THE LIGHTCONE GAUSS MAP AND THE LIGHTCONE DEVELOPABLE OF A SPACELIKE CURVE IN MINKOWSKI 3-SPACE

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Abstract. We define the notion of lightcone Gauss maps, lightcone pedal curves and lightcone developables of spacelike curves in Minkowski 3-space and establish the relationships between singularities of these objects and geometric invariants of curves under the action of the Lorentz group.

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1. Introduction. There are several articles concerning 'generic differential geometry' in Euclidean space [1–5, 8, 10, etc.]. The main tools in these articles are the distance-squared functions and the height functions on submanifolds. In this paper we introduce the notion of lightcone height functions and Lorentzian distance-squared functions on spacelike curves in Minkowski 3-space. We also define the notion of lightcone Gauss maps, lightcone pedal curves and lightcone developables of spacelike curves and establish the relationships between singularities of these subjects and geometric invariants of curves under the action of the Lorentz group as applications of standard techniques of singularity theory for the above functions.

On the other hand, there exists a special subject in Minkowski 3-space, called a *lightlike surface,* which is defined to be a surface tangent to the light cone at any point. We remark that any lightlike surface is the lightcone developable of a spacelike curve. Thus the main theorem (Theorem B) gives a generic classification of singularities of lightlike surfaces. For the basic notions in Lorentzian geometry, see [9].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ two vectors in $\mathbb{R}^3$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3$. We call $(\mathbb{R}^3, \langle, \rangle)$ a 3-dimensional *pseudo Euclidean space*, or *Minkowski 3-space*. We write $\mathbb{R}^3_1$ instead of $(\mathbb{R}^3, \langle, \rangle)$.

We say that a vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively.

Let $\gamma : I \longrightarrow \mathbb{R}^3_1$; $\gamma(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in $\mathbb{R}^3_1$ (i.e., $\dot{\gamma}(t) \neq 0$ for any $t \in I$), where I is an open interval. The curve γ is called a *spacelike curve* if $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle > 0$, for any $t \in I$. The *norm* of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. The *arc-length* of a spacelike curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \, dt.$$

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Then a parameter s is determined such that $\|\gamma'(s)\| = 1$, where $\gamma'(s) = d\gamma/ds(s)$. Consequently we say that a spacelike curve γ is *parameterized by arc-length* if it satisfies $\|\gamma'(s)\| = 1$. Throughout the remainder of this paper we assume the parameter s of γ is the arc-length parameter. Let us denote $\boldsymbol{t}(s) = \gamma'(s)$, and we call $\boldsymbol{t}(s)$ a *unit tangent vector* of γ at s. We define the *curvature* by $k(s) = \sqrt{|\langle \gamma''(s), \gamma''(s) \rangle|}$. If $k(s) \neq 0$ then the *unit principal normal vector* $\mathbf{n}$ of the curve γ at s is given by $\gamma''(s) = k(s) \cdot \boldsymbol{n}(s)$. The *signature* of $\boldsymbol{x}$ is defined to be

$$\mathrm{sign}(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \text{ is spacelike,} \\ 0 & \boldsymbol{x} \text{ is lightlike,} \\ -1 & \boldsymbol{x} \text{ is timelike.} \end{cases}$$

We write $\delta(\gamma(s)) = \mathrm{sign}(\boldsymbol{n}(s))$.

For any $\boldsymbol{x} = (x_1, x_2, x_3)$, $\boldsymbol{y} = (y_1, y_2, y_3) \in \mathbb{R}_1^3$, the *pseudo vector product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2 y_3 - x_3 y_2), x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).$$

The unit vector $\boldsymbol{b}(s) = \boldsymbol{t}(s) \wedge \boldsymbol{n}(s)$ is called a *unit binormal vector* of the curve γ at s. Since $\boldsymbol{t}(s)$ is spacelike, we have $\langle \boldsymbol{b}(s), \boldsymbol{b}(s) \rangle = -\delta(\gamma(s))$ and $\mathrm{sign}(\gamma'(s)) = 1$. Then the following Frenet-Serret type formula holds:

$$\begin{cases} \boldsymbol{t}'(s) = k(s) \cdot \boldsymbol{n}(s) \\ \boldsymbol{n}'(s) = -\delta(\gamma(s)) \cdot k(s) \cdot \boldsymbol{t}(s) + \tau(s) \cdot \boldsymbol{b}(s) \\ \boldsymbol{b}'(s) = \tau(s) \cdot \boldsymbol{n}(s), \end{cases}$$

where $\tau(s)$ is the torsion of the curve γ at s (cf. [6]). This is the fundamental formula for the study of spacelike curves in $\mathbb{R}_1^3$; it is, however, useless at the point $\gamma(s)$ with $k(s) = 0$. We now denote $N(s) = \gamma''(s)$ and $B(s) = \boldsymbol{t}(s) \wedge N(s)$. We simply call $N(s)$ a *principal normal vector* and $B(s)$ a *binormal vector*. If $k(s) \neq 0$, then we have $N(s) = k(s)\boldsymbol{n}(s)$ and $B(s) = k(s)\boldsymbol{b}(s)$. It follows that

$$\langle N(s) \pm B(s), N(s) \pm B(s) \rangle = k^2(s)(\delta(\gamma(s)) - \delta(\gamma(s))) = 0.$$

If $k(s) = 0$, then $N(s)$ is a lightlike vector, so that any pseudo perpendicular vector in the normal plane of $\gamma(s)$ is parallel to $N(s)$. We can prove that $N(s) \pm B(s) \neq \boldsymbol{0}$. This means that $N(s) \pm B(s)$ is a lightlike vector which is parallel to the vector $N(s)$ for $s \in I$ with $k(s) = 0$. Define

$$S_+^1 = \{x \in \mathbb{R}_1^3 | x = (1, x_2, x_3), x_2^2 + x_3^2 = 1\},$$

$$C_p = \{x = (x_1, x_2, x_3) \in \mathbb{R}_1^3 | -(x_1 - p_1)^2 + (x_2 - p_2)^2 + (x_3 - p_3)^2 = 0\},$$

where $p = (p_1, p_2, p_3)$. We call S_+^1 a *lightlike unit circle* and $C_p^* = C_p - \{p\}$ a *lightcone at the vertex* p. For any lightlike vector $\boldsymbol{x} = (x_1, x_2, x_3)$, we write $\tilde{\boldsymbol{x}} = (1, \frac{x_2}{x_1}, \frac{x_3}{x_1}) \in S_+^1$. With this notation, we have $\widetilde{N(s) \pm B(s)} = \widetilde{N(s)}$ if $k(s) = 0$.

We now define a map $LG_\gamma^+ : I \longrightarrow S_+^1$ by

$$LG_\gamma^+(s) = N(s) \widetilde{+} B(s)$$

and a curve $LP_\gamma^+ : I \longrightarrow C^*$ by

$$LP_\gamma^+(s) = \langle \gamma(s), N(s) \widetilde{+} B(s) \rangle \cdot (N(s) \widetilde{+} B(s)),$$

where we may assume that $\gamma(s) \neq 0$. Under the assumption that $k(s) \neq 0$, we also define a ruled surface $LD_\gamma^+ : I \times \mathbb{R} \longrightarrow \mathbb{R}_1^3$ by

$$LD_\gamma^+(s, u) = \gamma(s) + u(\boldsymbol{n}(s) + \boldsymbol{b}(s)).$$

We call LG_γ^+ the *lightcone Gauss map* and LP_γ^+ the *lightcone pedal curve* (or *lightcone dual curve*) of γ. We also call LD_γ^+ the *lightcone developable* of γ. The geometric properties of these subjects will be discussed in §3. We can also define LG_γ^-, LP_γ^- and LD_γ^- exactly in the same way as the above. Since these have the same properties as those of the above, we do not consider these here.

Let $\gamma : S^1 \longrightarrow \mathbb{R}_1^3$ be a spacelike curve with $\gamma''(s) \neq 0$. We consider the following properties of γ.

(A 1) The number of points p of $\gamma(S^1)$ where the lightcone at p has two-point contact with the principal normal curve γ'' is finite.

(A 2) There is no point p of $\gamma(S^1)$ where the lightcone at p has greater than two-point contact with the principal normal curve γ''.

In §3, under the assumption that $k(s_0) \neq 0$, we shall show that there exist just two lightcones at points

$$v_\pm = \gamma(s_0) + \frac{1}{k(s_0)\delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$$

such that γ has three-point contact with the lightcones at $v_\pm$. We call each lightcone an *osculating lightcone* of γ.

(A 3) The number of points at which γ has four-point contact with the osculating lightcone is finite.

(A 4) There is no osculating lightcone with which γ has at least five-point contact at a point.

Our main results are formulated as follows.

THEOREM A. (1) *Let $Imm_s(S^1, \mathbb{R}_1^3)$ be a space of spacelike curves equipped with Whitney C^∞-topology. Then the set of curves which satisfy* (A 1) *and* (A 2) *is a residual set in $Imm_s(S^1, \mathbb{R}_1^3)$.*

(2) *Let $Imm_s^+(S^1, \mathbb{R}_1^3)$ be a space of spacelike curves with $k(s) \neq 0$ equipped with Whitney C^∞-topology. Then the set of curves which satisfy* (A 3) *and* (A 4) *is a residual set in $Imm_s^+(S^1, \mathbb{R}_1^3)$.*

THEOREM B. (1) *Under the assumption of* (A 1) *and* (A 2)*,*

(a) *the lightcone Gauss map LG_γ^+ has a fold point at s_0 if and only if $k(s_0) = 0$,*

(b) *the lightcone pedal curve LP_γ^+ has a cusp point at $LP_\gamma^+(s_0)$ if and only if $k(s_0) = 0$.*

(2) *Assume* (A 3) *and* (A 4)*.*

(a) *The lightcone developable LD_γ^+ is nonsingular along the curve $\gamma(s)$. Moreover, if we consider another lightcone developable LD_γ^-, these surfaces intersect transversally along the curve $\gamma(s)$.*

(b) *The lightcone developable LD_γ^+ is locally diffeomorphic to the cuspidal edge at $\gamma(s_0) + u_0(\mathbf{n}(s_0) + \mathbf{b}(s_0))$ if and only if $u_0 = \frac{1}{k(s_0)\delta(\gamma(s_0))}$. Moreover, the locus of the vertices of the osculating lightcones $\gamma(s) + \frac{1}{k(s)\delta(\gamma(s))}(\mathbf{n}(s) + \mathbf{b}(s))$ is the cuspidal edge.*

(c) *The lightcone developable LD_γ^+ is locally diffeomorphic to the swallow tail at $\gamma(s_0) + u_0(\mathbf{n}(s_0) + \mathbf{b}(s_0)))$ if and only if $u_0 = \frac{1}{k(s_0)\delta(\gamma(s_0))}$ and $(k' - \tau \cdot k)(s) = 0$.*

If we consider a curve γ on the (x_2, x_3)-plane, it is always a spacelike curve. In this case the (x_2, x_3)-plane is a Euclidean plane, so that the curvature $k(s)$ is the ordinary Euclidean curvature and the torsion $\tau(s)$ is constantly equal to zero. Then $(k' - \tau \cdot k)(s) = k'(s)$. This means that the swallowtail point of the rightcone developable of γ corresponds to the ordinary vertex of γ. Figure 1 is the picture of the rightcone developable of the ellipse:

$$\gamma(t) = (0, 2\cos t, \sin t) \quad (0 \le t \le 2\pi).$$

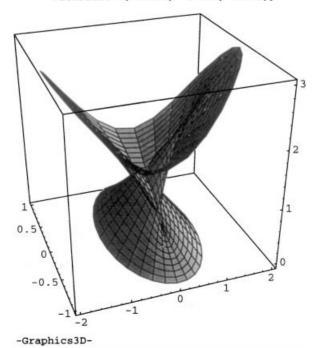We can easily recognize that there exist four swallowtails.

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ParametricPlot3D[
{Cos[t]*(1-2*n),
Sin[t]*(1-n),
n*(Sin[t]*Sin[t]+4*Cos[t]*Cos[t])^(1/2)},
{n,-0.01,1.5},{t,-Pi,Pi},
PlotPoints->30,
        BoxRatios->{5,5,5},
        ViewPoint->{-1.004, -2.808, 1.600}]
```


-Graphics3D-

Figure 1

In [**7**], M. Kossowski introduced the notion of $S^1 \times S^1$-valued Gauss maps associated with spacelike curves. The authors are much inspired by his paper. Especially, the authors learned the fact that $\boldsymbol{n}(s) \pm \boldsymbol{b}(s)$ is lightlike in his paper. As a matter of fact, the notion of lightcone Gauss maps in this paper is almost the same as that of $S^1 \times S^1$-valued Gauss maps. The basic techniques in this paper depend heavily on those in the attractive book of Bruce and Giblin [**4**]. In §2 we introduce the notion of lightcone height functions and Lorentzian distance-squared functions on spacelike curves and study these properties. The Lorentzian distance-squared function is just a direct analogy of the distance squared function in Euclidean 3-space. We can consider the Lorentzian focal surfaces and the Lorentzian canal surface of a spacelike curve by using Lorentzian distance squared functions. It is, however, a simple analogy of the Euclidean case. Here, we consider the notion of lightcone developables, which is a special subject in Lorentzian geometry. We study some Lorentzian invariants in §3. These invariants are squeezed out by the study of lightcone height functions and Lorentzian distance squared functions in §2. The proof of Theorem B is given in §4. In §5 we consider the generic properties by using the Lorentzian analogy of the notion of Monge-Taylor maps of curves in [**4**].

2. Lorentzian invariant functions on spacelike curves.   In this section we introduce two different families of functions on a spacelike curve which are useful in the study of Lorentzian invariants of spacelike curves.

For a spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$, we now define a function

$$H : I \times S^1_+ \longrightarrow \mathbb{R}$$

by $H(s, v) = \langle \gamma, v \rangle$. We call H a *lightcone height function* on the spacelike curves γ. We denote this $h(s) = H_{v_0}(s) = H(s, v_0)$, for any fixed $v_0 \in S^1_+$. We have the following proposition.

PROPOSITION 2.1. *Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a unit speed spacelike curve with $\gamma''(s) \neq 0$.*
(1) $h'(s_0) = 0$ if and only if v is in the normal plane at $\gamma(s_0)$. Especially, if $k(s_0) \neq 0$, then $v = N(s_0) \widetilde{\pm B}(s_0)$.
(2) $h'(s_0) = h''(s_0) = 0$ if and only if $v = N(s_0) \widetilde{\pm B}(s_0) = \widetilde{N(s_0)}$ and $k(s_0) = 0$.
(3) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = 0$ if and only if $v(s_0) = N(s) \widetilde{\pm B}(s_0) = \widetilde{N(s_0)}$, and $k(s_0) = \langle N'(s_0), N(s_0) \rangle = 0$.
(4) $h'(s_0) = h''(s_0) = h^{(3)}(s_0) = h^{(4)}(s_0) = 0$ if and only if $v = N(s_0) \widetilde{\pm B}(s_0) = \widetilde{N(s_0)}$, and $k(s_0) = \langle N'(s_0), N(s_0) \rangle = \langle N''(s_0), N(s_0) \rangle = 0$.

Proof. Let $H : I \times S^1_+ \longrightarrow \mathbb{R}$ be the lightcone height function on the spacelike curve $\gamma : I \longrightarrow \mathbb{R}^3_1$. Then we have $\frac{\partial H}{\partial s} = \langle \gamma'(s), v \rangle = \langle \boldsymbol{t}(s), v \rangle$, where $v = (1, x_2, x_3) \in S_+{}^1$. It follows that $\frac{\partial H}{\partial s} = 0$ if and only if $\langle v, \boldsymbol{t} \rangle = 0$. Especially, if $k(s) \neq 0$, then there exist λ, μ such that $v = \lambda(\boldsymbol{n} + \mu \boldsymbol{b})$. Since $\langle v, v \rangle = 0$, we have $\delta(\gamma) \cdot \lambda^2 - \delta(\gamma) \cdot \mu^2 = 0$, so that we have $\lambda^2 = \mu^2$. It follows that we have $v = \boldsymbol{n} \pm \boldsymbol{b}$.

On the other hand, we have $\frac{\partial^2 H}{\partial s^2} = \langle N(s), v \rangle$. If $k(s) \neq 0$, then $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = 0$ if and only if $v = \boldsymbol{n} \pm \boldsymbol{b} = N(s) \widetilde{\pm B}(s)$ and $\langle k \cdot \boldsymbol{n}, v \rangle = 0$. This is equivalent to the condition that $v = \boldsymbol{n} \pm \boldsymbol{b}$ and $k\langle \boldsymbol{n}, \boldsymbol{n} \pm \boldsymbol{b} \rangle = k\delta(\gamma) = 0$. Since $\delta(\gamma) = \pm 1$, the above condition

means that $v = \boldsymbol{n} \widetilde{\pm} \boldsymbol{b}$ and $k = 0$. This is a contradiction, so that we have $k(s) = 0$; then $N(s)$ is a lightlike vector. Since $\langle v, N(s) \rangle = 0$, v is parallel to $N(s)$. It is equivalent to the fact that $v = \widetilde{N(s)} = N(s) \widetilde{\pm} B(s)$.

Since $\frac{\partial^3 H}{\partial s^3} = \langle N'(s), v \rangle$, $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 F}{\partial s^3} = 0$ if and only if $v = \widetilde{N(s)} = N(s) \widetilde{\pm} B(s)$, $k(s) = 0$ and $\langle N'(s), N(s) \rangle = 0$.

Moreover, we have $\frac{\partial^4 H}{\partial s^4} = \langle N''(s), v \rangle$. Then $\frac{\partial H}{\partial s} = \frac{\partial^2 H}{\partial s^2} = \frac{\partial^3 H}{\partial s^3} = \frac{\partial^4 H}{\partial s^4} = 0$ if and only if $v = N(s_0) \widetilde{\pm} B(s_0) = \widetilde{N(s_0)}$, $k = \langle N', N \rangle = 0$ and $\langle N'', v \rangle = 0$. Since v is parallel to N, we have $\langle N'', N \rangle = 0$. □

We now define a function

$$G : I \times \mathbb{R}_1^3 \longrightarrow \mathbb{R}$$

by

$$G(s, v) = \langle \gamma - v, \gamma - v \rangle.$$

We call G the *Lorentzian distance-squared function* on a spacelike curve γ. We note that $g(s) = G_{v_0}(s) = G(s, v_0)$, for any fixed $v_0 \in \mathbb{R}_1^3$. We also have the following proposition.

PROPOSITION 2.2. *Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a unit speed spacelike curve with $k(s) \neq 0$.*
(1) $g(s_0) = g'(s_0) = 0$ *if and only if* $\gamma(s_0) - v(s_0) = \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ $(\lambda \in \mathbb{R} - \{0\})$.
(2) $g(s_0) = g'(s_0) = g''(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{1}{k(s_0) \cdot \delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ *and $k(s_0) \neq 0$.*
(3) $g(s_0) = g'(s_0) = g''(s_0) = g^{(3)}(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{1}{k(s_0) \cdot \delta(\gamma(s_0))}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$, $k(s_0) \neq 0$ *and* $(k' \mp k \cdot \tau)(s_0) = 0$.
(4) $g(s_0) = g'(s_0) = g''(s_0) = g^{(3)}(s_0) = g^{(4)}(s_0) = 0$ *if and only if* $v = \gamma(s_0) + \frac{(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))}{k(s_0) \cdot \delta(\gamma(s_0))}$, $k(s_0) \neq 0$ *and* $(k' \mp k \cdot \tau)(s_0) = (k' \mp k \cdot \tau)'(s_0) = 0$.

Proof. Let $G : I \times \mathbb{R}_1^3 \longrightarrow \mathbb{R}$ be a Lorentzian distance-squared function on the spacelike curve $\gamma : I \longrightarrow \mathbb{R}_1^3$. Then we have $\frac{\partial G}{\partial s} = 2\langle \gamma', \gamma - v \rangle$. It follows that $\frac{\partial G}{\partial s} = 0$ if and only if $\gamma - v = \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}$. Thus $G(s, v) = \frac{\partial G}{\partial s} = 0$ if and only if $\gamma - v = \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}$ and $\langle \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b}, \lambda \cdot \boldsymbol{n} + \mu \cdot \boldsymbol{b} \rangle = 0$. This is equivalent to the condition that $\gamma - v = \lambda(\boldsymbol{n} \pm \boldsymbol{b})$.

On the other hand, since $\frac{\partial^2 G}{\partial s^2} = 2\langle \boldsymbol{t}', \gamma - v \rangle + 2\langle \boldsymbol{t}, \boldsymbol{t} \rangle = 2\{\langle k \cdot \boldsymbol{n}, \gamma - v \rangle + 1\}$, $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = 0$ if and only if $\gamma - v = \lambda(\boldsymbol{n} \pm \boldsymbol{b})$ and $\langle k \cdot \boldsymbol{n}, \lambda(\boldsymbol{n} \pm \boldsymbol{b}) \rangle = -1$. The last equality is equivalent to the relation that $k \cdot \lambda \cdot \delta(\gamma) = k \cdot \lambda \langle \boldsymbol{n}, \boldsymbol{n} \rangle = -1$. This means that $v = \gamma + \frac{1}{k \cdot \delta(\gamma)}(\boldsymbol{n} \pm \boldsymbol{b})$ and $k \neq 0$.

Since $\frac{\partial^3 G}{\partial s^3} = 2\langle (k \cdot \boldsymbol{n})', \gamma - v \rangle$, $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = 0$ if and only if $k \neq 0$ and $\langle k' \cdot \boldsymbol{n} + k \cdot \boldsymbol{n}', -\frac{1}{k \cdot \delta(\gamma)}(\boldsymbol{n} \pm \boldsymbol{b}) \rangle = 0$, which is equivalent to the condition that $k \neq 0$ and $k' \cdot \delta(\gamma) \pm k \cdot \tau \langle \boldsymbol{b}, \boldsymbol{b} \rangle = 0$. This can be reduced to the condition that $k \neq 0$ and $k' - \pm k \cdot \tau = 0$.

Finally, we have $\frac{\partial^4 G}{\partial s^4} = 2\langle (k \cdot \boldsymbol{n})'', \gamma - v \rangle + 2\langle (k \cdot \boldsymbol{n})', \boldsymbol{t} \rangle$. Then $\frac{\partial^4 G}{\partial s^4} = 0$ if and only if

$$\langle k'' \cdot \boldsymbol{n} + k' \cdot \boldsymbol{n}' + \tau \boldsymbol{b}' k + \tau \boldsymbol{b} k' - 2\delta k k' \boldsymbol{t} - \delta k^2 \boldsymbol{t}' + k\tau' \boldsymbol{b}, \gamma - v \rangle$$
$$+ \langle k' \boldsymbol{n} + \tau \boldsymbol{b} k - \delta k^2 \boldsymbol{t}, \boldsymbol{t} \rangle = 0.$$

This is equivalent to the condition that

$$\langle k''\boldsymbol{n} + k'\boldsymbol{n}' + k\tau\boldsymbol{b}' + \tau\boldsymbol{b}k' - 2\delta kk'\boldsymbol{t} - \delta k^3\boldsymbol{n} + k\tau'\boldsymbol{b}, \gamma - v\rangle - \delta k^2 = 0.$$

Therefore $G = \frac{\partial G}{\partial s} = \frac{\partial^2 G}{\partial s^2} = \frac{\partial^3 G}{\partial s^3} = \frac{\partial^4 G}{\partial s^4} = 0$ if and only if

$$\langle k''\boldsymbol{n} + k'\boldsymbol{n}' + \tau\boldsymbol{b}'k + \tau\boldsymbol{b}k' - \delta k^3\boldsymbol{n} + k\tau'\boldsymbol{b}, -\frac{(\boldsymbol{n} \pm \boldsymbol{b})}{k \cdot \delta}\rangle - \delta k^2 = 0.$$

By the Frenet-Serret type formula, we can translate this condition into the condition that

$$\langle k''\boldsymbol{n} + k'\tau\boldsymbol{b} + k\tau^2\boldsymbol{n} + k'\tau\boldsymbol{b} - \delta k^3\boldsymbol{n} + k\tau'\boldsymbol{b}, -\frac{(\boldsymbol{n} \pm \boldsymbol{b})}{k \cdot \delta}\rangle - \delta k^2 = 0.$$

Hence we have the condition that $k'' \mp 2k'\tau \mp k\tau + k\tau^2 = 0$. Under the condition that $k \neq 0, k' = \pm k \cdot \tau$, the above condition is equivalent to $(k' \mp k \cdot \tau)' = 0$. □

3. Lorentzian invariants of spacelike curves. In this section we study the geometric properties of the lightcone Gauss maps, the lightcone pedal curve and the lightcone developables of spacelike curves. By the propositions in the last section, we can recognize that the functions $k(s)$ and $(k' \mp \tau k)(s)$ have special meanings. Firstly we have the following proposition.

PROPOSITION 3.1. (1) *Let* $\gamma : I \longrightarrow \mathbb{R}_1^3$ *be a spacelike curve with* $\gamma''(s) \neq 0$. *Then* $k(s) \equiv 0$ *if and only if* $\gamma''(s) \in C_0^*$.

(2) *Let* $\gamma : I \longrightarrow \mathbb{R}_1^3$ *be a spacelike curve with* $k(s) \neq 0$. *Then* $(k' \mp \tau k)(s) \equiv 0$ *if and only if* $p_\pm = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))$ *are constant vectors. Under this condition* $\gamma(s) \in C_{p_\pm}^*$.

Proof. By definition, the assertion (1) holds. In order to prove the assertion (2), we put

$$P_\pm(s) = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s));$$

then we have

$$P_\pm'(s) = \frac{-(k' \mp \tau k)(s)}{k(s)\delta(s)}(\boldsymbol{n}(s) \pm \boldsymbol{b}(s)).$$

It follows that $P_\pm'(s) \equiv 0$ if and only if $(k' \mp \tau k)(s) \equiv 0$. Since $(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))$ is lightlike, $\gamma(s) \in C_{p_\pm}$, where $p_\pm = P_\pm(s)$. □

COROLLARY 3.2. (1) *Let* $\gamma : I \longrightarrow \mathbb{R}_1^3$ *be a spacelike curve with* $\gamma''(s) \neq 0$. *If* $k(s) \equiv 0$, *then the lightcone Gauss map of* γ *is constant and the lightcone pedal curve of* γ *is a lightlike line.*

(2) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a spacelike curve with $k(s) \neq 0$. If $(k' - \tau k)(s) \equiv 0$, then the lightcone developable is the lightcone C_{p_+} with vertices p_+, where $p_+ = \gamma(s) + \frac{1}{k(s)\delta(\gamma)}(\boldsymbol{n}(s) + \boldsymbol{b}(s))$ are the constant points.*

Proof. (1) Since $N(s) = \gamma''(s)$ is lightlike for any $s \in I$, the locus of $\boldsymbol{t}(s) = \gamma'(s)$ is a lightlike line on the *pseudo sphere* S^2_1, where

$$S^2_1 = \{x \in \mathbb{R}^3_1 | \langle x, x \rangle = 1\}.$$

It follows that $\widetilde{N}(s)$ is constant and the locus of $\langle \gamma(s), \widetilde{N(s)} \rangle \widetilde{N(s)}$ is a lightlike line.

(2) By definition, we have

$$LD^+_\gamma(s, u) = \gamma(s) + u(\boldsymbol{n}(s) + \boldsymbol{b}(s)) = p_+ + (u - \frac{1}{k(s)\delta(\gamma)})(\boldsymbol{n}(s) + \boldsymbol{b}(s))$$

so that the image of $LD^+_\gamma(s, u)$ is in the pair of lightcones with vertices C_{p_+}. Since $\frac{\partial LD^+_\gamma}{\partial s}(s, u) = (1 - u\delta(\gamma)k(s))\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s)$ and $\frac{\partial LD^+_\gamma}{\partial u}(s, u) = (\boldsymbol{n}(s) + \boldsymbol{b}(s))$, we have

$$\frac{\partial LD^+_\gamma}{\partial s}(s, u) \wedge \frac{\partial LD^+_\gamma}{\partial u}(s, u) = (u\delta(\gamma)k(s) - 1)(\boldsymbol{n}(s) + \boldsymbol{b}(s)) + \tau(s)\boldsymbol{t}(s).$$

It follows that $LD^+_\gamma(s, u)$ is an immersed surface at (s, u) with $u \neq \frac{1}{k(s)\delta(\gamma)}$. The image of the set $u = \frac{1}{k(s)\delta(\gamma)}$ consists of the vertices p_+ of the lightcones. This completes the proof. $\qquad\square$

Let $F : \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ be a submersion and $\gamma : I \longrightarrow \mathbb{R}^3_1$ a spacelike curve. We say that γ and $F^{-1}(0)$ have *k-point contact* for $t = t_0$ if the function $g(t) = F \circ \gamma(t)$ satisfies $g(t_0) = g'(t_0) = \cdots = g^{(k-1)}(t_0) = 0$, $g^{(k)}(t_0) \neq 0$. By Propositions 2.1, 2.2 and 3.1, we have the following proposition.

PROPOSITION 3.3. (1) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $\gamma''(s) \neq 0$. Then γ'' and the lightcone C^*_0 have 2-point contact for $s = s_0$ if and only if $k(s_0) = 0$ and $\langle N'(s_0), N(s_0) \rangle \neq 0$.*

(2) *Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $k(s) \neq 0$. Then γ and the lightcone C^*_{p+} have 4-point contact for $s = s_0$ if and only if $(k' - k\tau)(s_0) = 0$ and $(k' - k\tau)'(s_0) \neq 0$.*

4. Unfoldings of functions of one-variable.

In this section we use some general results on the singularity theory for families of function germs. Detailed descriptions are found in the book [4]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ be a function germ. We call F an *r-parameter unfolding* of f, where $f(s) = F_{x_0}(s, x_0)$. We say that f has *A_k-singularity* at s_0 if $f^{(p)}(s_0) = 0$, for all $1 \leq p \leq k$, and $f^{(k+1)}(s_0) \neq 0$. We also say that f has *$A_{\geq k}$-singularity* at s_0 if $f^{(p)}(s_0) = 0$, for all $1 \leq p \leq k$. Let F be an unfolding of f and $f(s)$ have A_k-singularity ($k \geq 1$) at s_0. We denote the $(k-1)$-jet of the partial derivative $\frac{\partial F}{\partial x_i}$ at s_0 by $j^{(k-1)}(\frac{\partial F}{\partial x_i}(s, x_0))(s_0) = \sum_{j=1}^{k-1} \alpha_{ji}s^j$ for $i = 1, \ldots, r$. Then F is called a *(p) versal unfolding* if the $(k-1) \times r$ matrix of coefficients (α_{ji}) has rank $k-1$ ($k-1 \leq r$). Under the same condition as the above, F is called *versal unfolding* if the $k \times r$ matrix of coefficients $(\alpha_{0i}, \alpha_{ji})$ has rank k ($k \leq r$), where $\alpha_{0i} = \frac{\partial F}{\partial x_i}(s_0, x_0)$.

We now introduce important sets concerning the unfoldings relative to the above notions. The *singular set* of F is the set

$$S_F = \{(s, x) | \tfrac{\partial F}{\partial s}(s, x) = 0\}.$$

The *bifurcation set* $\mathcal{B}_F$ of F is the critical value set of the restriction to S_F of the canonical projection $\pi : \mathbb{R} \times \mathbb{R}^r \longrightarrow \mathbb{R}$:

$$\mathcal{B}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } \frac{\partial F}{\partial s} = \frac{\partial^2 F}{\partial s^2} = 0 \text{ at } (s, x)\}.$$

The *discriminant set* of F is the set

$$\mathcal{D}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

Then we have the following well-known result (cf. [**4**]).

THEOREM 4.1. *Let* $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \to \mathbb{R}$ *be an r-parameter unfolding of $f(s)$ which has A_k singularity at s_0.*
(1) *Suppose that F is a (p) versal unfolding.*
    (a) *If $k = 2$, then (s_0, x_0) is the fold point of $\pi|S_F$ and $\mathcal{B}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
    (b) *If $k = 3$, then $\mathcal{B}_F$ is diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
(2) *Suppose that F is a versal unfolding.*
    (a) *If $k = 1$, then $\mathcal{D}_F$ is locally diffeomorphic to $\{0\} \times \mathbb{R}^{r-1}$.*
    (b) *If $k = 2$, then $\mathcal{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.*
    (c) *If $k = 3$, then $\mathcal{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.*

Here, $C = \{(x_1, x_2) | x_1{}^2 = x_2{}^3\}$ is the ordinary cusp and

$$SW = \{(x_1, x_2, x_3) | x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$$

is the swallow tail. We also say that a point $x_0 \in \mathbb{R}^r$ is a *fold point* of a map germ $f : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, f(x_0))$ if there exist diffeomorphism germs $\phi : (\mathbb{R}^r, x_0) \longrightarrow (\mathbb{R}^r, 0)$ and $\psi : (\mathbb{R}^r, f(x_0)) \longrightarrow (\mathbb{R}^r, 0)$ such that $\psi \circ \phi(x_1, , x_r) = (x_1, \ldots, x_{r-1}, x_r^2)$.

For a unit speed spacelike curve $\gamma : I \longrightarrow \mathbb{R}_1^3$, we now define a function

$$\widetilde{H} : I \times S_+^1 \times \mathbb{R} \to \mathbb{R}$$

by $\widetilde{H}(s, v, u) = H(s, v) - u = \langle \gamma(s), v \rangle - u$, where H is the lightcone height function. Then we have the following fundamental theorem in this paper.

THEOREM 4.2. *Let* $\gamma : I \longrightarrow \mathbb{R}_1^3$ *be a unit speed spacelike curve with $\gamma''(s) \neq 0$ and $H : I \times S_+^1 \longrightarrow \mathbb{R}$ the lightcone height function on γ.*
(1) *If $h(s) = H_{v_0}(s)$ has A_2-singularity at s_0, then H is the (p) versal unfolding of h.*

(2) *If $h(s) = \widetilde{H}_{(v_0,u_0)}(s)$ has A_k-singularity ($k = 1, 2$) at s_0, then $\widetilde{H}$ is the versal unfolding of h.*

Let $\gamma : I \longrightarrow \mathbb{R}^3_1$ be a unit speed spacelike curve with $k(s) \neq 0$ and $G : I \times \mathbb{R}^3_1 \longrightarrow \mathbb{R}$ the Lorentzian distance-squared function. We consider the point $(s_0, v_0) \in I \times \mathbb{R}^3_1$ with $G(s_0, v_0) = 0$ and $v_0 \neq \gamma(s_0)$.

(3) *If $g(s)$ has A_k-singularity ($k = 1, 2, 3$) at s_0, then G is the versal unfolding of $g := G_{v_0}$.*

Proof. (1) We have shown that H_{v_0} has A_2-singularity at s_0 if and only if there exists a non-zero real number λ such that $v = \lambda N(s_0)$ and $k(s_0) = 0$. We denote by $\gamma(s) = (x_1(s), x_2(s), x_3(s))$, $v = (1, \cos\theta, \sin\theta)$. By definition, we have $H(s, \theta) = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta$. It follows that $\frac{\partial H}{\partial \theta} = -x_2(s) \cdot \sin\theta + x_3(s) \cdot \cos\theta$ and the 1-jet of $\frac{\partial H}{\partial \theta}$ at s_0 is given by $s \cdot -x'_2(s_0) \cdot \sin\theta + s \cdot x'_3(s_0) \cdot \cos\theta$. So we require the 1×2 matrix $(-x_2(s) \cdot \sin\theta + x_3(s) \cdot \cos\theta, -x'_2(s_0) \cdot \sin\theta + x'_3(s_0) \cdot \cos\theta)$ to have rank 1 which it always does since $-x'_2(s_0) \cdot \sin\theta + x'_3(s_0) \cdot \cos\theta \neq 0$. In fact, $-x'_2(s_0) \cdot \sin\theta + x'_3(s_0) \cdot \cos\theta$ is equal to the first component of $\gamma' \wedge v$. Suppose that $-x'_2(s_0) \cdot \sin\theta + x'_3(s_0) \cdot \cos\theta = 0$. Since $\langle \gamma' \wedge v, \gamma' \wedge v \rangle = \langle t \wedge \lambda_1 N, t \wedge \lambda_1 N \rangle = \lambda^2 \langle B, B \rangle = \lambda^2 \langle N, N \rangle = 0$, we have

$$\langle (0, x'_3 - x'_1 \sin\theta, x'_1 \cos\theta - x'_2), (0, x'_3 - x'_1 \sin\theta, x'_1 \cos\theta - x'_2) \rangle = 0.$$

This is equivalent to the condition that

$$x'^2_3 - 2x'_1 x'_3 \sin\theta + x'^2_1 \sin^2\theta + x'^2_1 \cos^2\theta - 2x'_1 x'_2 \cos\theta + x'^2_2 = 0.$$

It follows that

$$x'^2_3 + x'^2_2 + x'^2_1 - 2x'_1(x'_2 \sin\theta + x'_3 \cos\theta) = 0.$$

Since $-x'_1 + x'_2 \sin\theta + x'_3 \cos\theta = \langle \gamma', v \rangle = 0$, we have $-x'^2_1 + x'^2_2 + x'^2_3 = 0$. On the other hand, $-x'^2_1 + x'^2_2 + x'^2_3 = \langle \gamma', \gamma' \rangle = 1$. This is a contradiction. Hence H is (p) versal.

(2) In this case we have

$$\widetilde{H}(s, \theta, u) = H(s, \theta) - u = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta - u.$$

By Proposition 2.1, h has A_1-singularity at s_0 if and only if v is in the normal plane at $\gamma(s_0)$ and A_2-singularity at s_0 if and only if there exists a non-zero real number λ_1 such that $v = \lambda_1(s_0)N(s_0)$ and $k(s_0) = 0$.

Since $\widetilde{H}(s, v, u) = H(s, v) - u = \langle \gamma(s), v \rangle - u = -x_1(s) + x_2(s) \cdot \cos\theta + x_3(s) \cdot \sin\theta - u$. we have $\frac{\partial \widetilde{H}}{\partial \theta} = -x_2 \cdot \sin\theta + x_3 \cdot \cos\theta$ and $\frac{\partial \widetilde{H}}{\partial u} = 1$ at s_0. So the rank of the 1×2 matrix $(-x_2 \cdot \sin\theta + x_3 \cdot \cos\theta, 1)$ is 1. By the same reason as the case (1), we have $-x'_2(s_0) \cdot \sin\theta + x'_3(s_0) \cdot \cos\theta \neq 0$. It follows that the rank of the 2×2 matrix

$$\begin{pmatrix} -x_2 \cdot \sin\theta + x_3 \cdot \cos\theta & 1 \\ -x'_2 \cdot \sin\theta + x'_3 \cdot \cos\theta & 0 \end{pmatrix}$$

is 2. This means that $\widetilde{H}$ is versal when $h(s)$ has the A_k-singularity ($k = 1, 2$) at s_0.

(3) In this case we have

$$G(s, v) = -(x_1(s) - v_1)^2 + (x_2(s) - v_2)^2 + (x_3(s) - v_3)^2,$$

where $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ and $v = (v_1, v_2, v_3)$.

Thus we have $\frac{\partial G}{\partial v_1}(s) = 2(x_1(s) - v_1)$, and so the 2-jet at s_0 is $2(sx_1'(s_0) + \frac{1}{2}s^2 x_1''(s_0))$. We also have $\frac{\partial G}{\partial v_i} = -2(x_i(s) - v_i)$ $(i = 2, 3)$, and so the 2-jet at s_0 is $-2(sx_i'(s_0) + \frac{1}{2}s^2 x_i''(s_0))$. The condition for versality can be checked as follows.

(i) By Proposition 2.2, g has A_1-singularity at s_0 if and only if there exists a non-zero real number λ such that $v = \gamma(s_0) - \lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ and $k(s) \cdot \lambda \cdot \delta \neq -1$. When g has A_1-singularity at s_0, we require the 1×3 matrix $(2(x_1(s_0) - v_{0,1}), -2(x_2(s_0) - v_{0,2}), -2(x_3(s_0) - v_{0,3}))$ to have rank 1, which it always does since $v_0 \neq \gamma(s_0)$.

(ii) It also follows from Proposition 2.2 that g has the A_2-singularity at s_0 if and only if $v = \gamma + \frac{1}{k(s_0) \cdot \delta(\gamma)}(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0))$ and $k(s) \neq 0$, $k'(s) \neq \pm k(s) \cdot \tau$. When g has A_2-singularity at s, we require the 2×3 matrix

$$\begin{pmatrix} 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \end{pmatrix}$$

to have rank 2, which follows from the proof of the case (iii).

(iii) By Proposition 2.2, g has A_3-singularity at s_0 if and only if $v = \gamma + \frac{(\boldsymbol{n}\pm\boldsymbol{b})}{k\cdot\delta}$ and $k(s_0) \neq 0$, $(k' \mp k \cdot \tau)(s_0) = 0$ and $(k' \mp k \cdot \tau)'(s_0) = 0$. When g has A_3-singularity at s, we require the 3×3 matrix

$$\begin{pmatrix} 2(x_1(s_0) - v_1) & -2(x_2(s_0) - v_2) & -2(x_3(s_0) - v_3) \\ 2x_1'(s_0) & -2x_2'(s_0) & -2x_3'(s_0) \\ x_1''(s_0) & -x_2''(s_0) & -x_3''(s_0) \end{pmatrix}$$

to be nonsingular. The determinant of this matrix is $4\det((v - \gamma(s_0))\ \gamma'(s_0)\ \gamma''(s_0)) = 4\langle(v - \gamma(s_0)) \wedge \boldsymbol{t}(s_0), k(s_0) \cdot \boldsymbol{n}(s_0)\rangle = 4\langle\lambda(\boldsymbol{n}(s_0) \pm \boldsymbol{b}(s_0)) \wedge \boldsymbol{t}(s_0), k(s_0) \cdot \boldsymbol{n}(s_0)\rangle = 4\langle-\lambda\boldsymbol{b}(s_0) \mp \lambda\boldsymbol{n}(s_0), k(s_0)\boldsymbol{n}(s_0)\rangle = \pm 4\lambda k(s_0)\delta(\gamma) \neq 0$. But this just says $k(s_0) \neq 0$. This completes the proof. $\square$

We now give the proof of Theorem B.

Proof of Theorem B. For the proof of the assertion (1) (a), we consider the set S_H associated with the lightcone height function H given by $S_H = \{(s, v) \in I \times S_+^1 | h'(s) = 0\}$. By Proposition 2.1, we have $S_H = \{(s, v) | v = N(s) \widetilde{\pm} B(s)\}$. We also consider the canonical projection $\pi : I \times S_+^1 \longrightarrow S_+^1$ and we can identify $\pi|S_H$ and the lightcone Gauss map $LG_\gamma^\pm$. By the assumption and Propositions 2.1 and 3.3, h has A_2-singularity at s_0 if and only if $k(s_0) = 0$. It follows from Theorem 4.2 that H is the (p)versal unfolding of h at s_0.

Therefore Lemma 4.1, (1) (a) asserts that $\pi|S_H$ has a fold point at s_0.

In order to prove the assertion (1) (b), we define a map

$$\widetilde{H} : I \times S_+^1 \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$\widetilde{H}(s, v, u) = H(s, v) - u = \langle \gamma(s), v \rangle - u.$$

The discriminant set of $\widetilde{H}$ is

$$\mathcal{D}_{\widetilde{H}} = \{(v, u) | u = \langle \gamma, v \rangle, \ v = N(s) \widetilde{\pm} B(s)\}.$$

Applying a Lorentzian motion to the curve γ, we may assume that $\langle \gamma(s), \gamma(s) \rangle \neq 0$. This means that $u = \langle \gamma(s), N(s) \widetilde{\pm} B(s) \rangle \neq 0$.

We now define a map $\Phi : S_+^1 \times \mathbb{R}_+ \longrightarrow C_+$ by $\Phi((1, x_2, x_3), \mu) = (\mu, \mu \cdot x_2, \mu \cdot x_3)$, where $\mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$ and $C_+ = \{(x_1, x_2, x_3) | x_1^2 = x_2^2 + x_3^2, \ x_1 > 0\}$. Since $\Phi^{-1}(x_1, x_2, x_3) = ((1, \frac{x_2}{x_1}, \frac{x_3}{x_1}), x_1)$, Φ is a diffeomorphism. By the above arguments we may assume that $\mathcal{D}_{\tilde{H}} \subset S_+^1 \times \mathbb{R}_+$. It is clear that

$$\Phi(\mathcal{D}_{\tilde{H}}) = \{(u \cdot v) | v = \langle \gamma, N(s) \widetilde{\pm} B(s) \rangle \cdot (N(s) \widetilde{\pm} B(s)), \ s \in I\}.$$

By Lemma 4.1 and Theorem 4.2 (2), the discriminant set $\mathcal{D}_{\tilde{H}}$ of $\widetilde{H}$ is locally diffeomorphic to a line or the cusp. It follows from Proposition 2.1 that the proof of Theorem B (1) (b) is completed.

The assertions (2) (a) is trivial by the definition of the lightcone developable.

For the proof of the assertion (2) (b), (c), we consider the discriminant set of G. By Proposition 2.2, the discriminant set of G is given by

$$\mathcal{D}_G = \{v \in \mathbb{R}_1^3 | \text{ there exists } s \text{ and non-zero real number } \lambda \text{ with } v = \gamma(s) - \lambda(\boldsymbol{n}(s) \pm \boldsymbol{b}(s))\}.$$

Hence the assertions follow from Proposition 2.2 and Theorem 4.2. $\square$

5. Generic properties of spacelike curves.

In this section we consider the notion of Lorentzian Monge-Taylor maps for spacelike curves analogous to the ordinary notion of Monge-Taylor maps for space curves in Euclidean space (cf. [4]).

Firstly we try to choose a pseudo orthonormal frame along γ like the Euclidean case. In this case the tangent vector of γ is always spacelike, so that the normal plane contains lightlike vectors. We cannot choose a lightlike normal vector as a member of the basis of the frame along γ. If we construct such a frame, all arguments in the sequel are parallel to those of the Euclidean case. Let $\gamma : I \longrightarrow \mathbb{R}_1^3$ be a (regular) spacelike curve, with I an open connected subset of the unit circle S^1, increasing t corresponding to the anticlockwise orientation of S^1. We now choose a smooth family of unit vectors $\boldsymbol{n}(t)$, with $\boldsymbol{n}(t)$ pseudo-normal to γ at t, so that $\|\boldsymbol{n}(t)\| = 1$ and $\langle \boldsymbol{n}(t), \boldsymbol{t}(t) \rangle = 0$, for all $t \in I$. Such $\boldsymbol{n}(t)$ can be obtained as follows: consider the smooth map $\boldsymbol{t} : I \to S_1^2$ which takes t to the unit tangent vector $\boldsymbol{t}(t)$. If V is any vector in H_1^2 (where $H_1^2 = \{p \in \mathbb{R}_1^3 \mid \langle p, p \rangle = -1\}$) we can obtain the vector field $\boldsymbol{n}(t)$ by pseudo-orthogonally projecting V onto each of the normal planes and normalizing. Thus $\boldsymbol{n}(t) = \frac{V - \langle V, \boldsymbol{t}(t) \rangle \boldsymbol{t}(t)}{\|V - \langle V, \boldsymbol{t}(t) \rangle \boldsymbol{t}(t)\|}$; then we have $\langle \boldsymbol{n}, \boldsymbol{n} \rangle = -1$ and $\langle \boldsymbol{n}, \boldsymbol{t} \rangle = 0$. This means that

we can choose $\boldsymbol{n}(t)$ as a timelike vector field along γ. We can obtain a second smooth family of unit vectors $\boldsymbol{b}(t) = \boldsymbol{t}(t) \wedge \boldsymbol{n}(t)$ normal to γ at t. We remark that the triple $\boldsymbol{t}(t)$, $\boldsymbol{n}(t)$, $\boldsymbol{b}(t)$ is the Lorentzian frame along γ and $\boldsymbol{b}(t)$ is spacelike. We now use the pseudo-perpendicular lines spanned by $\boldsymbol{t}(t), \boldsymbol{n}(t), \boldsymbol{b}(t)$ as axes at $\gamma(t)$ with the unit points on the axes corresponding to the three given vectors. We remark that the Minkowski metric is invariant under the Lorentz transformation. Note that the curve $\gamma(t)$ is not necessarily of unit speed, with $\gamma(t_0) = 0$. Then the coordinates η, ζ and χ of $\gamma(t)$ relative to axes $\boldsymbol{t}, \boldsymbol{n}$ and $\boldsymbol{b}$ are functions of t: $\zeta(t) = \gamma(t) \cdot \boldsymbol{t}(t_0)$, $\eta(t) = \gamma(t) \cdot \boldsymbol{n}(t_0)$, $\chi(t) = \gamma(t) \cdot \boldsymbol{b}(t_0)$, $\eta(t) = f(\zeta(t))$, $\chi(t) = g((\zeta(t))$ where $f = f_0$, $g = g_0$. So $j^k f_t(0) = a_2(t)\zeta^2 + a_3(t)\zeta^3 + \cdots + a_k(t)\zeta^k$, $j^k g_t(0) = b_2(t)\zeta^2 + b_3(t)\zeta^3 + \cdots + b_k(t)\zeta^k$ in the neighbourhood $(0, 0, 0)$. Locally then $\gamma(I)$ can be written in the form $\{(f_t(\zeta), \zeta, g_t(\zeta))\}$, with $f(0) = g(0) = j^1 f_t(0) = j^1 g_t(0) = 0$. If V_k denotes the space of polynomials in ζ of degree ≥ 2 and $\leq k$, we have the *Lorentzian Monge-Taylor map* for the spacelike curve γ, $\mu_\gamma : I \longrightarrow V_k \times V_k$ given by $\mu_\gamma(t) = (j^k f_t(0), j^k g_t(0))$. ($V_k \times V_k$ can be identified with $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = \mathbb{R}^{2(k-1)}$ via the coordinates $(a_2, \ldots, a_k, b_2, \ldots, b_k)$.) Of course μ_γ depends rather heavily on our choice of unit normals $\boldsymbol{n}(t)$, where $a_i(t) = \frac{f_t(0)^{(i)}}{i!}$, $b_i(t) = \frac{g_t(0)^{(i)}}{i!}$ $(2 \leq i \leq k)$; that is

$$V_k \times V_k = \{(a_2\zeta^2 + b_3\zeta^3 + \cdots + b_k\zeta^k), (b_2\zeta^2 + b_3\zeta^3 + \cdots + a_k\zeta^k)\}.$$

Let P_k denote the set of maps $\psi : \mathbb{R}_1^3 \to \mathbb{R}_1^3$ of the form

$$(\psi(x, y, z) = (\psi_1(x, y, z), \psi_2(x, y, z), \psi_3(x, y, z)),$$

where $\psi_i(x, y, z)$ is a polynomial in x, y and z of degree $\leq k$. An element $\psi \in P_k$ is determined by the coefficients of the various monomials $x^l y^i z^j$ occurring in ψ_1, ψ_2 and ψ_3. There are altogether $1 + 3 + \cdots + (2k - 1) + 2k + 1 = (k + 1)^2$ monomials of degree $\leq k$, so that P_k can be thought of as a Euclidean space $\mathbb{R}^{(k+1)^2}$. It is this space which will provide the required deformations of the curve.

To simplify matters we now assume that the curve $\gamma(I)$ is compact; that is $I = S^1$. The identity map $1_{\mathbb{R}_1^3} : \mathbb{R}_1^3 \to \mathbb{R}_1^3$, is of course an element of P_k (provided $k \geq 1$), and using the compactness of $\gamma(S^1)$ it easily follows that there is a open neighbourhood U of $1_{\mathbb{R}_1^3}$ in P_k with the property that if $\psi \in U$ then the linear map $T\psi(\gamma(t)) : \mathbb{R}_1^3 \to \mathbb{R}_1^3$; $v \mapsto D\psi(\gamma(t)) \cdot v$ is such that $D\psi(\gamma(t)) \cdot \boldsymbol{n}(t)$ is a timelike vector and $D\psi(\gamma(t)) \cdot \boldsymbol{t}(t)$ is a spacelike vector, where $D\psi(\gamma(t))$ denotes the derivative of ψ at $\gamma(t)$. If we deform the original curve by the map ψ, then we can also obtain the required new smooth family of normal vectors $\boldsymbol{n}_\psi(t)$ as follows. Since the map $\psi : \mathbb{R}_1^3 \to \mathbb{R}_1^3$ is a diffeomorphism on some open set containing $\gamma(I)$, the vector $\boldsymbol{n}(t)$ will be sent to some new timelike vector $D\psi(\gamma(t))\boldsymbol{n}(t)$ which will be neither zero nor tangent to $\psi \circ \gamma$ at t. Pseudo-orthogonally project this vector onto the psudo-normal plane to $\psi \circ \gamma$ at t and normalize, that is

$$\boldsymbol{n}_\psi(t) = \frac{D\psi(\gamma(t))\boldsymbol{n}(t) - \langle D\psi(\gamma(t))\boldsymbol{n}(t), \boldsymbol{t}_\psi\rangle \boldsymbol{t}_\psi}{\|D\psi(\gamma(t))\boldsymbol{n}(t) - \langle D\psi(\gamma(t))\boldsymbol{n}(t), \boldsymbol{t}_\psi\rangle \boldsymbol{t}_\psi\|}, \langle \boldsymbol{n}_\psi(t), \boldsymbol{n}_\psi(t)\rangle = -1,$$

where $\boldsymbol{t}_\psi$ denotes the tangent vector of the curve $\psi \circ \gamma$ at t. Assuming as before that $I = S^1$, we choose an open neighbourhood U of $1_{S^1} \in P_k$ consisting of polynomial

maps which map an open set containing $\gamma(S^1)$ diffeomorphically to its image. We have now shown that there is a smooth map

$$\mu : S^1 \times U \longrightarrow V_k \times V_k$$

defined by $\mu(-, \psi) =$ Monge-Taylor map for the curve $\psi \circ \gamma$ using the family of normal vectors $\boldsymbol{n}_\psi(t)$. By exactly the same arguments as in the proof of Theorem 9.9 in [**4**], we have the following theorem.

THEOREM 5.1. *Let Q be a manifold in $V_k \times V_k = \mathbb{R}^{2k-2}$. For some open set $U_1 \subset U$ containing the identity map, the map $\mu : S^1 \times U_1 \to V_k$, defined by $\mu(t, \psi) = \mu_{\psi \circ \gamma}(t)$, is transverse to Q. (In fact we can prove that μ is a submersion so that Q does not enter the argument at all.)*

By the direct calculations, we have the following lemmas. The calculations are rather long and tedious and so we omit the details.

LEMMA 5.2. *Let $\gamma : S^1 \to \mathbb{R}^3_1$ be a spacelike curve defined by*

$$\gamma(t) = (f_t(\zeta), \zeta, g_t(\zeta)) = (a_2\zeta^2 + a_3\zeta^3 + \cdots, \zeta, b_2\zeta^2 + b_3\zeta^3 + \cdots)$$

with $\zeta(t_0) = 0$.

(1) *$k = 0$ and $\langle N', N \rangle = 0$ at t_0 if and only if $a_2^2 - b_2^2 = 0$ and $a_2a_3 - b_2b_3 = 0$.*
(2) *$k'(s_0) - k(s_0) \cdot \tau(s_0) = 0$ and $(k' - k \cdot \tau)' = 0$ at t_0 if and only if*

$$\begin{cases} \pm(a_2a_3 - b_2b_3) - (a_3b_2 - a_2b_3) = 0, \\ -3(a_2a_3 - b_2b_3)^2 + \{4(a_2a_4 - b_2b_4) + 3(a_3^2 - b_3^2)\} \cdot |a_2^2 - b_2^2|, \\ -4(a_2^2 - b_2^2) \cdot |a_2^2 - b_2^2|^2 \mp 3(a_2a_3 - b_2b_3) \cdot (a_3b_2 - a_2b_3), \\ -4|a_2^2 - b_2^2| \cdot (a_4b_2 - a_2b_4) \pm 6(a_2a_3 - b_2b_3) \cdot (a_3b_2 - a_2b_3) = 0, \end{cases}$$

where ζ is the coordinate along the $\boldsymbol{t}$-direction, $f_t(\zeta)$ the coordinate along the $\boldsymbol{n}$-direction and $g_t(\zeta)$ the coordinate along the $\boldsymbol{b}$-direction. Moreover, $(a_2, a_3, \cdots, a_k, b_2, b_3, \cdots, b_k) \in \mathbb{R}^{2k}$, k, k' as in §2.

LEMMA 5.3 *We consider smooth maps $\rho_i : V_3 \times V_3 = \mathbb{R}^4 \longrightarrow \mathbb{R}$ $(i = 1, 2)$ given by*

$$\begin{cases} \rho_1 = a_2^2 - b_2^2, \\ \rho_2 = -b_2b_3 + a_2a_3, \end{cases}$$

and $\phi_i : V_4 \times V_4 = \mathbb{R}^6 \longrightarrow \mathbb{R}$ $(i = 1, 2)$ given by

$$\begin{cases} \phi_1 = \pm(a_2a_3 - b_2b_3) - (a_3b_2 - a_2b_3), \\ \phi_2 = -3(a_2a_3 - b_2b_3)^2 + \{4(a_2a_4 - b_2b_4) + 3(a_3^2 - b_3^2)\} \cdot |a_2^2 - b_2^2|, \\ \quad -4(a_2^2 - b_2^2) \cdot |a_2^2 - b_2^2|^2 \mp 3(a_2a_3 - b_2b_3) \cdot (a_3b_2 - a_2b_3), \\ \quad -4|a_2^2 - b_2^2| \cdot (a_4b_2 - a_2b_4) \pm 6(a_2a_3 - b_2b_3) \cdot (a_3b_2 - a_2b_3). \end{cases}$$

(1) *The set $Q_1 = \{(a_2, a_3, b_2, b_3) \in \mathbb{R}^4 | \rho_1 = \rho_2 = 0\}$ is a codimension two submanifold in $\mathbb{R}^4$.*

(2) *The set $Q_2 = \{(a_2, a_3, a_4, b_2, b_3, b_4) \in \mathbb{R}^6 | \phi_1 = \phi_2 = 0\}$ is a codimension two submanifold in $\mathbb{R}^6$.*

We can use Theorem 5.1, Lemmas 5.2 and 5.3 for the proof of Theorem A in exactly the same way as the proof of Corollary 9.7 in [**4**]. We omit the details here.

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