in Q_2^+$ and $r_- a \in Q_2^-$ for some arrow $a \in Q_1$:

$$\mathtt{Jac}\,(Q) := \frac{\mathbb{C}Q}{\langle r_a \mid a \in Q_1 \rangle}.$$

Every Jacobi algebra has a central element

$$\ell = \sum_{v \in Q_0} c_v$$

where, for each vertex v, c_v is a cyclic path with $h(c_v) = t(c_v) = v$ that forms a cycle in Q_2. Using the relations, one can show that this element is indeed central and does not depend on the cycles we chose to sum.

Remark 2.2. In a more general context, Jacobi algebras can be defined using a quiver and a superpotential, which is a linear combination of cycles. Jacobi algebras are important because, if certain consistency conditions are satisfied, they provide the noncommutative analogues of Calabi–Yau-3 manifolds: their derived categories are Calabi–Yau-3, meaning that the third shift is a Serre functor. Therefore, in many cases, endomorphism rings of tilting sheaves on Calabi–Yau-3 manifolds are Jacobi algebras; we will see a specific instance in Theorem 2.8. For more information on this topic we refer to [Boc08, Gin06].

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Fix a dimer model Q and denote its universal cover by $\tilde{Q}$. The universal cover of Q embedded in S is obtained by lifting all arrows and vertices in all possible ways to the universal cover of S. This may result in an infinite quiver.

For any arrow $\tilde{a} \in \tilde{Q}_1$ we can construct its *zig ray* $\mathcal{Z}^+_{\tilde{a}}$. This is an infinite path

$$\ldots \tilde{a}_2 \tilde{a}_1 \tilde{a}_0$$

such that $\tilde{a}_0 = \tilde{a}$ and $\tilde{a}_{i+1}\tilde{a}_i$ sits in a positive cycle if i is even and in a negative cycle if i is odd. Similarly, the *zag ray* $\mathcal{Z}^-_{\tilde{a}}$ is the path where $\tilde{a}_{i+1}\tilde{a}_i$ sits in a positive cycle if i is odd and in a negative cycle if i is even. The projection of a zig ray or a zag ray down to Q will give us a cyclic path because Q is finite. Such a cyclic path will be called a *zigzag cycle*. A dimer model is said to be *zigzag consistent* if for every arrow $\tilde{a}$ the zig ray and the zag ray meet only in $\tilde{a}$:

$$(\mathcal{Z}^-_{\tilde{a}})_i = (\mathcal{Z}^+_{\tilde{a}})_j \implies i = j = 0.$$

In Example 2.1, the first and third quivers are consistent, but the second quiver is not because $(\mathcal{Z}^-_{\tilde{x}})_3 = (\mathcal{Z}^+_{\tilde{x}})_3 = \tilde{y}$.

For dimer models on a torus, there are two extra characterizations of consistency, which we will use later on.

Theorem 2.3 [Boc12b]. *For a dimer model* Q *on a torus, the following statements are equivalent.*

(i) Q *is zigzag consistent.*

(ii) Q *admits a consistent* $\mathcal{R}$*-charge; this is a map* $\mathcal{R} : Q_1 \to (0,2)$ *such that every positive or negative cycle has degree* 2 *and for every vertex* v *we have*

$$\sum_{h(a)=v} (1 - \mathcal{R}_a) + \sum_{t(a)=v} (1 - \mathcal{R}_a) = 2.$$

(iii) $\mathtt{Jac}\,(Q)$ *embeds in* $\widehat{\mathtt{Jac}}\,(Q) := \mathtt{Jac}\,(Q) \otimes_{\mathbb{C}[\ell]} \mathbb{C}[\ell, \ell^{-1}]$.

Other characterizations of consistency and similar results can be found in [Bro11, Dav11, IU08, IU11, MR10].

Remark 2.4. The algebra $\widehat{\mathtt{Jac}}\,(Q)$ can be written as the path algebra of the double quiver with relations. The double quiver contains Q as a subquiver but adds for each arrow $a \in Q_1$ an extra arrow a^{-1} in the opposite direction. The new relations are the original relations together with $aa^{-1} = h(a)$ and $a^{-1}a = t(a)$ for each arrow in Q_1. The isomorphism between the double quiver with relations and $\widehat{\mathtt{Jac}}\,(Q)$ gives the identification $a^{-1} = p\ell^{-1}$ if $ap = h(a)\ell$.

We will call paths and cycles in the double quiver *weak paths* and *weak cycles*. If we want to stress that a weak path contains only arrows of the original quiver, we will call it *real*.

Another ingredient that we need consists of the *perfect matchings*. These are subsets $\mathcal{P} \subset Q_1$ such that every positive and every negative cycle contains precisely one arrow of $\mathcal{P}$. Every perfect matching can also be seen as a degree function on $\widehat{\mathtt{Jac}}\,(Q)$ that assigns a degree 1 if $a \in \mathcal{P}$ and degree 0 otherwise. We will write $\mathcal{P}(p)$ for the degree of a weak path p. Note that for the element ℓ we have $\mathcal{P}(\ell) = 1$.

Later on we will need the following two lemmas.

Lemma 2.5 [HHV06]. *For a consistent dimer model on a torus, two paths represent the same element if they have the same homotopy class and the same degree for at least one perfect matching.*

LEMMA 2.6 [Bro11, Boc12b]. *For a consistent dimer model on a torus, there is for every homotopy class a path that has degree zero for at least one perfect matching.*

Remark 2.7. A path as in Lemma 2.6 is said to be 'minimal' because every path with a given homology class is of the form $p\ell^k$ where p is the minimal path and $k \geqslant 0$.

Fix a vertex o, which we will call the trivial vertex, and two weak cycles x and y that span the homology of the torus. To every perfect matching $\mathcal{P}$ we can associate a point $\vec{\mathcal{P}} := (\mathcal{P}(x), \mathcal{P}(y), \mathcal{P}(\ell)) \in \mathbb{Z}^3$. A set of matchings $\{\mathcal{P}_1, \ldots, \mathcal{P}_u\}$ is said to be o-stable if there is a real path from o to every other vertex that has $\mathcal{P}_i$-degree zero for all matchings in the set. The notion of o-stable coincides with that of θ-stable [Kin94] if θ is negative on o and positive on all other vertices. If it is clear which vertex we have fixed, we will abbreviate o-stable to stable. For each stable set S we can define a cone $\sigma_S = \sum_{\mathcal{P} \in S} \mathbb{R}^+ \vec{\mathcal{P}}$.

The technique of perfect matchings can be used to relate dimer models to the geometry of crepant resolutions of Gorenstein singularities.

THEOREM 2.8 (Ishii and Ueda [IU09]; Mozgovoy and Bender [BM09, Moz09]). *If* Q *is a consistent dimer on a torus, then the following hold.*

(i) *The collection of cones σ_S, where S is a stable set of matchings, forms a fan; moreover, the toric variety of this fan, $\tilde{X}$, is a crepant resolution of $X = \mathsf{Spec}\, Z(A)$.*

(ii) *If we intersect the fan with the plane at height $z = 1$, we get a convex polygon* $\mathtt{P}$ *which is subdivided into elementary triangles, and on each integral point of the polygon sits a unique stable perfect matching. These lattice points form a basis for the toric divisors of $\tilde{X}$; so any $\mathbb{Z}$-linear combination of stable perfect matchings gives us a line bundle.*

(iii) *Fix a set of paths $\{p_v\}$ from o to every other vertex v. The direct sum $\mathscr{T}$ of the line bundles $\mathscr{L}_v$ with divisors $\sum \mathcal{P}_i(p_v)\mathcal{P}_i$ (where the sum runs over all stable perfect matchings) is a tilting bundle on $\tilde{X}$, and $\mathsf{End}(\mathscr{T}) = \mathtt{Jac}\,(\mathrm{Q})$.*

The above theorem has some implications which we will need further on.

COROLLARY 2.9. *If* Q *is a consistent dimer on a torus, then the following hold.*

(i) *The number of vertices in* Q *is the number of elementary triangles in* $\mathtt{P}$.

(ii) *For each weak path p in* Q*, we can split the stable matchings into two sets, $M^+ = \{\mathcal{P} \mid \mathcal{P}(p) \geqslant 0\}$ and $M^- := \{\mathcal{P} \mid \mathcal{P}(p) < 0\}$. The subset of triangles, line segments and lattice points spanned by lattice points in* $\mathtt{P}$ *that correspond to stable sets in $M^{\pm}$ constitutes a simply connected simplicial set.*

(iii) *For each weak path p in* Q*, we can split the stable* boundary matchings *into two sets, $B^{\pm} := M^{\pm} \cap \partial\mathtt{P}$. The subset of line segments and lattice points spanned by lattice points in $\partial\mathtt{P}$ that correspond to stable sets in $B^{\pm}$ constitutes a connected simplicial set (i.e. a circle segment or the whole circle).*

(iv) *The number of zigzag paths in* Q *is the number of elementary line segments on the boundary of* $\mathtt{P}$*. More precisely, if $\{\mathcal{P}_1, \mathcal{P}_2\}$ is a stable set of two perfect matchings on the boundary, then the arrows contained in either $\mathcal{P}_1$ or $\mathcal{P}_2$ but not in both form a zigzag path; all zigzag paths arise in this way.*

(v) *The stable perfect matchings on the boundary that contain a given arrow are precisely the ones that lie between the elementary line segments of this arrow's zig path and zag path.*

Proof. Statement (i) follows because the derived equivalence gives an equality between the ranks of the Grothendieck groups; for the Jacobi algebra this rank is the number of vertices in the quiver, and for the crepant resolution it is the number of elementary triangles needed to subdivide $\mathtt{P}$.

Statement (ii) can be proved using simplicial homology. Let p be any weak path in the dimer with $h(p)=v$ and $t(p)=w$; then $\sum \mathcal{P}_i(p)\mathcal{P}_i$ is a line bundle equivalent to $\mathscr{L}_v\mathscr{L}_w^{-1}$. Because $\mathscr{T}$ is tilting, we have that $\mathscr{L}_v\mathscr{L}_w^{-1}$ has no higher homology. The homology can be calculated from the complex $\mathcal{F}^\bullet, \delta$ where

$$\mathcal{F}^r := \bigoplus_{|S|=r} \mathbb{C}\big[X^{m_1}Y^{m_2}Z^{m_3} \mid m_1\mathcal{P}(x)+m_2\mathcal{P}(y)+m_3 \geqslant -\mathcal{P}(p)\ \forall \mathcal{P}\in S\big],$$

with the sums being taken over the stable sets of matchings, and

$$\delta(X^{m_1}Y^{m_2}Z^{m_3})_S = \sum_{\mathcal{P}\in S} \pm (X^{m_1}Y^{m_2}Z^{m_3})_{S\setminus\{\mathcal{P}\}}$$

is the boundary map between these simplicial sets.

If we look at the summand corresponding to $X^{m_1}Y^{m_2}Z^{m_3}=1$, we see that this complex calculates the simplicial homology of the simplicial subcomplex containing only the $S\subset M^+$. This is acyclic if and only if the subcomplex is simply connected. To get the statement for M^-, we need to look at the weak path $p^{-1}\ell^{-1}$.

Assertion (iii) is an easy consequence of (ii): if B^+ is not connected, then either M^+ or M^- is not connected.

Statement (iii) also implies that for two consecutive arrows a and b in a cycle, the segments B^+ are disjoint (because they sit in the same cycle) and adjacent to each other (because their union forms the segment $B^+(ab)$). This implies, furthermore, that if one looks at the segments of the arrows in a positive or negative cycle, they will cover the boundary of the polygon in a cyclic way.

Suppose we have two stable boundary matchings $\mathcal{P}_1$ and $\mathcal{P}_2$ next to each other. In each cycle, either both matchings mark the same arrow or, by the previous discussion, they mark consecutive arrows.

Therefore the arrows in $(\mathcal{P}_1\cup\mathcal{P}_2)\backslash(\mathcal{P}_1\cap\mathcal{P}_2)$ must form a union of zigzag cycles. If they formed more than one zigzag cycle, then $\{\mathcal{P}_1,\mathcal{P}_2\}$ could not be stable because the complement of $\mathcal{P}_1\cup\mathcal{P}_2$ would consist of more than one connected component and hence there could not be a path from the trivial vertex to every other vertex.

Now, given a zigzag path $\mathcal{Z}$, we can construct a stable set of two perfect matchings. If q is a minimal path from the trivial vertex to any other vertex, we define $\mathcal{P}^+(q)$ to be the number of times q intersects $\mathcal{Z}$ in an even arrow (i.e. $(\mathcal{Z})_i$ with $i\in 2\mathbb{Z}$), while $\mathcal{P}^-$ is defined analogously using the odd arrows.

For any other path p, let q_1 and q_2 be two minimal paths connecting the trivial vertex with the endpoints of p such that $q_1^{-1}pq_2$ is contractible. If $q_1^{-1}pq_2=\ell^k o$, then we set $\mathcal{P}^\pm(p)= k+\mathcal{P}^\pm(q_1)-\mathcal{P}^\pm(q_2)$. It is easy to check that $\mathcal{P}^\pm$ are well-defined perfect matchings. They form a stable pair because the zigzag path cuts the torus in a cylinder, which is still strongly connected, so there is a minimal path from o to any other vertex that does not intersect the zigzag path. The $\mathcal{P}^\pm$ are boundary matchings because there is a cycle that has degree zero for both, namely the opposite cycle to the zigzag path. This completes the proof of statement (iv).

Statement (v) follows easily from statements (iii) and (iv): it is clear that the arrow sits in just one of the two perfect matchings of the stable pair corresponding to its zig ray (and zag ray). The fact that $B^+(a)$ is a segment does the rest. □

3. Mirror symmetry for dimers

In general, homological mirror symmetry conjectures an equivalence between two categories, one constructed from algebraic geometry and one from symplectic geometry. Originally, Kontsevich conjectured in [Kon95] an equivalence between the derived Fukaya category of a compact symplectic Calabi–Yau manifold and the derived category of coherent sheaves of a mirror manifold which is compact complex Calabi–Yau. Over the years it turned out that this conjecture is part of a set of equivalences which are much broader than the compact Calabi–Yau setting [Abo09, AKO06, AKO08, HV07, Kat07]. Removing the compactness or the Calabi–Yau condition often makes the mirror a singular object, which physicists call a Landau–Ginzburg model [Orl04, Orl06].

In the case that we will be considering, both categories can be constructed explicitly from a dimer model. In this section we will summarize the main results of [Boc11], which are a generalization of those in [AAEKO11].

3.1 The Fukaya category

If Q is a dimer model, we can look at its wrapped Fukaya category. For more information on general wrapped Fukaya categories we refer to [AS10]. Objects in this category are the arrows of the quiver, which we consider as Lagrangian submanifolds of the underlying surface punctured by the vertices. Morphisms between Lagrangians are given by time-one flow curves of a Hamiltonian flow on the surface that connect these Lagrangians. The products are given by counting certain maps from the disc to the surface such that the boundary of the disc lies on the Lagrangians and the time-one flow curves. This produces an $\mathtt{A}_\infty$-category $\mathtt{fuk}(\mathrm{Q})$.

3.2 Matrix factorizations

To each arrow a in a dimer we can associate a matrix factorization of $\ell \in \mathtt{Jac}\,(\mathrm{Q})$, which is a diagram of the form

$$\bar{P}_a := \mathtt{Jac}\,(\mathrm{Q})h(a) \underset{\bar{a}}{\overset{a}{\rightleftarrows}} \mathtt{Jac}\,(\mathrm{Q})t(a)$$

where $\bar{a}$ is defined such that $a\bar{a} \in \mathrm{Q}_2^+$. Note that $\bar{P}_a$ can also be viewed as a $\mathbb{Z}_2$-graded projective $\mathtt{Jac}\,(\mathrm{Q})$-module with a curved differential d such that $d^2 = \ell$.

Given two such matrix factorizations $\bar{P}$ and $\bar{Q}$, the space $\mathtt{Hom}_{\mathtt{Jac}\,(\mathrm{Q})}(\bar{P}, \bar{Q})$ becomes equipped with an ordinary (non-curved) differential δ. The category $\mathtt{H\,mf}(\mathrm{Q})$ contains as objects the matrix factorizations $\bar{P}_a$ and as hom-spaces the homology of δ on $\mathtt{Hom}(\bar{P}, \bar{Q})$. The dg-structure on $\mathtt{Hom}_{\mathtt{Jac}\,(\mathrm{Q})}(\bar{P}, \bar{Q})$ can be turned into an $\mathtt{A}_\infty$-structure on $\mathtt{H\,mf}(\mathrm{Q})$.

3.3 Dimer duality

The two categories defined above can be related by a certain duality on the level of dimers. Let Q be any, not necessarily consistent, dimer. We define its mirror dimer $\check{\mathrm{Q}}$ as follows.

(i) The vertices of $\check{\mathrm{Q}}$ are the zigzag cycles of Q.

(ii) The arrows of $\check{\mathrm{Q}}$ are the arrows of Q, $h(a)$ is the zigzag cycle coming from the zig ray, and $t(a)$ is the cycle coming from the zag ray.

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(iii) The positive faces of $\check{Q}$ are the positive faces of Q.

(iv) The negative faces of $\check{Q}$ are the negative faces of Q in reverse order.

Some relevant examples of dimer duality can be found in Table 1 in the final section of the paper.

Remark 3.1. The dual can also be obtained by cutting the dimer along the arrows, flipping over the clockwise faces, reversing their arrows, and then gluing everything back again. This construction is basically the same as the one introduced by Feng *et al.* in [FHKV08], but applied to all possible dimers. It is also important to note that the genus of the surface in which the dual dimer lives often differs from the genus of the surface of the original dimer.

Theorem 3.2 [Boc11]. *If* Q *is a consistent dimer, then the categories* $\mathtt{H\,mf}(Q)$ *and* $\mathtt{fuk}(\check{Q})$ *are* $\mathtt{A}_\infty$*-isomorphic.*

An $\mathtt{A}_\infty$-category can be completed by adding twisted objects, which are morally complexes of the old ones (see [Kel01]). We also need to add in projectors to ensure that the category is idempotent complete. If we apply this process to both categories above, we get two categories $\mathtt{D}^\pi\mathtt{Fuk}(Q)$ and $\mathtt{D}^\pi\mathtt{mf}(Q)$. The first one only depends on the surface in which Q is embedded and the number of punctures (i.e. vertices in Q); we call it the idempotent completed wrapped Fukaya category of the punctured surface.

If Q sits on a torus, then, by a theorem of Ishii and Ueda [IU09], the second category is equivalent to the idempotent completion of a category of singularities

$$\mathtt{DSing} f^{-1}(0) := \frac{\mathtt{D}^b\mathtt{Coh}\, f^{-1}(0)}{\mathtt{Perf}\, f^{-1}(0)},$$

where $f : \tilde{X} \to \mathbb{C} : p \mapsto \ell(p)$ and $\tilde{X}$ is a crepant resolution of $\mathsf{Spec}\, Z(\mathtt{Jac}\,(Q))$.

In this way we recover a version of mirror symmetry which equates the idempotent completed category of singularities of $f^{-1}(0)$ to the idempotent completed wrapped Fukaya category of a surface with genus $\frac{1}{2}(2 - \#\check{Q}_0 + \#Q_0)$ and $\#\check{Q}_0$ punctures. Details of this equivalence will be provided in a follow-up paper to [Boc11].

4. Weak del Pezzo dimers

4.1 From surface to dimer

Now we return to the setting of weak toric del Pezzo surfaces. Let $v_1, \ldots, v_k \in \mathbb{Z}^2$ be the vectors in cyclic order that define a weak toric del Pezzo surface X, and let $(\mathscr{L}_i)_{i\in\mathbb{Z}}$ be a cyclic full strongly exceptional sequence on X. We can represent each $\mathscr{L}_i$ by a divisor $l_{i1}E_1 + \cdots + l_{ik}E_k$. Here E_j is the divisor corresponding to the vector v_j and $\mathscr{E}_i = \mathcal{O}(E_i)$ will be its corresponding line bundle.

How do we get a dimer model out of these data? Let $\tilde{Y}$ be the total space of the canonical bundle $\mathscr{K}$ on X, and denote the natural projection by $\pi : \tilde{Y} \to X$. From toric geometry we know that the fan of Y can be constructed from the fan of X in the following way. We lift every vector v_i to a vector $\tilde{v}_i = (v_{i1}, v_{i2}, 1)$; these points form a polygon $\mathtt{P}$ in the plane with third coordinate equal to 1. Let $z = (0, 0, 1)$ be the unique internal lattice point of this polygon. The maximum-dimensional cones of the fan of $\tilde{Y}$ are then

$$[\tilde{v}_1, \tilde{v}_2, z], [\tilde{v}_2, \tilde{v}_3, z], \ldots, [\tilde{v}_{k-1}, \tilde{v}_k, z], [\tilde{v}_k, \tilde{v}_1, z].$$

Because all the vectors $\tilde{v}_i$ and z lie in the same plane, $\tilde{Y}$ is a local Calabi–Yau 3-fold. Moreover, this variety is a crepant resolution of an affine variety Y generated by the cone $[\tilde{v}_1, \ldots, \tilde{v}_k]$.

To generate this cone, we only need the $\tilde{v}_i$ that lie on the corners of the polygon. In terms of the (a_i) sequence of the polygon, these are the $\tilde{v}_i$ for which $a_i \neq -2$. This gives us the following formula for the coordinate ring of Y:

$$\mathbb{C}[Y] := \mathbb{C}\big[x^{m_1}y^{m_2}z^{m_3} \mid \forall\mu : a_\mu = -2 : \langle m, \tilde{v}_\mu\rangle \geqslant 0\big].$$

A theorem of Bridgeland (see [Bri05]) states that if $(\mathscr{L}_i)_{i\in\mathbb{Z}}$ is a cyclic full strongly exceptional sequence on X, then for any $i \in \mathbb{Z}$ the direct sum

$$\mathscr{T} = \bigoplus_{j=i}^{i+k-1} \pi^*\mathscr{L}_j$$

forms a tilting bundle on $\tilde{Y}$. Different choices of i will give isomorphic tilting bundles. The Picard group of $\tilde{Y}$ is generated by the toric divisors $\tilde{E}_i$ corresponding to the $\tilde{v}_i$ and an extra divisor Z coming from z. The pullback of a line bundle over X with divisor $l_{i1}E_1 + \cdots + l_{ik}E_k$ results in a line bundle with divisor $l_{i1}\tilde{E}_1 + \cdots + l_{ik}\tilde{E}_k$.

Because $\tilde{Y}$ is a toric Calabi–Yau-3 variety and $\mathscr{T}$ is a direct sum of line bundles, $\mathsf{B} = \mathsf{End}\,\mathscr{T}$ is a toric Calabi–Yau-3 order in the sense of [Boc09]. The main theorem in [Boc09] implies that B is the Jacobi algebra of a consistent dimer model Q on a torus and a noncommutative crepant resolution of Y in the sense of [vdB02]. This is an endomorphism ring of reflexive $\mathbb{C}[Y]$-modules with global dimension equal to the dimension of Y. For every $\pi^*\mathscr{L}_j$ we get a reflexive $\mathbb{C}[Y]$-module

$$L_j := \mathrm{Span}_{\mathbb{C}}\big[x^{m_1}y^{m_2}z^{m_3} \mid \forall\mu : a_\mu = -2 : \langle m, \tilde{v}_\mu\rangle + l_{j\mu} \geqslant 0\big].$$

The endomorphism ring of the direct sum of these reflexives, $\mathsf{End}_{\mathbb{C}[Y]} \bigoplus_{j=i}^{i+k-1} L_j$, is isomorphic to B. The algebra B is also called the rolled-up helix algebra; see [BS09].

Analogously to [Boc12a], we can construct the dimer for B in the following way. Let u be the number of corner vertices of the polygon and identify the lattice $\mathbb{Z}^u \subset \mathbb{R}^u$ with the set of reflexive $\mathbb{C}[Y]$-modules of the form

$$T_a := \mathbb{C}\big[x^{m_1}y^{m_2}z^{m_3} \mid \forall\mu : a_\mu = -2 : \langle m, \tilde{v}_\mu\rangle + a_{j\mu} \geqslant 0\big] \quad \text{where } a \in \mathbb{Z}^u.$$

Now let $\tilde{Q}_0$ be the subset of $\mathbb{Z}^u$ that corresponds to modules isomorphic to L_i for some i. Draw an arrow between a and b in $\tilde{Q}_0$ if $T_a \subset T_b$ (or, equivalently, if $a_i \leqslant b_i$ for all $i \leqslant u$) and there is no other $c \in \tilde{Q}_0$ with $T_a \subset T_c \subset T_b$. Factor out the equivalence relation generated by $a \equiv b \iff T_a \cong T_b$. This projects our infinite quiver down to a finite quiver Q. The positive and negative cycles of Q are all paths of $\mathbb{Z}^u$-degree $(1,\ldots,1)$.

Example 4.1 (The projective plane blown up in three points). The toric fan of this del Pezzo surface has six two-dimensional cones as shown in Figure 1. The full cyclic strongly exceptional sequence is obtained by extending the six line bundles to an infinite sequence by tensoring with powers of $\mathscr{K}$. An arrow corresponding to an embedding $T_a \subset T_b$ is labelled by the vector $b - a$.

4.2 From dimer to surface

A dimer on a torus is said to be *weak del Pezzo* if it is consistent and its polygon has one internal lattice point. This means that the number of elementary triangles in the polygon P equals the number of elementary segments on the boundary, which, by Corollary 2.9, is equivalent to the property that the number of zigzag paths equals the number of vertices. By choosing the weak paths x and y (used to construct the lattice points for the perfect matchings) carefully, we can

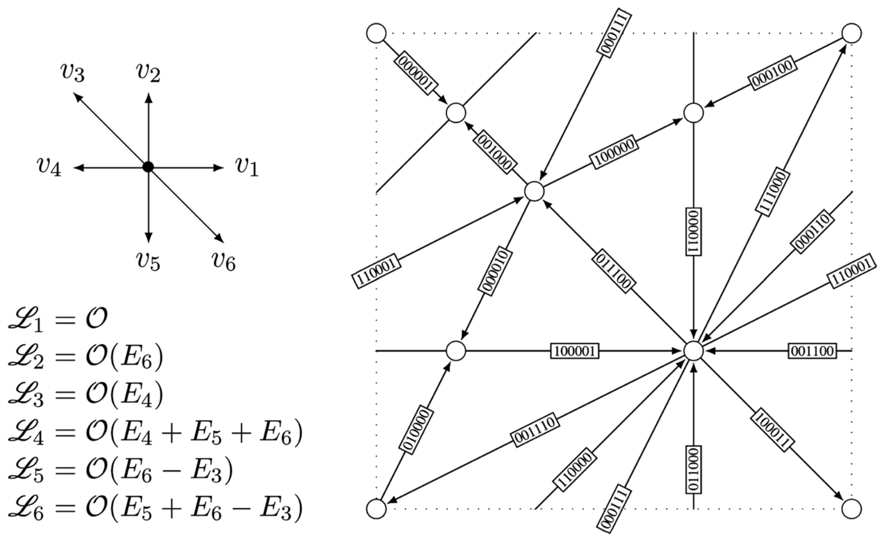

Figure 1. A dimer model for the projective plane blown up in three points.

assume that the internal vertex of the polygon is $(0, 0, 1)$. We order the boundary lattice points of the polygon cyclically and assume they have coordinates $\tilde{v}_i := (v_i, 1)$ where $v_i \in \mathbb{Z}^2$.

To any weak del Pezzo dimer Q we will associate the toric del Pezzo surface X_{Q} defined by the v_i. This gives us a sequence (a_i) which can be reconstructed from the zigzag cycles. Fix a trivial vertex o and use this to assign to each $(v_i, 1)$ a stable perfect matching $\mathcal{P}_i$ and to each elementary line segment between $(v_i, 1)$ and $(v_{i+1}, 1)$ a zigzag path $\mathcal{Z}_i$ (see Theorem 2.8 and Corollary 2.9).

Proposition 4.2. *The sequence (a_i) of the weak toric del Pezzo surface X_{Q} is the same as the sequence $(k_i - 2)$ where k_i is the number of common arrows between the ith and $(i+1)$th zigzag cycles.*

Proof. The ith zigzag path that corresponds to the boundary segment $\tilde{v}_{i-1}\tilde{v}_i$ points in the direction perpendicular to $v_i - v_{i-1}$. Therefore the intersection number on the torus between the ith and $(i+1)$th zigzag cycles can be calculated as

$$\begin{aligned}\det \begin{pmatrix} v_i - v_{i-1} \\ v_{i+1} - v_i \end{pmatrix} &= \det \begin{pmatrix} v_i \\ v_{i+1} \end{pmatrix} + \det \begin{pmatrix} -v_{i-1} \\ v_{i+1} \end{pmatrix} + \det \begin{pmatrix} v_i \\ -v_i \end{pmatrix} + \det \begin{pmatrix} -v_{i-1} \\ -v_i \end{pmatrix} \\ &= 1 + \det \begin{pmatrix} a_i v_i + v_{i+1} \\ v_{i+1} \end{pmatrix} + 0 + 1 \\ &= 2 + a_i.\end{aligned}$$

The intersection number between two consecutive zigzag paths is always equal to the number of joint arrows. If this were not the case, the zigzag paths would have to cross each other in different directions (both from left to right and from right to left), which would further imply that there is an arrow a_1 for which the first zigzag cycle is a zig ray and the second a zag ray, as well as an arrow a_2 for which the first zigzag cycle is a zag ray and the second a zig ray. By Corollary 2.9(v), one of these arrows would be contained in all stable boundary perfect matchings except one (while the other is only contained in a single one). Now each arrow is contained in at least one boundary matching and each perfect matching contains just one arrow of a positive or

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negative cycle. Therefore a_1 or a_2 would sit in a positive cycle of at most two arrows, but this is forbidden by our definition of dimer model. □

We now construct a full strongly exceptional sequence on this weak del Pezzo surface. Fix a trivial vertex o and let E_i be the divisor corresponding to the vector v_i and o-stable perfect matching $\mathcal{P}_i$. Choose a degree function $\mathcal{R} = \lambda_1\mathcal{P}_1 + \cdots + \lambda_n\mathcal{P}_n$ with $\lambda_i > 0$ such that $(\mathcal{R}(x), \mathcal{R}(y), \mathcal{R}(\ell)) = (0, 0, 2)$ (this is possible because $(0, 0, 2)$ is in the cone spanned by the polygon).

All cyclic paths in the dimer have an $\mathcal{R}$-degree that is an even integer. Each vertex v can be given a number $\mathcal{R}(p)$ with $h(p) = o$ and $t(p) = v$. This number is uniquely defined in $\mathbb{R}/2\mathbb{Z}$, and we can use it to give a cyclic order to the vertices of Q. Let $w_1, \ldots, w_k$ be these vertices in cyclic order, starting with $w_1 = o$.

Now put $\mathscr{L}_1 = \mathcal{O}_X$ and define $\mathscr{L}_{i\pm1}$ inductively from $\mathscr{L}_i$ as follows. If p is a weak path from w_i to $w_{i\pm1}$ with $0 \leqslant \mathcal{R}(p) < 2$, then set

$$\mathscr{L}_{i\pm1} = \mathscr{L}_i \otimes \mathscr{E}_1^{\pm\mathcal{P}_1(p)} \otimes \cdots \otimes \mathscr{E}_k^{\pm\mathcal{P}_k(p)}.$$

Proposition 4.3. *For any weak del Pezzo dimer* Q *with trivial vertex* o*, the sequence of line bundles* $(\mathscr{L}_i)$ *as constructed above is a well-defined cyclic strongly exceptional sequence on* X_{Q}*, and its sequence* (b_i) *is given by* $(\#\{a \in \mathrm{Q}_1 \mid h(a) = w_{i+1}, t(a) = w_i\} - 2)$.

Proof. We first have to show that the sequence is well-defined. If we had chosen a different weak path p', then $\mathcal{R}(p') = \mathcal{R}(p)$ because $\mathcal{R}(c)$ of any cycle is an even integer. Therefore

$$\begin{aligned}\mathscr{L}_{i+1}^{-1}\mathscr{L}'_{i+1} &= \mathscr{E}_1^{\mathcal{P}_1(p')-\mathcal{P}_1(p)} \otimes \cdots \otimes \mathscr{E}_k^{\mathcal{P}_k(p')-\mathcal{P}_k(p)} \\ &= \mathscr{E}_1^{\langle v_1,(i,j)\rangle} \otimes \cdots \otimes \mathscr{E}_k^{\langle v_k,(i,j)\rangle} \cong \mathcal{O} \quad \text{in } \mathsf{Pic}X_{\mathrm{Q}}.\end{aligned}$$

To show that the sequence is strongly exceptional, it suffices to show that for any weak path p with $0 \leqslant \mathcal{R}(p) < 2$, the bundle

$$\mathscr{L}(p) := \mathscr{E}_1^{\mathcal{P}_1(p)} \otimes \cdots \otimes \mathscr{E}_k^{\mathcal{P}_k(p)}$$

has $H^i(\mathscr{L}(p)) = 0$ for all $i \neq 0$ and $H^i(\mathscr{L}(p^{-1})) = 0$ for all i.

The cohomology of $\sum_i \mathcal{P}_i(p)E_i$ can be computed using the complex $\mathcal{F}^2 \xrightarrow{\delta} \mathcal{F}^1 \xrightarrow{\delta} \mathcal{F}^0$ where

$$\mathcal{F}^r := \bigoplus_{|S|=r, S\subset\partial\mathrm{P}} \mathbb{C}[X^iY^j \mid i\mathcal{P}(x) + j\mathcal{P}(y) \geqslant -\mathcal{P}(p)\ \forall\mathcal{P} \in S],$$

with the sums being taken over the stable sets of boundary matchings, and

$$\delta(X^iY^j)_S = \sum_{\mathcal{P}\in S} \pm(X^iY^j)_{S\setminus\{\mathcal{P}\}}$$

is the boundary map between these simplicial sets. There is homology with $\mathbb{Z}^2$-degree (i, j) if and only if the simplicial subcomplex of boundary perfect matchings for which $i\mathcal{P}(x) + j\mathcal{P}(y) \geqslant -\mathcal{P}(p)$ has homology.

We can rephrase these conditions in terms of $p' = pX^iY^j$: the homology $H^u(\mathscr{L}(p))$ is nonzero if and only if there is a weak path p' with $h(p) = h(p')$, $t(p) = t(p')$ and $\mathcal{R}(p) = \mathcal{R}(p')$ such that

$$\begin{cases} \mathcal{P}_r(p') \geqslant 0 \text{ for all } r & \text{if } u = 0, \\ \text{the } r \text{ for which } \mathcal{P}_r(p') \geqslant 0 \text{ do not form a connected segment in } \mathbb{Z}/k\mathbb{Z} & \text{if } u = 1, \\ \mathcal{P}_r(p') < 0 \text{ for all } r & \text{if } u > 1. \end{cases}$$

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So $H^2(p') = 0$ because $\mathcal{R}(p') \geqslant 0$, while $H^1(p') = 0$ because of Corollary 2.9(iii). If $0 < \mathcal{R}(p) < 1$, then $H^0(p^{-1}) = 0$ because, by Poincaré duality, $H^0(p^{-1}) = H^2(p\ell) = 0$.

To prove the statement about the sequence (b_i), we need to show that $\dim \mathtt{Hom}(\mathscr{L}_i, \mathscr{L}_{i+1})$ equals the number of arrows between w_i and w_{i+1}. In other words, none of the morphism factors. This is indeed true because the $\mathcal{R}$-degree of every morphism is positive but there is no vertex intermediate between w_i and w_{i+1}. □

Remark 4.4. If we start with a weak del Pezzo dimer, construct its surface and full cyclic strongly exceptional sequence and use this to construct a dimer again, we end up with the dimer we started with. This is true because $L_j := \Gamma(\pi^* \mathcal{L}_j) \cong T_{a^p}$, where $a^p_\mu = \mathcal{P}_\mu(p)$ for any path p from o to the vertex w_j. So, for every path p starting at o, we get a module T_{a^p}, and $T_{a^p} \subset T_{a^q}$ if $a^{p^{-1}q} \geqslant 0$. The latter condition implies that $p^{-1}q$ must be a real path in the dimer. If not, let $k > 0$ be the minimal power for which $pq^{-1}\ell^k$ is real. From [Boc12b, Theorem 8.7] we know that there is a corner perfect matching $\mathcal{P}_\mu$ for which $\mathcal{P}_\mu(pq^{-1}\ell^k) = 0$, but this would imply that $a^{pq^{-1}}_\mu < 0$. We see that there is a bijection between the real paths in the dimer and the embeddings $T_{a^p} \subset T_{a^q}$, and hence the embeddings that do not factor correspond to the arrows of the original dimer.

Remark 4.5. Normally one would first construct a tilting bundle on the total space Y using an appropriate stability condition and then restrict this to a strongly exceptional sequence on the zero fibre X. However, we prefer not to do this, because in general it is not so straightforward to cook up the stability condition that does the trick. This can be done using work by Craw and Ishii [CI04]. In general, the notion of o-stability does not give a moduli space of representations that is isomorphic to the total space Y. But the o-stable perfect matchings are sufficient to generate the strongly exceptional sequence without constructing a moduli space of θ-stable representations.

Theorem 4.6. *If* Q *is weak del Pezzo, then* $\check{\mathrm{Q}}$ *is also weak del Pezzo with* $(a_i)^{\check{\mathrm{Q}}} = (b_i)^{\mathrm{Q}}$ *and* $(b_i)^{\check{\mathrm{Q}}} = (a_i)^{\mathrm{Q}}$.

Proof. Because we already know that the number of zigzag paths equals the number of vertices and dimer duality interchanges the two quantities, the Euler characteristic does not change. So both the dimer and its dual are embedded in a torus.

Now we prove that $\check{\mathrm{Q}}$ is consistent. Take a zigzag path $\mathcal{Z}$ in the original dimer Q. By Corollary 2.9(iv), we can find o-stable perfect matchings $\mathcal{P}_1$ and $\mathcal{P}_2$ on the boundary of the polygon such that $\mathcal{Z} = (\mathcal{P}_1 \cup \mathcal{P}_2) \backslash (\mathcal{P}_1 \cap \mathcal{P}_2)$. Let $\mathcal{O}$ be the left opposite path of $\mathcal{Z}$; this consists of all the rest of the arrows of the positive cycles that meet $\mathcal{Z}$, except for those in $\mathcal{Z}$ itself. This path has degree-zero $\mathcal{P}_1$ and $\mathcal{P}_2$, and hence it is not a multiple of ℓ. It can identified with the monomial $X^iY^jZ^k \in Z(\mathtt{Jac}\,(\mathrm{Q}))$ with $(-i, -j)$ being the homology class of $\mathcal{Z}$ and k as small as possible. Because $\mathtt{P}$ is a polygon with one internal lattice point, the dual cone must also be generated by a polygon with one internal lattice point; so $k = 1$.

Now $\mathcal{Z} = \mathcal{O}^{-1}\ell^{z/2}$ where z is the length of $\mathcal{Z}$, and the zigzag path corresponds to the element $X^{-i}Y^{-j}Z^{-1+z/2}$. Choose a degree function $\mathcal{R} = \lambda_1\mathcal{P}_1 + \cdots + \lambda_n\mathcal{P}_n$ with $\lambda_i > 0$ such that $(\mathcal{R}(x), \mathcal{R}(y), \mathcal{R}(\ell)) = (0, 0, 2)$ (remember that this is possible because $(0, 0, 2)$ is in the cone spanned by the polygon). Then $\mathcal{R}(\mathcal{Z}) = 2 + z$ and we can write this as

$$\sum_{a \in \mathcal{Z}} \mathcal{R}(a) = -2 + z \quad \text{or} \quad \sum_{a \in \mathcal{Z}} (1 - \mathcal{R}(a)) = 2.$$

If we look at this condition from the point of view of the dual dimer $\check{\mathrm{Q}}$, it is precisely the condition for a consistent $\mathcal{R}$-charge.

The (a_i) and (b_i) sequences are interchanged because the duality interchanges vertices and zigzag paths. □

4.3 A categorical point of view

Using the dimer duality and the fact that both the dimer and its dual are consistent, we get two A_∞-isomorphisms of A_∞-categories,

$$\mathtt{fuk}(\mathrm{Q}) \cong_\infty \mathtt{H\,mf}(\check{\mathrm{Q}}), \quad \mathtt{fuk}(\check{\mathrm{Q}}) \cong_\infty \mathtt{H\,mf}(\mathrm{Q}).$$

Following [Boc11], we can go over to the derived versions of the twisted completions of all these categories. In this way we get equivalences between the derived wrapped Fukaya category of the surface with punctures $|\mathrm{Q}|\backslash\mathrm{Q}_0$ and the category of singularities of $\mathtt{Jac}\,(\mathrm{Q})/(\ell)$. The latter is, by definition,

$$\mathtt{DSing\,Jac}\,(\mathrm{Q})/(\ell) := \frac{\mathtt{D}^b\mathtt{Mod\,Jac}\,(\mathrm{Q})/(\ell)}{\mathtt{Perf\,Jac}\,(\mathrm{Q})/(\ell)}.$$

By a theorem of Ishii and Ueda (see [IU09]), the category of singularities of $\mathtt{Jac}\,(\mathrm{Q})/(\ell)$ is also equivalent to the category of singularities of $f_{\mathrm{Q}}^{-1}(0)$, where $f_{\mathrm{Q}} : \tilde{Y} \to \mathbb{C}$ corresponds to the coordinate function $\ell \in \mathbb{C}[Y] = Z(\mathtt{Jac}\,(\mathrm{Q}))$. So

$$\mathtt{DFuk}(|\mathrm{Q}|\backslash\mathrm{Q}_0) \cong \mathtt{DSing}\, f_{\check{\mathrm{Q}}}^{-1}(0), \quad \mathtt{DFuk}(|\check{\mathrm{Q}}|\backslash\check{\mathrm{Q}}_0) \cong \mathtt{DSing}\, f_{\mathrm{Q}}^{-1}(0).$$

Now both $|\mathrm{Q}|\backslash\mathrm{Q}_0$ and $|\check{\mathrm{Q}}|\backslash\check{\mathrm{Q}}_0$ are tori with the same number of punctures, so we get that the four completed categories above are all equivalent. But, as there is no prescribed isomorphism between the two tori, there seems to be no canonical isomorphism between $\mathtt{DSing}\, f_{\check{\mathrm{Q}}}^{-1}(0)$ and $\mathtt{DSing}\, f_{\mathrm{Q}}^{-1}(0)$.

One expects, however, to be able to identify two objects in $\mathtt{DSing\,Jac}\,(\mathrm{Q})/(\ell)$, one with ext-algebra equal to $\mathtt{H\,mf}(\mathrm{Q})$ (considered as an algebra) and one with ext-algebra $\mathtt{H\,mf}(\check{\mathrm{Q}})$. The former is $\mathtt{Jac}\,(\mathrm{Q})/\bar{\mathcal{J}} = \bigoplus_{v\in\mathrm{Q}_0} S_v$ viewed as a $\mathtt{Jac}\,(\mathrm{Q})/(\ell)$-module, because one can easily check that the resolution of $S_v = v\mathtt{Jac}\,(\mathrm{Q})/\bar{\mathcal{J}}$ stabilizes to

$$\bigoplus_{a,h(a)=v} \bar{P}_a \otimes_{\mathtt{Jac}\,(\mathrm{Q})} \mathtt{Jac}\,(\mathrm{Q})/(\ell).$$

We expect the latter to be the direct sum

$$\bigoplus_{a,\mathcal{Z}_a^+=\mathcal{Z}} \bar{P}_a \otimes_{\mathtt{Jac}\,(\mathrm{Q})} \mathtt{Jac}\,(\mathrm{Q})/(\ell)$$

where $\mathcal{Z}$ is a zigzag path. At the moment it is not clear whether one can find objects $S_{\mathcal{Z}}$ in $\mathtt{Mod\,Jac}\,(\mathrm{Q})/(\ell)$ (or $\mathtt{Coh}\, f_{\mathrm{Q}}^{-1}(0)$) that stabilize to these.

Remark 4.7. The categorical interpretation also indicates why the construction only works for dimers coming from toric weak del Pezzo surfaces. For other dimers, the dual lives on a surface with different genus, and hence there cannot be an equivalence between all four categories.

On the other hand, while tilting objects for all rational surfaces have been constructed in [HP11b], these usually do not come from cyclic full strongly exceptional sequences; therefore a generalization outside the weak del Pezzo case does not seem likely. Even for non-toric weak del Pezzo surfaces it is not obvious, because then the rolled-up helix algebra does not come from a dimer model any more, and hence its quiver does not embed in a torus. Also, it is not even clear what the mirror of the singular locus of $\mathtt{B}/\ell$ is.

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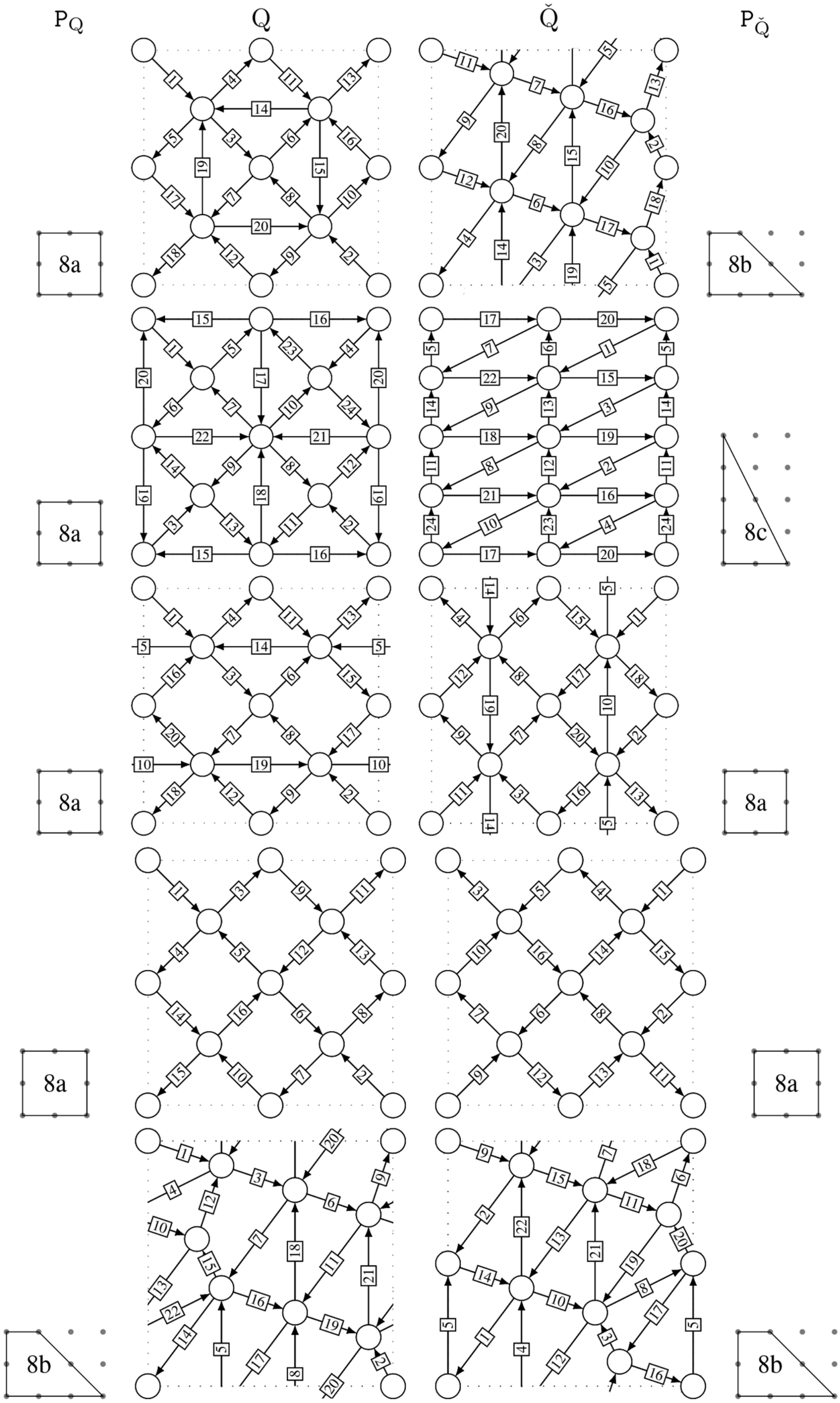
Table 1. Dimers and their corresponding duals for reflexive polygons of size 8.



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The best chance of generalizing this result is to increase the dimension and consider full strongly exceptional sequences on toric Fano 3-folds.

5. An example: dimers for reflexive polygons of size 8

We conclude this paper with an illustration of the main theorem for reflexive polygons of size 8. There are three such polygons: a square, a trapezoid and a triangle. The square has four dimers, the trapezoid two, and the triangle one. In Table 1 we show all dimers and their corresponding duals. Next to each dimer is its corresponding polygon. We draw attention to the following remarks.

(i) Because a dimer Q and its dual $\check{\mathrm{Q}}$ have the same arrows and the same faces but different vertices, we labelled the arrows in such a way that the corresponding arrows in a dimer and its dual have the same number. One can easily check that each face in Q also appears in $\check{\mathrm{Q}}$ (although sometimes the orientation of the arrows around the face is reversed). For example, in the first row we can spot a triangle composed of arrows $7, 8, 20$, which is slightly below centre in the dimer Q and slightly left above centre in its dual dimer $\check{\mathrm{Q}}$.

(ii) We did not label the vertices because there is no correspondence between the vertices of a dimer and those of its dual. There is, however, a correspondence between the vertices of a dimer and the zigzag paths of its dual dimer. For instance, in the second row, the arrows incident with the vertex at the centre of the dimer Q are $7, 22, 9, 18, 8, 21, 10, 17$; in the dual dimer $\check{\mathrm{Q}}$, one can find a zigzag path going downwards that contains the same arrows.

(iii) From Corollary 2.9 we know that there is a one-to-one correspondence between the zigzag paths in the dimer and the line segments of the polygon. We drew the polygons next to their dimers in such a way that the zigzag paths point in the same direction as the outward normals to the boundary segments in the polygon. For example, the zigzag path in the second row that we looked at in (ii) points downwards and corresponds to a horizontal line segment on the boundary of polygon 8c, because its normal points downwards. On the other hand, the zigzag path composed of the arrows $24, 21, 12, 19, 14, 22, 6, 20$ is directed upwards and to the right, and it corresponds to a line segment of the hypotenuse of polygon 8c.

(iv) Three of these dimers are self-dual: two for the square and one for the trapezoid. The self-dual dimers can be found in the last three rows of the table. Note that in these cases there is no canonical isomorphism between the dimer and its dual.

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Raf Bocklandt raf.bocklandt@gmail.com
School of Mathematics and Statistics, Herschel Building,
Newcastle University, Newcastle upon Tyne, NE1 7RU, UK

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