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A PROBABILISTIC APPROACH TO GRADIENT ESTIMATES

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ABSTRACT. Suppose u is a harmonic function on a domain D and x, x' are in D. We estimate $|u(x) - u(x')|$ using two Brownian motions started at x and x' and killed on exiting a cube $Q \subset D$. By selecting appropriate versions of the two Brownian motions, a classical gradient estimate for u is easily derived.

0. Introduction. This note is motivated by an idea of Lindvall, Rogers [5] on the coupling of diffusion processes. The authors constructed two diffusions X_t and X'_t begun at x and x', respectively, where X and X' share a common generator L. Their processes are coupled, i.e., $X_t = X'_t$ for all t after $T = \inf\{t > 0 : X_t = X'_t\}$. Those results were concerned with the construction of a successful coupling of two diffusion processes, i.e. one for which $P(T < \infty) = 1$. Our work rests in estimating the probability of a coupling being unsuccessful. These probability estimates can yield gradient estimates for the harmonic functions for the diffusion in question. We treat the case of Brownian motion in Euclidean space and obtain well-known basic gradient estimates for classical harmonic functions. These gradient estimates are the starting point of the Schauder estimates.

It is also quite easy to obtain gradient estimates when the Laplacian is perturbed by a linear first order term. For this method on manifolds, see Cranston [1] and for an application to the Schrödinger equation, Cranston and Zhao [3].

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1. The Couplings. For our probability space we take Ω, the space of continuous functions with values in $\mathbb{R}^d$. Given a generator L, $((X, X'), P^{(x,x')})$ will denote a pair of diffusions X and X' which have generator L under $P^{(x,x')}$ and $P^{(x,x')}(X_0 = x, X'_0 = x') = 1$. Furthermore, setting $T = T(X, X') = \inf\{t > 0 : X_t = X'_t\}$ we will set $X'_t = X_t$ for $t \geq T$, i.e. X' will couple with X at T. For arbitrary couplings, T will not be a stopping time for X or X' but one nice feature of the coupling of Lindvall-Rogers is that T is a stopping time. We describe the Lindvall-Rogers coupling for diffusions on $\mathbb{R}^d$. For σ an invertible, smooth, bounded $d \times d$-matrix valued function of $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and b an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$, X_t will satisfy the s.d.e.

$$dX_t = \sigma(t, X_t)dB_t + b(X_t)dt, \; X_0 = x.$$

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The idea now is to construct an $O(d)$-valued process ($O(d)$ = orthogonal $d \times d$ matrices) $H_t = H(t, X_t, X'_t)$ and X'_t so that if $dB'_t \equiv H_t dB_t$, then B' is, of course, a d-dimensional Brownian motion, $BM(\mathbb{R}^d)$, and X'_t solves

$$dX'_t = \sigma(t, X'_t)dB'_t + b(X'_t)dt, \ X'_0 = x'$$

until $T(X, X')$ after which X'_t is set equal to X_t. The process H_t is obtained by first setting

$$y = y(x, x') = x - x'$$
$$u(t, x, x') = \frac{\sigma(t, x')^{-1}y}{|\sigma(t, x')^{-1}y|}$$

and

$$H(t, x, x') = I - 2u(t, x, x')u(t, x, x')^T.$$

Then $H_t = H(t, X_t, X'_t)$. As remarked in Lindvall-Rogers, H is reflection in the plane orthogonal to $\sigma(t, x')^{-1}y$. Our only use of this particular coupling will be in the cases $\sigma \equiv I$ and $\sigma_t = e^{-tA}$ where A is a constant $d \times d$ matrix. In the first case the coupling has the following simple description. Let P be the hyperplane perpendicular to the line through x and x' with $\frac{x+x'}{2} \in P$. Run $X_t = B_t$ and then X'_t is simply X_t reflected in P. For this pair $T(B, B') = \sigma_P \equiv \inf\{t > 0 : B_t \in P\}$ is obviously a stopping time. In the second instance, we will take $\sigma_t = e^{-tA}$, start with

$$dZ_t = e^{-tA}dB_t \quad , \quad Z_0 = x$$

and apply the above construction. It is interesting to observe that $T = T(X, X')$ is a stopping time. Notice that since σ is invertible

$$dB_t = \sigma^{-1}(X_t)dX_t - \sigma^{-1}(X_t)b(X_t)dt.$$

Thus B is recovered from X and $\sigma(B_s : s \leq t) = \sigma(X_s : s \leq t)$ and a stopping time for one of X or B is a stopping time for the other. Since X, X' and therefore $H(X, X')$ are predictable functionals of B, $T(X, X')$ is a stopping time for $\sigma(B_s : s \leq t)$. Therefore, $T(X, X')$ is a stopping for X and similarly for X'.

2. Harmonic functions. This section is devoted to a probabilistic derivation of classical gradient estimates for solutions of Poissons equation $\frac{1}{2} \triangle u = f$ in a Euclidean domain D. The estimates also almost trivially extend to constant coefficient elliptic operators.

We begin with the coupling by reflection method of Lindvall, Rogers [5]. As in the previous paragraph, given two starting points $x, x' \in \mathbb{R}^d$ and a Brownian motion B with $B_0 = x$ construct another $BM(\mathbb{R}^d)$, B' with $B'_0 = x'$ by reflecting B in the hyperplane normal to the segment between x and x' and half way between the two. Then with $T = \inf\{t > 0 : B_t = B'_t\}$ we have $T = \sigma_P$. Also for a subset $Q \subseteq \mathbb{R}^d$ define $\tau_Q = \inf\{t > 0 : B_t \notin Q\}, \tau'_Q = \inf\{t > 0 : B'_t \notin Q\}$. For $x \in D \subseteq \mathbb{R}^d$ define $\delta_x = \operatorname{dist}(x, D^c)$ and for a function defined on Q, $\operatorname{osc}_Q u \equiv \sup_Q u - \inf_Q u$.

THEOREM 1. *Let u solve $\frac{1}{2}\Delta u = f$ on D. Then with*

$$Q = \{y : \max_{1\le i\le d} |x_i - y_i| < \frac{1}{2}\delta_x\}$$

we have

$$|\nabla u(x)| \le \frac{d}{\delta_x} \operatorname*{osc}_Q u + \frac{1}{4}\delta_x \operatorname*{osc}_Q f.$$

If b is a vector field on D with $|b(x)| \le m$ and $|b(x) - b(y)| \le m|x - y|$ for all x, y and $\frac{1}{2}\Delta u(x) + b(x)\cdot\nabla u(x) = 0$, $x \in D$, then there is a constant $c = c(m)$ such that

$$|\nabla u(x)| \le 2(1 + c/\delta_x) \operatorname*{osc}_Q u.$$

The next Corollary is a well-known consequence of the above and Harnack's inequality.

COROLLARY 2. *There is a positive constant c such that if $\frac{1}{2}\Delta u = 0$ on D and $u > 0$ then*

$$|\nabla u(x)| \le \frac{c}{\delta_x} u(x)\,.$$

PROOF (OF THEOREM 1). Given $x, x' \in D$ with $|x - x'| < \frac{\delta_x}{4}$, define the cube (slightly different from that in the Theorem but still denoted by Q)

$$Q = \{y : \max_{1\le i\le d} |y_i - \frac{(x_i + x_i')}{2}| \le \frac{\delta_x}{2}\}.$$

Notice that $\tau_Q = \tau_Q'$. Using the Lindvall-Rogers coupling and Itô's formula

$$|u(x) - u(x')| \le |E^{(x,x')}\left[u(B_{\tau_Q}) - u(B'_{\tau_Q'})\right]| + |E^{(x,x')}\int_0^{\tau_Q} (f(B_s) - f(B_s'))\,ds|.$$

In the first expectation on the right hand side, $u(B_{\tau_Q}) - u(B'_{\tau_Q'}) \ne 0$ only on the set $\{T > \tau_Q\}$. Similarly, the integrand inside the second expectation is only nonzero on $\{s : 0 < s < T \wedge \tau_Q\}$. Thus,

$$|u(x) - u(x')| \le \left(\operatorname*{osc}_Q u\right) P^{(x,x')}(T > \tau_Q) + \left(\operatorname*{osc}_Q f\right) E^{(x,x')}[T \wedge \tau_Q]$$

So it remains to prove the estimates

a) $P^{(x,x')}(T > \tau_Q) \le \frac{d|x-x'|}{\delta_x}$
b) $E^{(x,x')}[T \wedge \tau_Q] \le \frac{1}{4}|x - x'|\delta_x$.

These will be done by a reduction to a one-dimensional problem. Set $\hat{x} = \frac{x+x'}{2}$ and define

$$\hat{Q} = \{y \in Q : (y - \hat{x})\cdot(x - \hat{x}) > 0\},$$
$$L = \{y \in \partial\hat{Q} : (y - \hat{x})\cdot(x - \hat{x}) = 0\}$$
$$S = \{y \in \partial\hat{Q}\backslash L : |(y - \hat{x})\cdot\frac{(x - \hat{x})}{|x - \hat{x}|}| < \frac{\delta_x}{2}\}$$
$$U = \partial\hat{Q}\backslash(S \cup L)$$

the upper half of Q, its lower, side and upper boundary, respectively. Then

$$P^{(x,x')}(T > \tau_Q) = P^{(x,x')}(B_{\tau_{\hat{Q}}} \notin L).$$

As a preparation for the case of general d, we proceed to estimate the latter probability in $d = 2$. For $d = 2$ then, put $M = \left\{ y \in \hat{Q} : \frac{|(y-\hat{x})\cdot(x-\hat{x})|}{|y-\hat{x}||x-\hat{x}|} = \frac{\sqrt{2}}{2} \right\}$ (the union of the line segments from $\hat{x}$ to the endpoints of U) and $\sigma_M = \inf\{ t > 0 : B_t \in M \}$. Then

$$P^{(x,x')}(B_{\tau_{\hat{Q}}} \notin L) = P^{(x,x')}(B_{\tau_{\hat{Q}}} \in U) + P^{(x,x')}(B_{\tau_{\hat{Q}}} \in S)$$

and

$$\begin{aligned} P^{(x,x')}(B_{\tau_{\hat{Q}}} \in S) &= P^{(x,x')}\left(P^{B_{\sigma_M}}(B_{\tau_{\hat{Q}}} \in S); \sigma_M < \tau_{\hat{Q}})\right) \\ &\leq P^{(x,x')}\left(P^{B_{\sigma_M}}(B_{\tau_{\hat{Q}}} \in U); \sigma_M < \tau_{\hat{Q}})\right) \\ &\quad \text{(which holds by symmetry)} \\ &\leq P^{(x,x')}\left(B_{\tau_{\hat{Q}}} \in U\right) \end{aligned}$$

and so

$$P^{(x,x')}(B_{\tau_{\hat{Q}}} \notin L) \leq 2P^{(x,x')}(B_{\tau_{\hat{Q}}} \in U).$$

Next put $S' = \{ y \in \mathbb{R}^2 : 0 < (y - \hat{x}) \cdot \frac{(x-\hat{x})}{|x-\hat{x}|} < \frac{\delta_x}{2} \}$, the smallest slab containing $\hat{Q}$ and $V = \left\{ y \in \partial S' : (y - \hat{x}) \cdot \frac{x-\hat{x}}{|x-\hat{x}|} = \frac{\delta_x}{2} \right\}$. Then, in view of the previous inequality,

$$\begin{aligned} P^{(x,x')}(B_{\tau_{\hat{Q}}} \notin L) &\leq 2P^{(x,x')}(B_{\tau_{S'}} \in V) \\ &= \frac{2|x - x'|}{\delta_x} \end{aligned}$$

since $P^{(x,x')}(B_{\tau_{S'}} \in V)$ is the probability that a one-dimensional Brownian motion started at $\frac{|x-x'|}{2}$ leaves $[0, \frac{\delta_x}{2}]$ at $\frac{\delta_x}{2}$. This proves a) for $d = 2$. For $d \geq 3$, set $S' = \{ y \in \mathbb{R}^d : (y-\hat{x}) \cdot \frac{x-\hat{x}}{|x-\hat{x}|} < \frac{\delta_x}{2} \}$ and $V = \left\{ y \in \partial S' : (y - \hat{x}) \cdot \frac{x-\hat{x}}{|x-\hat{x}|} < \frac{\delta_x}{2} \right\}$. Assume, for convenience, that the axes are selected so that $x_i = x_i'$ for $i = 2, \cdots, d$. Define $A_1 = \{ B_{\tau_{S'}} \in V \}$ and for $2 \leq i \leq d$, $A_i = \{ |(B - \hat{x})_i|$ exceeds $\delta_x/2$ before $\tau_{S'} \}$. Then, proceeding as in the case $d = 2$, we have

$$P^{(x,x')}(A_i) \leq \left(\frac{|x - x'|}{2} \right) \Big/ (\delta_x/2) = \frac{|x - x'|}{\delta_x}, \quad i = 1, 2, \cdots, d.$$

Thus,

$$P^{(x,x')}(B_{\tau_{\hat{Q}}} \notin L) \leq \sum_{i=1}^{d} P^{(x,x')}(A_i) \leq \frac{d|x - x'|}{\delta_x},$$

and so

$$P^{(x,x')}(T > \tau_Q) \leq \frac{d|x - x'|}{\delta_x}.$$

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Turning to b), for any $d \geq 3$,

$$\begin{aligned} E^{(x,x')}[T \wedge \tau_Q] &= E^{(x,x')}[\tau_{\hat{Q}}] \\ &\leq E^{(x,x')}[\tau_{S'}] \\ &= \frac{|x-x'|}{2}\left(\frac{\delta_x}{2} - \frac{|x-x'|}{2}\right) \end{aligned}$$

where again the penultimate quantity is the expected time for a one-dimensional Brownian motion to exit $[0, \frac{\delta_x}{2}]$ when started at $\frac{|x-x'|}{2}$. That proves b).

With a) and b) proved,

$$|u(x) - u(x')| \leq d\frac{|x-x'|}{\delta_x}\left(\operatorname*{osc}_Q u\right) + \frac{1}{4}\delta_x|x-x'|\left(\operatorname*{osc}_Q f\right)$$

and consequently

$$|\nabla u(x)| \leq \frac{d}{\delta_x}\operatorname*{osc}_Q u + \frac{1}{4}\delta_x \operatorname*{osc}_Q f.$$

We now show how the method also handles $L = \frac{1}{2}\Delta + b(\cdot)$. Take two diffusions X and X' with $X_0 = x$, $X'_0 = x'$ as described in Section 1. That is, take $\sigma = I$, then

$$\begin{aligned} dX_t &= dB_t + b(X_t)dt \\ dX'_t &= dB'_t + b(X'_t)dt\,, \end{aligned}$$

where $\{B'_t\}$ is just $\{B_t\}$ reflected in the hyperplane P as in the case $b \equiv 0$. A short calculation with Ito's formula shows that there is a $BM(\mathbb{R})$, b, such that

$$\begin{aligned} d|X_t - X'_t| &= 2db_t + \left\langle \frac{X_t - X'_t}{|X_t - X'_t|}\,, b(X_t) - b(X'_t)\right\rangle dt\,, \\ |X_0 - X'_0| &= |x - x'|\,. \end{aligned}$$

Recall that we assume

$$|b(x) - b(y)| \leq m|x-y|,\ x, y \in D \text{ and } |b(x)| \leq m\,, \quad x \in D\,.$$

Then, to obtain a gradient estimate for a solution $Lu = 0$ in D, we must estimate, with Q as in Theorem 1,

$$P^{(x,x')}(T(X,X') > \tau_Q(X) \wedge \tau_Q(X'))\,.$$

Notice first if $\sigma_\delta = \inf\{t > 0 : |X_t - X'_t| = \delta_x\}$, then this probability is equal to $P^{(x,x')}(T(X,X') > \tau_Q(X) \wedge \tau_Q(X') \wedge \sigma_\delta)$.

This probability is smaller than

$$P^{(x,x')}(T(X,X') > \tau_Q(X) \wedge \sigma_\delta) + P^{(x,x')}(T(X,X') > \tau_Q(X') \wedge \sigma_\delta)\,.$$

Focusing on the first term we use two one-dimensional comparison theorems applied to $|X_t - X'_t|$ and $|X_t - x|$. Since

$$d|X_t - x| = dw_t + \frac{d-1}{2|X_t - x|}dt + \frac{X_t - x}{|X_t - x|}b(X_t)dt$$

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and in view of the equation for $|X_t - X'_t|$ we consider

$$d\rho_t = 2db_t + 2m\rho_t dt\ , \quad \rho_0 = |x - x'|$$
$$d\eta_t = dw_t + \frac{d-1}{2\eta_t}dt + mdt\ , \quad \eta_0 = 0$$

where b and w are the same $BM(\mathbb{R})$s appearing in $d|X_t - X'_t|$ and $d|X_t - x|$, respectively. Then a.s. from one-dimensional comparison theorems

$$|X_t - X'_t| \le \rho_t\ , \quad \forall t\ ,$$
$$|X_t - x| \le \eta_t\ , \quad \forall t\ ,$$

Consequently, if $\sigma_\delta(\rho) = \inf\{t > 0 : \rho_t \ge \delta_x\}$ then

$$T(X, X') < \inf\{t > 0 : \rho_t = 0\} = \sigma_0(\rho)$$
$$\tau_Q(X) > \inf\{t > 0 : \eta_t > \delta_x/2\} = \sigma_{\delta/2}(\eta)$$

and

$$\sigma_\delta(\rho) < \sigma_\delta\ .$$

Thus,

$$P^{(x,x')}(T(X, X') > \tau_Q(X) \wedge \sigma_\delta) \le P^{(\rho_0,0)}(\sigma_0(\rho) > \sigma_{\delta/2}(\eta) \wedge \sigma_\delta(\rho))$$

where we have used the superscript $(\rho_0, 0)$ to indicate the starting position for (ρ_t, η_t). The probability on the right hand side is the probability that (ρ, η) at time $\tau = \inf\{t > 0 : \rho_t \notin (0, \delta) \text{ or } \eta_t \notin (0, \delta/2)\}$ is in the set $A = \{(\rho, \eta) : \eta = \delta/2, 0 < \varphi \le \delta\} \cup \{(\rho, \eta) : 0 < \eta \le \delta/2, \rho = \delta\}$. That is we need to estimate $E^{(\rho_0,0)}1_A(\rho_\tau, \eta_\tau)$. Consider the function

$$g(\rho, \eta) = \begin{cases} 1, & 0 < \rho < \delta, 3\delta/8 < \eta < \delta/2, \\ \left[(\delta - \rho)\frac{1+\sin((\eta - \frac{\delta}{4})\frac{4\pi}{\delta})}{2} + \rho\right]\delta^{-1}, & 0 < \rho < \delta, \delta/8 < \eta < 3\delta/8, \\ \rho, & 0 < \rho < \delta, 0 < \eta < \delta/8\ . \end{cases}$$

Then $g(\rho, \eta) \ge 1_A(\rho, \eta)$ so $E^{(\rho_0,0)}g(\rho_\tau, \eta_\tau) \ge E^{(\rho_0,0)}1_A(\rho_\tau, \eta_\tau)$. By Ito's formula, writing $c_t dt = 2d < b, w >_t$, with $|c_t| \le 2$,

$$E^{(\rho_0,0)}g(\rho_\tau, \eta_\tau) =$$
$$g(\rho_0, 0) + E^{(\rho_0,0)}\int_0^\tau \left[2g_{\rho\rho} + 2m\rho_t g_\rho + \frac{1}{2}g_{\eta\eta} + \left[\frac{d-1}{2\eta} + m\right]g_\eta + c_t g_{\rho\eta}\right]dt\ .$$

But, $g_{\rho\rho} = 0$, $|g_\rho| \le 1 \wedge \delta^{-1}$, $|g_\eta| \le \frac{2\pi}{\delta}$ and $g_\eta = 0$ for $0 < \eta < \delta/8$, and $|g_{\eta\eta}| \leqq \frac{8\pi^2}{\delta^2}$, $|g_{\rho\eta}| \le \frac{2\pi}{\delta^2}$. Thus,

$$E^{(\rho_0,0)}g(\rho_\tau, \eta_\tau) \le g(\rho_0, 0) + \frac{c}{\delta^2}E^{(\rho_0,0)}\tau\ .$$

An easy comparison shows $E^{(\rho_0,0)}\tau \le E^{\rho_0}\sigma_\delta(\rho)$. Note $u(t) = E^t\sigma_\delta(\rho)$ solves the boundary value problem

$$2u'' + 2mtu' = -1$$
$$u(0) = u(\delta) = 0\ .$$

Thus

$$u(t) = \frac{1}{2}\int_0^t e^{\frac{-mr^2}{2}}\left[\int_0^\delta e^{\frac{-mu^2}{2}}\int_0^u e^{\frac{ms^2}{2}}ds\left(\int_0^\delta e^{\frac{-mu^2}{2}}du\right)^{-1}du - \int_0^r e^{\frac{mu^2}{2}}du\right]dr$$

so $u(\rho) \leq \frac{1}{2}\cdot e^{\frac{m\delta^2}{2}}\rho\delta$. Also, by definition $g(\rho_0, 0) = \rho_0$.

Thus,

$$P^{(x,x')}(T(X,X') > \tau_Q(X)\wedge\sigma_\delta) \leq \left(1+\frac{c}{\delta}\right)\rho_0$$

and since $\rho_0 = |x-x'|$

$$P^{(x,x')}(T(X,X') > \tau_Q(X)\wedge\tau_Q(X')) \leq 2\left(1+\frac{c}{\delta}\right)|x-x'|\,.$$

Thus, if $\frac{1}{2}\Delta u + b(\cdot)\cdot\nabla u = 0$ in D

$$|\nabla u(x)| \leq 2\left(1+\frac{c}{\delta_x}\right)\operatorname*{osc}_Q u\,.$$

This completes the proof. ∎

REMARKS. (1) The constants d and $\frac{1}{4}$ in Theorem 1 appear to be the correct ones (compare with Gilbarg, Trudinger [4], p.37.)

(2) Elliptic operators $L = a_{ij}\frac{\partial^2}{\partial x^i\partial x^j}$ with constant coefficients can be easily handled with this method. If $\sigma\sigma^t = a$ for a constant matrix σ, then the diffusion with generator L is $X_t = \sigma B_t$ with B a $BM(\mathbb{R}^d)$. To couple X with another such diffusion X' just couple B and B' and take $X' = \sigma B'$.

(3) Higher order derivatives might be handled by coupling many particles. We give a brief account of this for $\partial^2 u/\partial x_i^2$. Let

$$e = \frac{\partial}{\partial x_i} \text{ and observe that}$$

$$u(x+he) + u(x-he) - 2u(x) = h^2\frac{\partial^2 u}{\partial x_i^2} + o(h^2)\,.$$

Take Q to be a cube with center x, diameter δ_x and such that the line $x+te$ intersects the boundary of Q in a right angle.

Begin a $BM(\mathbb{R}^d)$, X, at $x+he$ and use the Lindvall, Rogers reflection to produce another $BM(\mathbb{R}^d)$, Z, beginning at $x-he$. For the third $BM(\mathbb{R}^d)$, Y, begun at x, we take (Y,e) to be a $BM(\mathbb{R})$ independent of (X,e) and for $(e,f) = 0$ take $(Y,f) = (X,f)$. Then with

$$\tau = \inf\{t > 0 : X_t \notin Q \text{ or } Y_t \notin Q \text{ or } Z_t \notin Q\}$$

$$\begin{aligned} u(x+he) + u(x-he) - 2u(x) &= E[u(X_\tau) + u(Z_\tau) - 2u(Y_\tau); T(X,Y) > \tau, T(Y,Z) > \tau] \\ &\quad + E[u(X_\tau) - u(Z_\tau); T(X,Y) \geq \tau, T(Y,Z) < \tau] \\ &\quad + E[u(Z_\tau) - u(X_\tau); T(X,Y) < \tau, T(Y,Z) \geq \tau]\,. \end{aligned}$$

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It is not too difficult to show that

$$P(T(X,Y) > \tau, T(Y,Z) > \tau) \leq \frac{ch^2}{\delta_x^2}\,.$$

What is a little surprising is that by symmetry of Y in the e direction, the last terms cancel one another. Thus,

$$|u(x+he) + u(x-he) - 2u(x)| \leq c\frac{h^2}{\delta_x^2}\sup_Q u$$

and consequently,

$$\left|\frac{\partial^2 u}{\partial x_i^2}(x)\right| \leq \frac{c}{\delta_x^2}\sup_Q u\,.$$

3. Operators with linear drift. In this section we derive gradient estimates for Ornstein-Uhlenbeck operators on $\mathbb{R}^d$. Let A be a $d \times d$ constant matrix with eigenvalues $\lambda_1, \cdots \lambda_d$ which satisfy

$$Re\lambda_1 \leq Re\lambda_2 \leq \cdots \leq Re\lambda_d.$$

Set $L = \frac{1}{2}\triangle + Ax \cdot \bigtriangledown$. The corresponding diffusion started at $X_0 = x$ is given by

$$X_t = e^{tA}\left(x + \int_0^t e^{-sA}dB_s\right)$$

These operators were studied in Cranston-Orey-Rössler [2] and March [6] and may have many bounded solutions $Lu = 0$. A coupled copy X_t' of X_t begun at x' is given by

$$X_t' = e^{tA}\left(x' + \int_0^t e^{-sA}dB_s'\right)$$

Thus $T(X,X') = T(Z,Z')$ where $Z_t = x + \int_0^t e^{-sA}dB_s$, $Z_t' = x' + \int_0^t e^{-sA}dB_s'$.

We recall the Lindvall-Rogers method for coupling in this situation. Set

$$Y = Z - Z',\ \sigma_s = e^{-sA}, U = \frac{\sigma^{-1}Y}{|\sigma^{-1}Y|}, V = \frac{Y}{|Y|}.$$

The idea again is that the Brownian motion driving Z' will have increments which are an orthogonal transformation of the increments of the Brownian motion B which drives Z. The transformation at time t is given by

$$H_t = I - 2U_tU_t^T = H(Z_t, Z_t')$$

$$\text{so} \qquad dB_t' = H_t dB_t.$$

Now, (see Lindvall-Rogers for details) simple computation shows that with

$$\alpha_t = 2\sigma_t U_t U_t^T$$

then

$$d(|Y_t|) = \alpha_t^T V_t dB_t.$$

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That is, $|Y_t|$ is a martingale with

$$d\langle |Y|\rangle_t = |\alpha_t^T V_t|^2 dt = \frac{4|Y_t|^2}{|\sigma_t^{-1} Y_t|^2} dt.$$

Since $\sigma_t^{-1} = e^{tA}$,

$$\frac{|\sigma_t^{-1} Y_t|^2}{|Y_t|^2} \leq e^{2Re\lambda_d t}.$$

As a result,

$$4e^{-2Re\lambda_d t} \leq \frac{d\langle |Y|\rangle_t}{dt}.$$

With the intention of time-changing, consider τ_t defined by

$$t = \int_0^{\tau_t} d\langle |Y|\rangle_s$$

so that $W_t = |Y|_{\tau_t}$ is now Brownian motion started at $|x - x'|$ up to $\zeta = \int_0^\infty d\langle |Y|\rangle_t$. By the bound on $d\langle |Y|\rangle_t$, a.s.

$$\frac{2}{Re\lambda_d} \leq \zeta$$

and the upper bound now gives us,

$$\begin{aligned}
P^{(x,x')}(T(Z,Z') = \infty) &= P^{(x,x')}\big(|Y_t| \neq 0 \text{ for all } t\big) \\
&= P^{(x,x')}\big(W_t \neq 0 \text{ for all } t \leq \zeta \,|\, W_0 = |x - x'|\big) \\
&\leq P^{(x,x')}\left(W_t \neq 0 \text{ for all } t \leq \frac{2}{Re\lambda_d} \,|\, W_0 = |x - x'|\right) \\
&= P^{(x,x')}\left(\sup_{0 \leq t \leq \frac{2}{Re\lambda_d}} W_t < |x - x'| \,|\, W_0 = 0\right) \\
&= P^{(x,x')}\left(\left|W_{\frac{2}{Re\lambda_d}}\right| < |x - x'| \,|\, W_0 = 0\right), \text{ by reflection principle} \\
&= P^{(x,x')}\left(|W_1| < \sqrt{\frac{Re\lambda_d}{2}}\,|x - x'| \,|\, W_0 = 0\right) \text{ by scaling} \\
&\leq \sqrt{\frac{Re\lambda_d}{2}}\,|x - x'|\ .
\end{aligned}$$

This leads us to

THEOREM 2. *Suppose A is a constant $d \times d$ matrix with $0 < Re\lambda_d$, λ_i the eigenvalues of A. Then if $Lu(x) = \frac{1}{2} \triangle u(x) + Ax \nabla u(x) = 0$ for all $x \in \mathbb{R}^d$ and $u \in L^\infty(\mathbb{R}^d)$ we have*

$$|\nabla u(x)| \leq \sqrt{2Re\lambda_d}\,\|u\|_\infty\ .$$

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PROOF. Given x take another x' nearby and consider $X_t = e^{tA}Z_t$, $X'_t = e^{tA}Z'_t$ with Z_t, Z'_t as above. Then $T(X, X') = T(Z, Z') = T$ is a stopping time and applying the optional sampling theorem at $T \wedge t$

$$|u(x) - u(x')| \leq \left|E^{(x,x')}(u(X_{T\wedge t}) - u(X'_{T\wedge t}))\right|$$

on letting $t \longrightarrow \infty$,

$$|u(x) - u(x')| \leq 2\|u\|_\infty P^{(x,x')}(T(X, X') = \infty)$$
$$\leq \sqrt{2Re\lambda_d}\|u\|_\infty.$$

REMARK. $Re\lambda_d$ gives the maximal rate of dispersion, so to speak, of the diffusion X. Thus the faster particles separate (with $Re\lambda_d$ large) the bigger the gradients will be for the corresponding "harmonic" functions.

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