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Langlands-Shahidi Method and Poles of Automorphic L-Functions: Application to Exterior Square L-Functions

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Abstract. In this paper we use Langlands-Shahidi method and the result of Langlands which says that non self-conjugate maximal parabolic subgroups do not contribute to the residual spectrum, to prove the holomorphy of several *completed* automorphic L-functions on the whole complex plane which appear in constant terms of the Eisenstein series. They include the exterior square L-functions of GL_n, n odd, the Rankin-Selberg L-functions of $\mathrm{GL}_n \times \mathrm{GL}_m$, $n \neq m$, and L-functions $L(s, \sigma, r)$, where σ is a generic cuspidal representation of SO_{10} and r is the half-spin representation of $\mathrm{GSpin}(10, \mathbb{C})$. The main part is proving the holomorphy and non-vanishing of the local normalized intertwining operators by reducing them to natural conjectures in harmonic analysis, such as standard module conjecture.

Introduction

Langlands' theory of Eisenstein series [La2] has been found very useful in the theory of automorphic L-functions. Langlands had the idea of studying automorphic L-functions using Eisenstein series [La1]. This was further developed and refined by Shahidi [Sh1-5]. This is known as Langlands-Shahidi method of studying automorphic L-functions (see [Ge-Sh] or [Sh6] for an excellent survey). This theory has been found very powerful in establishing functional equations and finiteness of poles of automorphic L-functions in the great generality which appear in the constant terms of Eisenstein series. On the other hand, it has been thought that the precise location of poles of L-functions is very hard to get by this method. Of course, the result of Moeglin-Waldspurger [M-W2] is the first instance, where they proved, using Eisenstein series, that the *completed* Rankin-Selberg L-function for $\mathrm{GL}_n \times \mathrm{GL}_m$ is holomorphic for $0 < \mathrm{Re}\, s < 1$.

In this paper we use Langlands-Shahidi method [Sh4] and the following simple result of Langlands [La] to prove the holomorphy of several *completed* automorphic L-functions which appear in constant terms of the Eisenstein series. Because of the functional equation $L(s, \sigma, r) = \epsilon(s, \sigma, r)L(1 - s, \tilde{\sigma}, r)$, it is enough to establish the holomorphy for $\mathrm{Re}\, s \geq \frac{1}{2}$.

Let G be a quasi-split reductive connected algebraic groups over a number field F and $\mathbb{A}$ is the ring of adeles of F. Let Z_d be the maximal F-split torus of the center of G. Fix a unitary character ξ of $Z_d(F) \setminus Z_d(\mathbb{A})$. Let

$$L^2\big(G(F) \setminus G(\mathbb{A}), \xi\big) = \big\{ f \in L^2\big(G(F)Z_d(\mathbb{A}) \setminus G(\mathbb{A})\big) \mid f(zg) = \xi(z) f(g),$$
$$\text{for all } z \in Z_d(\mathbb{A}), g \in G(\mathbb{A}) \big\}.$$

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If G is semi-simple, we do not have to consider the central characters. It is of great importance to decompose $L^2\big(G(F)\setminus G(\mathbb{A}),\xi\big)$. Langlands' theory tells us that it has an orthogonal decomposition according to the conjugacy classes of (M,σ), where M is a Levi subgroup of G and σ is a cuspidal representation of M. Its discrete part attached to (M,σ) is called the residual spectrum, denoted by $L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A}),\xi\big)_{(M,\sigma)}$. It is spanned by residues of Eisenstein series associated to (M,σ). Suppose P is a maximal parabolic subgroup generated by $\theta=\Delta-\{\alpha\}$, where Δ is a set of simple roots. Then there exists a unique Weyl group element w_0 such that $w_0\theta\subset\Delta$ and $w_0\alpha<0$. If $w_0\theta=\theta$, P is called self-conjugate.

***Proposition 0.1* (Langlands [La2, Lemma 7.5])** *Unless $P=\mathrm{MN}$ is self-conjugate and σ is a cuspidal representation which satisfies $w_0\sigma=\sigma$, $L^2_{\mathrm{dis}}\big(G(F)\setminus G(\mathbb{A}),\xi\big)_{(M,\sigma)}$ is zero.*

We apply the above result to the following situation: We follow [Sh4] and use the same notation. Let $P=\mathrm{MN}\subset G$ be a maximal parabolic subgroup and σ be a cuspidal representation of $M(\mathbb{A})$. We may and will assume that the poles of Eisenstein series may be on the real axis by normalizing σ so that the action of the maximal split torus in the center of M at the archimedean places is trivial (see Section 2). The poles of the Eisenstein series attached to (M,σ) coincide with those of its constant term which consists of automorphic L-functions and local normalized intertwining operators and the residue of the Eisenstein series for $s>0$ belongs to the residual spectrum. If P is not self-conjugate or $w_0\sigma\neq\sigma$, then the Eisenstein series does not have poles for $s>0$. If we can show that the local normalized intertwining operators are holomorphic and non-zero for $\operatorname{Re}s\geq\frac{1}{2}$, then the automorphic L-functions do not have a pole for $s\geq\frac{1}{2}$.

Up to isogeny or more generally central surjections, there are four non self-conjugate maximal parabolic subgroups in split groups whose derived groups are almost simple: (1) $G=\mathrm{GL}_{m+n}$ and $P=\mathrm{MN}$, $M=\mathrm{GL}_m\times\mathrm{GL}_n$ for $m\neq n$, (2) $G=\mathrm{SO}_{2n}$ and $P=\mathrm{MN}$, $M=\mathrm{GL}_n$ for n odd, (3) G is a simply-connected split group of type E_6 and $P=\mathrm{MN}$, the derived group of M is $\mathrm{SL}_2\times\mathrm{SL}_5$ and (4) G is a simply-connected split group of type E_6 and $P=\mathrm{MN}$, $M=\mathrm{GL}_1\cdot D_5$ (almost direct product), which is $\mathrm{GSpin}(10)$.

By using the classification of unitary representations of GL_n due to Tadic [Ta], we prove the result on local normalized intertwining operators in cases (1), (2) and (3). We have the following theorems. In the case of (1), it is a special case of [M-W2, Appendix] and [J-S1].

Theorem 0.2

1. *Let σ_1 (resp. σ_2) be a cuspidal representation of GL_m (resp. GL_n), $m\neq n$. Then the completed Rankin-Selberg L-function $L(s,\sigma_1\times\tilde{\sigma}_2)$ is entire.*
2. *Let σ be a cuspidal representation of GL_n, n odd. Then the completed exterior square L-function $L(s,\sigma,\wedge^2)$ is entire.*
3. *Let σ_1,σ_2 be cuspidal representations of PGL_2, PGL_5, resp. Suppose Conjecture 7.1 of [Sh1] holds for the exceptional group of type E_6. Then the completed L-function $L(s,\sigma_1\otimes\tilde{\sigma}_2,\rho_2\otimes\wedge^2\rho_5)$ is entire, where ρ_n is a standard representation of $\mathrm{GL}_n(\mathbb{C})$.*

Recall the definition of the above L-functions: Let S be a finite set of places, including all the archimedean places, such that for every $v\notin S$, σ_{1v}, σ_{2v}, are all unramified. For $v\notin S$, let $A(\sigma_{1v})=\{\operatorname{diag}(\alpha_{1v},\ldots,\alpha_{mv})\}$ be the semisimple conjugacy classes attached to σ_{1v}. Let $A(\sigma_{2v})=\{\operatorname{diag}(\beta_{1v},\ldots,\beta_{nv})\}$ be the one attached to σ_{2v}. Then the local L-functions are

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given by

$$L(s,\sigma_{1v}\times\tilde\sigma_{2v})=\prod_{1\le i\le m,1\le j\le n}(1-\alpha_{iv}\beta_{jv}^{-1}q_v^{-s})^{-1}$$

$$L(s,\sigma_v,\wedge^2)=\prod_{1\le i<j\le n}(1-\alpha_{iv}\alpha_{jv}q_v^{-s})^{-1}$$

$$L(s,\sigma_{1v}\otimes\tilde\sigma_{2v},\rho_2\otimes\wedge^2\rho_5)=\prod_{1\le i\le 2,1\le j<k\le 5}(1-\alpha_{iv}\beta_{jv}^{-1}\beta_{kv}^{-1}q_v^{-s})^{-1}.$$

The local L-functions at ramified places $v\in S$ are defined in [Sh1] in such a way that they agree with the ones defined by parametrization.

Proposition 0.3

1. *Let σ_1,σ_2 be cuspidal representations of* GL_n, *where $\sigma_1\not\cong\sigma_2\otimes|\det|^t$ for $t\in\mathbb{C}$. Then the Rankin-Selberg L-function $L(s,\sigma_1\times\sigma_2)$ is entire.*
2. *Let σ be a non self-dual cuspidal representation of* GL_n, *n even. Then the exterior square L-function $L(s,\sigma,\wedge^2)$ is entire.*

F. Shahidi encouraged us to consider the case (4) after his work with Muić [Mu-Sh]: we get an automorphic L-function $L(s,\sigma,r)$ where σ is a generic cuspidal representation of $M(\mathbb{A})$ and r is a representation of ${}^LM^0=\mathrm{GSpin}(10,\mathbb{C})$. Here r is one of the two 16-dimensional irreducible half-spin representations of $\mathrm{GSpin}(10,\mathbb{C})$. However, we were not able to prove that the local normalized intertwining operators are holomorphic and non-zero for $\mathrm{Re}\,s\ge\frac12$ at ramified places. One serious obstacle is that we do not have the standard module conjecture for $\mathrm{SO}(2n)$. Nevertheless, we obtain the result that the partial L-function $L_S(s,\sigma,r)$ is holomorphic for $\mathrm{Re}\,s>0$. In the same way, we see that the partial L-function $L_S(s,\sigma_1\otimes\tilde\sigma_2,\rho_2\otimes\wedge^2\rho_5)$ in Theorem 0.2 is holomorphic for $\mathrm{Re}\,s>\frac12$ without any assumption.

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1 Preliminaries

In this section, let F be a local field of characteristic zero. We follow the conventions of [C-Sh] or [Sh4]. Let $\mathbf{G}$ be a quasi-split connected reductive algebraic group over F. Fix a Borel subgroup $\mathbf{B}$ and write $\mathbf{B}=\mathbf{TU}$, where $\mathbf{T}$ is a maximal torus and $\mathbf{U}$ denotes the unipotent radical of $\mathbf{B}$.

Fix a F-parabolic subgroup $\mathbf{P}=\mathbf{MN}$ with $\mathbf{N}\subset\mathbf{U}$ and $\mathbf{T}\subset\mathbf{M}$, a Levi decomposition. Let $\mathbf{A}_0$ be the maximal F-split torus of $\mathbf{T}$ and denote by $W=W(\mathbf{A}_0)$ the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Let $\tilde w_0$ be the longest element in $W(\mathbf{A}_0)$ modulo that of the Weyl group of $\mathbf{A}_0$ in $\mathbf{M}$ and w_0 be a representative for $\tilde w_0$. If P is a maximal parabolic subgroup generated by $\theta=\Delta-\{\alpha\}$, then w_0 is the unique element in W such that $w_0(\theta)\subset\Delta$ while $w_0(\alpha)<0$.

Set

$$\mathfrak{a} = X(\mathbf{M})_F \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$\mathfrak{a}^*_{\mathbb{C}} = \mathfrak{a}^* \otimes_{\mathbb{R}} \mathbb{C},$$

where $X(\mathbf{M})_F$ is the group of F-rational characters of $\mathbf{M}$. As usual, we let

$$I(\nu,\sigma) = \mathrm{Ind}_{\mathrm{MN}\uparrow G}\, \sigma \otimes \exp^{\langle \nu, H_P(\ )\rangle} \otimes \mathbf{1},$$

where $\nu \in \mathfrak{a}^*_{\mathbb{C}}$.

Suppose ν is in the positive Weyl chamber and σ is tempered. Then $I(\nu,\sigma)$ has a unique irreducible quotient, denoted by $J(\nu,\sigma)$. Let $A(\nu,\sigma,w_0)$ be the standard intertwining operator from $I(\nu,\sigma)$ into $I(w_0\nu, w_0\sigma)$. Then $J(\nu,\sigma)$ is the image of $A(\nu,\sigma,w_0)$.

Now assume $\mathbf{P}$ is maximal and let α be the unique simple root in $\mathbf{N}$. As in [Sh1], let $\tilde{\alpha} = \langle \rho, \alpha\rangle^{-1} \cdot \rho$, where ρ is half the sum of roots in $\mathbf{N}$. We identify $s \in \mathbb{C}$ with $s\tilde{\alpha} \in \mathfrak{a}^*_{\mathbb{C}}$ and denote $I(s,\sigma) = I(s\tilde{\alpha},\sigma)$.

Remark 1.1 We have to pay attention to the normalization of $\tilde{\alpha}$ because it is crucial for our purpose. For example, if $G = \mathrm{Sp}_{2n}, P = \mathrm{MN}, M = \mathrm{GL}_n$, then $I(s,\sigma) = I(s\tilde{\alpha},\sigma) = \mathrm{Ind}_P^G(\sigma \otimes |\det|^s) \otimes 1$. But if $G = \mathrm{SO}_{2n}$ or SO_{2n+1}, $P = \mathrm{MN}$, $M = \mathrm{GL}_n$, then $I(s,\sigma) = I(s\tilde{\alpha},\sigma) = \mathrm{Ind}_P^G(\sigma \otimes |\det|^{\frac{s}{2}}) \otimes 1$. On the other hand, if $G = \mathrm{SO}_{2n}$ or SO_{2n+1}, $P = \mathrm{MN}$, $M = \mathrm{GL}_k \otimes G_l$, where $G_l = \mathrm{SO}_{2l}$ or SO_{2l+1}, $k < n$, then $I(s,\sigma) = I(s\tilde{\alpha},\sigma) = \mathrm{Ind}_P^G(\sigma \otimes |\det|^s \otimes \tau) \otimes 1$ for σ (resp. τ) tempered representation of GL_k (resp. G_l).

Let $A(s\tilde{\alpha},\sigma,w_0)$ be the standard intertwining operator from $I(s\tilde{\alpha},\sigma)$ into $I\big(w_0(s\tilde{\alpha}), w_0(\sigma)\big)$. Denote by ${}^LM$, the L-group of $\mathbf{M}$ and let ${}^L\mathfrak{n}$ be the Lie algebra of the L-group of $\mathbf{N}$. Let r be the adjoint action of ${}^LM$ on ${}^L\mathfrak{n}$ and decompose $r = \bigoplus_{i=1}^m r_i$, with ordering as in [Sh1]. For each i, $1 \le i \le m$, let $L(s,\sigma,r_i)$ be the local L-function defined in [Sh1]. It is defined to agree completely with Langlands definition of L-functions whenever there is a parametrization. In particular the L-function for arbitrary σ is just the analytic continuation of the one attached to the tempered inducing data through the product formula (*cf.* part 3 of Theorem 3.5 and equation 7.10 of [Sh1]). (See also Theorem 5.2 of [Sh2].)

Recall Conjecture 7.1 of [Sh1].

Conjecture *Assume σ is tempered and generic. Then each $L(s,\sigma,r_i)$ is holomorphic for* $\mathrm{Re}\, s > 0$.

Proposition 1.1 **[Sh1]** *If $m = 1$ or (2) $m = 2$ and $L(s,\sigma,r_2) = \prod_j (1-\alpha_j q^{-s})^{-1}$ for σ tempered and generic, possibly an empty product where each $\alpha_j \in \mathbb{C}$ is of absolute value one (in particular if r_2 is one-dimensional, this holds), then the conjecture holds.*

Proposition 1.2 **[C-Sh]** *If G is a classical group, then the conjecture holds.*

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2 Basic facts on Eisenstein series

From this section on, we work with a number field F. Let $\mathbf{P} = \mathbf{MN}$ be a maximal parabolic subgroup of $\mathbf{G}$ generated by $\theta = \Delta - \{\alpha\}$. We follow the convention of [Sh4]. Let $\sigma = \otimes \sigma_v$ be a unitary cuspidal representation of $M(\mathbb{A})$. We may and will assume that the poles of Eisenstein series may be on the real axis by assuming that σ is trivial on A part of $P(\mathbb{R})$, where $P(\mathbb{R}) = M^0AN$ is the Langlands decomposition. In the case of $M = \mathrm{GL}_n$, we can identify the A part of $P(\mathbb{R})$ with F_∞^+, where $\mathbb{A}_F^* = \mathbb{I}^1 \cdot F_\infty^+$ with $\mathbb{I}^1$ ideles of norm 1. So in this case the central character ω_σ of σ is trivial on F_∞^+. Given a K-finite function φ in the space of σ, we shall extend φ to a function $\tilde{\varphi}$ on $G(\mathbb{A})$ and set $\Phi_s(g) = \tilde{\varphi}(g) \exp\langle s + \rho_P, H_P(g)\rangle$, where H_P is the Harish-Chandra homomorphism. Define an Eisenstein series

$$E(s, \tilde{\varphi}, g, P) = \sum_{\gamma \in P(F) \backslash G(F)} \Phi_s(\gamma g).$$

It is known [La2] that $E(s, \tilde{\varphi}, g, P)$ converges for $\mathrm{Re}\, s \gg 0$ and extends to a meromorphic function of s in $\mathbb{C}$, with only a finite number of poles in the plane $\mathrm{Re}\, s \geq 0$, all simple and on the real axis if we normalize σ as above.

We also know that the space of Φ_s is isomorphic to $I(s, \sigma) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \sigma \otimes \exp\big(\langle s\tilde{\alpha}, H_P(\ )\rangle\big)$, the global induced representation from $P(\mathbb{A})$ to $G(\mathbb{A})$. Let $f \in I(s, \sigma)$. If $E(s, f, g, P)$ is defined by analytic continuation, then it is an automorphic form on G. Recall that the residual spectrum attached to (M, σ), $L^2_{\mathrm{dis}}\big(G(F) \setminus G(\mathbb{A}), \xi\big)_{(M,\sigma)}$ is spanned by the residues of the Eisenstein series $E(s, f, g, P)$ for $\mathrm{Re}\, s > 0$ and $f \in I(s, \sigma)$.

We know that the poles of the Eisenstein series coincide with those of its constant terms. Let M' be the subgroup of G generated by $w_0(\theta)$ and P' be a maximal parabolic subgroup which has M' as its Levi factor and $N' \subset U$ as its unipotent radical. Recall the definition of self-conjugate maximal parabolic subgroups [Sh3]: P is called self-conjugate if and only if $w_0(\theta) = \theta$. Given a parabolic subgroup $Q = M_Q N_Q$, the constant term of $E(s, f, g, P)$ along N_Q is zero if $Q \neq P$ and $Q \neq P'$. If P is not self-conjugate, then

$$E_N(s, f, g, P) = f(g)$$
$$E_{N'}(s, f, g, P) = M(s, \sigma, w_0) f(g).$$

If P is self-conjugate, then $E_N(s, f, g, P)$ is a sum of the above two terms. Here $M(s, \sigma, w_0)$ is the standard intertwining operator from the global induced representation $I(s, \sigma)$ to $I(w_0 s, w_0 \sigma)$. Let $M(s, \sigma, w_0) = \otimes_v A(s, \sigma_v, w_0)$. We normalize the intertwining operator $A(s, \sigma_v, w_0)$ as follows:

$$A(s, \sigma_v, w_0) = r(s, \sigma_v, w_0) N(s, \sigma_v, w_0),$$

(2.1)
$$r(s, \sigma_v, w_0) = \prod_{i=1}^{m} \frac{L(is, \sigma_v, r_i)}{L(1 + is, \sigma_v, r_i) \epsilon(s, \sigma_v, r_i, \psi_v)},$$

where $L(is, \sigma_v, r_i)$ and $\epsilon(s, \sigma_v, r_i, \psi_v)$ are defined in [Sh1]. Let $N(s, \sigma, w_0) = \otimes_v N(s, \sigma_v, w_0)$, $r(s, \sigma, w_0) = \prod_v r(s, \sigma_v, w_0)$ and $\epsilon(s, \sigma, r_i) = \prod_v \epsilon(s, \sigma_v, r_i, \psi_v)$. Then we have, for $f \in$

$I(s,\sigma)$,

(2.2)

$$M(s,\sigma,w_0)f = r(s,\sigma,w_0)N(s,\sigma,w_0)f, \quad r(s,\sigma,w_0) = \prod_{i=1}^{m} \frac{L(is,\sigma,r_i)}{L(1+is,\sigma,r_i)\epsilon(s,\sigma,r_i)}.$$

Recall Langlands' theory in this case: Let $\phi_f = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=s_0} E(s,f,g,P)\,ds$. Then ϕ_f spans a dense subspace of $L^2\bigl(G(F)\setminus G(\mathbb{A}),\xi\bigr)_{(M,\sigma)}$. The L^2-norm of ϕ_f is given by

$$\begin{aligned}\langle \phi_f,\phi_f\rangle_{L^2(G(F)\setminus G(\mathbb{A}),\xi)} &= \int_{Z_d(\mathbb{A})G(F)\setminus G(\mathbb{A})} |\phi_f|^2\,dx \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=s_0} \sum_{w\in\Omega(\theta,\theta)} \bigl(M(s,\sigma,w)f(s), f(-w\bar{s})\bigr)\,ds,\end{aligned}$$

where $\Omega(\theta,\theta)=\{\mathrm{id}\}$ if P is not self-conjugate and $\Omega(\theta,\theta)=\{\mathrm{id},w_0\}$ if P is self-conjugate. However, when P is self-conjugate and $w_0\sigma\neq\sigma$, $\bigl(M(s,\sigma,w_0)f(s), f(-w_0\bar{s})\bigr)$ is identically zero since $M(s,\sigma,w_0)f(s)\in I\bigl(-s,w_0(\sigma)\bigr)$ and $f(-w_0\bar{s})\in I(\bar{s},\sigma)$. Therefore we have

Proposition 2.1 **(Langlands)** *Unless* $P=\mathrm{MN}$ *is self-conjugate and* $w_0\sigma=\sigma$, *the residual spectrum attached to* (M,σ), $L^2_{\mathrm{dis}}\bigl(G(F)\setminus G(\mathbb{A}),\xi\bigr)_{(M,\sigma)}$, *is zero.*

Proof Under the assumption, in the L^2-norm formula, the integrand is holomorphic. Therefore, we can move the contour to $\mathrm{Re}\, s=0$, *i.e.*, ϕ_f does not contribute to the discrete spectrum. ■

Since the poles of Eisenstein series are contained in the constant terms, we have

Corollary 2.2 *If* $P=\mathrm{MN}$ *is not self-conjugate or* $w_0\sigma\neq\sigma$, *then the global intertwining operator* $M(s,\sigma,w_0)$ *is holomorphic for* $\mathrm{Re}\, s>0$.

We know that $\epsilon(s,\sigma,r_i)$ is an exponential factor and so it has neither a zero nor a pole. So in (2.2), we need to know that $\prod_{i=1}^m L(1+is,\sigma,r_i)$ has no zeros for $\mathrm{Re}\, s>0$. However this is an easy consequence of [Sh3]:

Lemma 2.3 *If* $P=\mathrm{MN}$ *is not self-conjugate or* $w_0\sigma\neq\sigma$, *then* $\prod_{i=1}^m L(1+is,\sigma,r_i)$ *has no zeros for* $\mathrm{Re}\, s>0$.

Proof Consider χ-Fourier coefficient of $E(s,f,g,P)$ [Sh3]: it is given by

$$E_\chi(s,f,e,P) = \prod_{v\notin S} W_{f_v}(s,e_v)\prod_{i=1}^{m} L_S(1+is,\sigma,r_i)^{-1},$$

where W_{f_v} is the Whittaker model of $I(s,\sigma_v)$. Then W_{f_v} is holomorphic for $\mathrm{Re}\, s>0$ and non-vanishing. If P is not self-conjugate or $w_0\sigma\neq\sigma$, then $E(s,f,g,P)$ is holomorphic for $\mathrm{Re}\, s>0$ and so $\prod_{i=1}^m L_S(1+is,\sigma,r_i)$ has no zero for $\mathrm{Re}\, s>0$. ■

From (2.2), we have to analyze the local intertwining operators $N(s, \sigma_v, w_0)$. Suppose we have the following:

Assumption (A) *$N(s, \sigma_v, w_0)$ is holomorphic and non-zero for* $\operatorname{Re} s \geq \frac{1}{2}$ *for any v.*

Let $\sigma = \otimes \sigma_v$ be a globally generic unitary cuspidal representation of M. Then for all v, σ_v is generic and unitary. Suppose σ_v is non-tempered. The following standard module conjecture is proved for various cases including GL_n and also Sp_{2n}, SO_{2n+1} [Mu2]. In [C-Sh], it is proved when G is an arbitrary quasi-split classical group and π_0 is supercuspidal.

Standard module conjecture *Given a non-tempered, generic σ_v, there is a tempered data π_0 and a complex parameter Λ_0 which is in the corresponding positive Weyl chamber so that* $\sigma_v = I_{M_0}(\Lambda_0, \pi_0) = \operatorname{Ind}_{M_0}^{M}(\pi_0 \otimes q^{\langle \Lambda_0, H_{P_0}^{M}(\ ) \rangle})$.

Let σ_v be as above in the conjecture and let $P_0 = M_0 N_0 \subset P$ be another parabolic subgroup with $M_0 \subset M$. Then $I(s, \sigma_v) = I(s\tilde{\alpha} + \Lambda_0, \pi_0)$. By inducing in stages and the factorization property of intertwining operators, we have

$$A(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w}) = A_{M_0}(\Lambda_0, \pi_0, w_{P_0}) A(s, \sigma_v, w_0),$$

where $\tilde{w} = w_{P_0} w_0$ and w_0 is the longest element of the Weyl group of the split component of M in G, $\tilde{w}$ is that of M_0 in G and w_{P_0} is the longest element of the Weyl group of the split component of M_0 in M. Here the operator $A_{M_0}(\Lambda_0, \pi_0, w_{P_0})\colon I_{M_0}(\Lambda_0, \pi_0) \mapsto I_{M_0}(w_0\Lambda_0, w_0\pi_0)$ establishes an isomorphism since $I_{M_0}(\Lambda_0, \pi_0)$ is irreducible, and is identified with its induced map.

Lemma 2.4 *Suppose $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber for* $\operatorname{Re} s \geq \frac{1}{2}$ *together with standard module conjecture and Conjecture 7.1 of [Sh1], then Assumption (A) holds.*

Proof By definition, the normalizing factor $r(s, \sigma_v, w_0)$ in (2.1) is the product of the normalizing factors given by the rank-one intertwining operators attached to the positive roots $\{\beta > 0, \tilde{w}\beta < 0\}$ [Sh3]. However, $\langle s\tilde{\alpha} + \Lambda_0, \beta^\vee \rangle > 0$ since $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber for $\operatorname{Re} s \geq \frac{1}{2}$. So by Proposition 1.2, the normalizing factor $r(s, \sigma_v, w_0)$ is holomorphic and non-zero. Since π_0 is tempered, $A(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w})$ is holomorphic and so $N(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w})$ is holomorphic and non-zero. The image of $N(s\tilde{\alpha} + \Lambda_0, \pi_0, \tilde{w})$ is irreducible by Langlands' classification theorem. Therefore, $N(s, \sigma_v, w_0)$ is holomorphic and the image of $N(s, \sigma_v, w_0)$ is irreducible. ∎

We classify all non self-conjugate maximal parabolic subgroups of split groups whose derived groups are almost simple. Let $\theta = \Delta - \{\alpha\}$. Note that $w_0 = w_l w_{l,\theta}$ and $w_{l,\theta}(\theta) = -\theta$. Therefore, P_θ is self-conjugate if and only if $w_l(\alpha) = -\alpha$. Note that $w_0 = -1$ except in the case of type A_n, D_n (n odd), E_6. So in those cases all maximal parabolic subgroups are self-conjugate. By checking case by case in the case of type A_n, D_n (n odd), E_6, we see

Lemma 2.5 *The only non self-conjugate maximal parabolic subgroups of split groups whose derived groups are almost simple, are the following:*

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1. *Type A_n: n even, all maximal parabolic subgroups, or n odd, all except $\theta = \Delta - \{e_{\frac{n-1}{2}} - e_{\frac{n+1}{2}}\}$. This is the case $\mathrm{GL}_n \times \mathrm{GL}_m \subset \mathrm{GL}_{n+m}$, where $n \neq m$.*
2. *Type D_n: n odd and $\theta = \Delta - \{\alpha_n\}$. This is the case $\mathrm{GL}_n \subset \mathrm{SO}_{2n}$.*
3. *Type E_6: $\theta = \Delta - \{\alpha_3\}$. This is the case $P = \mathrm{MN}$, where the derived group of M is $\mathrm{SL}_2 \times \mathrm{SL}_5$.*
4. *Type E_6: $\theta = \Delta - \{\alpha_1\}$. This is the case $\mathrm{GL}_1 \cdot D_5 \subset E_6$ (almost direct product).*

3 Main Theorems

We look at four cases in Lemma 2.5 separately. Due to Langlands' result (Corollary 2.2) and (2.2) and Lemma 2.3, we only have to establish Assumption (A).

3.1 $G = \mathrm{SO}_{2n}$, $P = \mathrm{MN}$, $M = \mathrm{GL}_n$, n odd

Recall the following facts from [Sh4], [Sh5]. Let $\sigma = \otimes_v \sigma_v$ be a unitary cuspidal representation of GL_n. Then in (2.2), $r = r_1 = \wedge^2 \rho_n$, the irreducible $\frac{1}{2}n(n-1)$-dimensional representation of $\mathrm{GL}_n(\mathbb{C})$ on the space $\wedge^2 \mathbb{C}^n$ of alternating tensors of rank 2. Suppose σ_v is unramified. Then there exists n unramified quasi-characters $\mu_1, \dots, \mu_n$ of F^* such that $\sigma_v \subset \mathrm{Ind}_B^{\mathrm{GL}_n} \mu_1 \otimes \cdots \otimes \mu_n$ (actually it is an equality since σ_v is generic). Let A_{σ_v} be the (semisimple) conjugacy class of the matrix $\mathrm{diag}\big(\mu_1(\varpi), \dots, \mu_n(\varpi)\big)$ in $\mathrm{GL}_n(\mathbb{C}) = {}^L M$. Then the local Langlands' L-function for the representations $\wedge^2 \rho_n$ and σ_v is given by

$$L(s, \sigma_v, \wedge^2 \rho_n) = \det\big(I - \wedge^2 \rho_n(A_{\sigma_v}) q_v^{-s}\big)^{-1} = \prod_{1 \le i < j \le n} \big(1 - \mu_i(\varpi)\mu_j(\varpi) q_v^{-s}\big)^{-1}.$$

We recall the following well-known facts.

Proposition 3.1

1. *[Sh1] For each v, the local Langlands' L-function $L(s, \sigma_v, \wedge^2 \rho_n)$ can be defined. We use the one in [Sh1] given inductively; For tempered σ_v, the L-function is well-defined and both definitions in [Sh1] and [Sh4] agree. For a non-tempered σ_v, we find the Langlands' data and define the L-function inductively from the Langlands' data.*
2. *[Sh4] The completed L-function $L(s, \sigma, \wedge^2 \rho_n) = \prod_v L(s, \sigma_v, \wedge^2 \rho_n)$ can be continued meromorphically to all of $\mathbb{C}$ and satisfies the standard functional equation*

$$L(s, \sigma, \wedge^2 \rho_n) = \epsilon(s, \sigma, \wedge^2 \rho_n) L(1 - s, \tilde{\sigma}, \wedge^2 \rho_n).$$

3. *[J-S2] Let S be a finite set of places including archimedean places such that σ_v is unramified for $v \notin S$. The partial L-function $L_S(s, \sigma, \wedge^2 \rho_n) = \prod_{v \notin S} L(s, \sigma_v, \wedge^2 \rho_n)$ is absolutely convergent for $\mathrm{Re}\, s > 1$ and hence has no zero there.*
4. *[J-S2], [Sh3] The completed L-function $L(s, \sigma, \wedge^2 \rho_n)$ has no zeros and no poles on the line $\mathrm{Re}\, s = 1$.*

We note that in [J-S2], [Sh3], it is proved that only the partial L-function $L_S(s, \sigma, \wedge^2 \rho_n)$ is holomorphic for $\mathrm{Re}\, s \ge 1$. We prove in Proposition 3.4 that each of the local L-function $L(s, \sigma_v, \wedge^2 \rho_n)$ is holomorphic for $\mathrm{Re}\, s \ge 1$.

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Recall that any cuspidal representation σ of GL_n is globally generic and therefore σ_v is generic for all v. Recall the classification of unitary representations of GL_n [Ta], [Vo]: Any generic non-tempered representation σ_v of GL_n, n odd, can be written as follows:

$$\sigma_v = \mathrm{Ind}_{M_0}^{\mathrm{GL}_n}\big(\pi_1(x_1)\otimes\cdots\otimes\pi_m(x_m)\otimes\tau_1\otimes\cdots\otimes\tau_k\otimes\pi_m(-x_m)\otimes\cdots\otimes\pi_1(-x_1)\big),$$

where $\frac{1}{2} > x_1 \geq \cdots \geq x_m > 0$ with $\pi_1,\ldots,\pi_m,\tau_1,\ldots,\tau_k$ discrete series representations. Here $\pi_i(x_i) = \pi_i \otimes |\det|^{x_i}$.

Recall that we are identifying s with $s\tilde{\alpha}$, and $\tilde{\alpha} = \frac{1}{2}(e_1+\cdots+e_n)$, where $e_1-e_2,\ldots,e_{n-1}-e_n, e_{n-1}+e_n$ are positive simple roots. Therefore $I(s,\sigma_v) = \mathrm{Ind}_{\mathrm{GL}_n}^{G}\big(\sigma_v\otimes|\det(\ )|^{\frac{s}{2}}\big)\otimes 1$. Notice $\frac{s}{2}$ instead of s. Then

$$I(s,\sigma_v) = \mathrm{Ind}_{M_0}^{G}\pi_1\otimes\cdots\otimes\pi_m\otimes\tau_1\otimes\cdots \otimes\tau_k\otimes\pi_m\otimes\cdots\otimes\pi_1\exp\big(\langle s\tilde{\alpha}+\Lambda_0, H_{M_0}(\ )\rangle\big), \tag{3.1}$$

where $\Lambda_0 = (x_1,\ldots,x_m,0,\ldots,0,-x_m,\ldots,-x_1)$ and $s\tilde{\alpha} = (\frac{s}{2},\ldots,\frac{s}{2})$.

Lemma 3.2 *Let π_{1v} (resp. π_{2v}) be a supercuspidal representation of GL_k (resp. GL_l). Then the normalized rank-one intertwining operators $N(s,\pi_{1v}\otimes\pi_{2v},w_0)$ of GL_{k+l}, $N(s,\pi_{1v},w_0)$ of SO_{2k} and $N(s,\pi_{1v},w_0)$ of SO_{2k+1} are holomorphic and non-zero except possibly at $\mathrm{Re}\, s=-1$.*

Proof By the general theory in [Sh1], for a supercuspidal representation π_v, in (2.1), $\prod_{i=1}^m L(is,\pi_v,r_i)^{-1}A(s,\pi_v,w_0)$ is entire and non-zero. Therefore the poles of $N(s,\pi_v,w_0)$ come from zeros of $\prod_{i=1}^m L(1+is,\pi_v,r_i)^{-1}$. However, by [Sh1, Proposition 7.3], each $L(s,\pi_v,r_i)^{-1}$ is a product (possibly empty) of $(1-\alpha_i q_v^{-s})^{-1}$ with $|\alpha_i| = 1$. From this, our assertion follows since $m=1$ in all of the above cases. ∎

Lemma 3.3 *Let v be any place, archimedean or non-archimedean.*

1. *For two discrete series representations π_v (resp. π_v') of GL_k (resp. GL_l), the normalized rank-one intertwining operator $N(s,\pi_v\otimes\pi_v',w_0)$ of GL_{k+l} is holomorphic and non-zero for $\mathrm{Re}\, s > -\frac{1}{2}$.*
2. *For a discrete series representation π_v of GL_k, k odd or even, the normalized rank-one intertwining operator $N(s,\pi_v,w_0)$ of SO_{2k} is holomorphic and non-zero for $\mathrm{Re}\, s > -1$.*

Proof Assume first that v is a non-archimedean place.

(1) If $\mathrm{Re}\, s > 0$, then both $A(s,\pi_v\otimes\pi_v',w_0)$ and $L(s,\pi_v\times\pi_v')$ are holomorphic and non-zero. So $N(s,\pi_v\otimes\pi_v',w_0)$ is holomorphic and non-zero for $\mathrm{Re}\, s>0$. If $\mathrm{Re}\, s = 0$, then this is well-known (see, for example, [Sh1]). Therefore we only need to consider for $-\frac{1}{2} < \mathrm{Re}\, s < 0$.

Note that any discrete series representation π_v of GL_k is the unique subrepresentation of $I(\nu,\tau_v) = |\det|^{\frac{a-1}{2}}\rho_v\otimes|\det|^{\frac{a-3}{2}}\rho_v\otimes\cdots\otimes|\det|^{-\frac{a-1}{2}}\rho_v$ with $\tau_v=\rho_v\otimes\cdots\otimes\rho_v$ and $\nu = (\frac{a-1}{2},\frac{a-3}{2},\ldots,-\frac{a-1}{2})$ and ρ_v a supercuspidal representation of GL_b. Another discrete series representation π_v' of GL_l is the unique subrepresentation of $I(\nu',\tau_v')$ with $\tau_v' = \rho_v'\otimes\cdots\otimes\rho_v'$ and $\nu' = (\frac{a'-1}{2},\frac{a'-3}{2},\ldots,-\frac{a'-1}{2})$. Then $I(s,\pi_v\otimes\pi_v')$ is a subrepresentation

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of $I(\lambda,\tau_v\otimes\tau_v')$, where $\lambda=(\frac{s}{2}+\frac{a-1}{2},\dots,\frac{s}{2}-\frac{a-1}{2},-\frac{s}{2}+\frac{a'-1}{2},\dots,\frac{s}{2}-\frac{a-1}{2})$. Then by the inductive property of intertwining operators, we have

$$N(s,\pi_v\otimes\pi_v',w_0)=N(\lambda,\tau_v\otimes\tau_v',w_0)|_{I(s,\pi_v\otimes\pi_v')}.$$

$N(\lambda,\tau_v\otimes\tau_v',w_0)$ is a product of the rank-one operators associated to supercuspidal representations (see [Sh3]) attached to the positive roots $\{\beta>0\mid w_0\beta<0\}$. For those positive roots, $\langle\lambda,\beta^\vee\rangle=(\frac{s}{2}+\frac{a-1}{2}-i)-(-\frac{s}{2}+\frac{a'-1}{2}-j)$, $i=0,\dots,a$, $j=0,\dots,a'$. But for $-\frac{1}{2}<\operatorname{Re}s<0$, $\operatorname{Re}\bigl((\frac{s}{2}+\frac{a-1}{2}-i)-(-\frac{s}{2}+\frac{a'-1}{2}-j)\bigr)$ cannot be -1. So by Lemma 3.2, each rank-one intertwining operators associated to supercuspidal representations are holomorphic and thus $N(\lambda,\tau_v\otimes\tau_v',w_0)$ is holomorphic. Note that for $-\frac{1}{2}<\operatorname{Re}s<0$, $w_0(s\tilde\alpha)$ is in the positive Weyl chamber and id $=N\bigl(w_0(s\tilde\alpha),w_0(\pi_v\otimes\pi_v'),w_0\bigr)N(s,\pi_v\otimes\pi_v,w_0)$. We showed that $N\bigl(w_0(s\tilde\alpha),w_0(\pi_v\otimes\pi_v'),w_0\bigr)$ and $N(s,\pi_v\otimes\pi_v,w_0)$ are holomorphic and therefore $N(s,\pi_v\otimes\pi_v,w_0)$ cannot be zero.

(2) As in the above, we only need to consider the interval $-1<\operatorname{Re}s<0$. A discrete series representation π_v of GL_k is the unique subrepresentation of $I(\nu,\sigma_v)$ with $\sigma_v=\rho_v\otimes\cdots\otimes\rho_v$ and $\nu=(\frac{a-1}{2},\frac{a-3}{2},\dots,-\frac{a-1}{2})$. Then $I(s,\pi_v)$ is a subrepresentation of $I(\lambda,\sigma_v)$, where $\lambda=(\frac{s}{2}+\frac{a-1}{2},\frac{s}{2}+\frac{a-3}{2},\dots,\frac{s}{2}-\frac{a-1}{2})$. Then by the inductive property of intertwining operators, we have

$$N(s,\pi_v,w_0)=N(\lambda,\sigma_v,w_0)|_{I(s,\pi_v)}.$$

$N(\lambda,\sigma_v,w_0)$ is a product of rank-one operators associated to supercuspidal representations attached to the positive roots $\{\beta>0\mid w_0\beta<0\}$ (see [Sh3]). For those positive roots, $\langle\lambda,\beta^\vee\rangle=\frac{s}{2}+\frac{a-1}{2}-i$, $i=0,\dots,a$ or $(\frac{s}{2}+\frac{a-1}{2}-i)\pm(\frac{s}{2}+\frac{a-1}{2}-j)$, $0\le i<j\le a$. If $-1<\operatorname{Re}s<0$, $\operatorname{Re}(\frac{s}{2}+\frac{a-1}{2}-i)$, $\operatorname{Re}\bigl((\frac{s}{2}+\frac{a-1}{2}-i)\pm(\frac{s}{2}+\frac{a-1}{2}-j)\bigr)$ cannot be -1. So the rank-one operators are holomorphic and non-zero. Therefore, $N(\lambda,\sigma_v,w_0)$ is holomorphic and so $N(s,\pi_v,w_0)$ is holomorphic and non-zero by the same argument as in (1).

Now let v be an archimedean place. Then the discrete series exist only for GL_1 or GL_2 over a real place. Note that the discrete series for GL_2 over a real place is given by the subrepresentation $\sigma(\mu,\nu)$ of the principal series $\pi(\mu,\nu)$ when $\mu(x)=|\ |^{\frac{p+it}{2}}\operatorname{sgn}(x)$ and $\nu(x)=|\ |^{\frac{-p+it}{2}}$, where p is a positive integer and t is a real number. We go exactly the same way as non-archimedean places as above. ■

Remark 3.1 Moeglin-Waldspurger [M-W2, Proposition I.10] proved much stronger result that the normalized rank-one intertwining operator $N(s,\pi_v\otimes\pi_v',w_0)$ of GL_{k+l} is holomorphic and non-zero for $\operatorname{Re}s>-1$ for two discrete series representations π_v (resp. π_v') of GL_k (resp. GL_l). It also follows from [C-Sh] by noting that $N(s,\pi_v\otimes\pi_v',w_0)=\frac{L(s+1,\pi_v\times\pi_v')}{L(s,\pi_v\times\pi_v')}A(s,\pi_v\otimes\pi_v',w_0)$. By [C-Sh], $L(s+1,\pi_v\times\pi_v')$ is holomorphic for $\operatorname{Re}s>-1$ and $\frac{A(s,\pi_v\otimes\pi_v',w_0)}{L(s,\pi_v\times\pi_v')}$ is entire.

From Lemma 3.3, we have

Proposition 3.4

1. *Each local L-function $L(s,\sigma_v,\wedge^2\rho_n)$ is holomorphic for* $\operatorname{Re}s\ge1$.

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2. *Let* Re $s \geq \frac{1}{2}$. *Assumption (A) holds in the case in consideration, i.e.,* $N(s, \sigma_v, w_0)$ *is holomorphic and non-zero for* Re $s \geq \frac{1}{2}$ *for all* v.

Proof In (3.1), we identify $N(s, \sigma_v, w_0)$ with $N(s\tilde{\alpha} + \Lambda_0, \pi_1 \otimes \cdots \otimes \pi_m \otimes \tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_m \otimes \cdots \otimes \pi_1, w_0)$. $s\tilde{\alpha} + \Lambda_0 = (\frac{s}{2} + x_1, \frac{s}{2} + x_2, \ldots, \frac{s}{2} + x_m, \frac{s}{2}, \ldots, \frac{s}{2}, \frac{s}{2} - x_m, \ldots, \frac{s}{2} - x_1)$. Note that if Re $s \geq 1$, Re$(\frac{s}{2} - x_i) > 0$. Therefore, $s\tilde{\alpha} + \Lambda_0$ is in the positive Weyl chamber. Therefore, as in the proof of Lemma 2.4, the normalized intertwining operator $N(s, \sigma_v, w_0)$ is holomorphic and non-zero for Re $s \geq 1$. The holomorphy of $L(s, \sigma_v, \wedge^2 \rho_n)$ for Re $s \geq 1$ follows from (2.2) by noting that $L(s, \sigma, \wedge^2 \rho_n)$ has no zeros for Re $s \geq 1$. (Since $L(s, \sigma_v, \wedge^2 \rho_n)^{-1}$ is a polynomial in q_v^{-s}, if $L(s, \sigma_v, \wedge^2 \rho_n)$ has a pole, it has infinitely many poles.) This proves (1).

Note that for $\frac{1}{2} \leq$ Re $s < 1$, $-\frac{1}{4} <$ Re$(\frac{s}{2} - x_i) < \frac{1}{2}$. Therefore the rank-one normalized intertwining operators attached to permutations among $\{\frac{s}{2} - x_1, \ldots, \frac{s}{2} - x_m\}$ and the sign changes $\frac{s}{2} - x_i \mapsto -\frac{s}{2} + x_i$, are holomorphic and non-zero due to Lemma 3.3. Actually they are isomorphisms. So there is an isomorphism by a normalized intertwining operator which sends (3.1) to $I(\Lambda_1, \pi \otimes \cdots \otimes \pi_m \otimes \tau_1 \otimes \cdots \otimes \tau_k \otimes \pi_m \otimes \cdots \otimes \pi_1)$, where Λ_1 is in the positive Weyl chamber of the split component of a Levi subgroup. The normalized intertwining operator attached to the latter induced representation is holomorphic and non-zero by Proposition 1.2. So the same thing is true for $N(s, \sigma_v, w_0)$. ∎

Therefore we obtain the following theorem.

Theorem 3.5 *Let* σ *be a unitary cuspidal representation of* GL_n, *where* n *is odd. Then the exterior square L-function* $L(s, \sigma, \wedge^2 \rho_n)$ *is entire.*

Proof By (2.2), Corollary 2.2 and Proposition 3.4, $\frac{L(s,\sigma,\wedge^2\rho_n)}{L(s+1,\sigma,\wedge^2\rho_n)}$ is holomorphic for $s \geq \frac{1}{2}$. However, $L(s, \sigma, \wedge^2 \rho_n)$ does not have zeros for Re $s \geq 1$ by Lemma 2.3. So $L(s, \sigma, \wedge^2 \rho_n)$ is holomorphic for $s \geq \frac{1}{2}$. The functional equation of $L(s, \sigma, \wedge^2 \rho_n)$ implies that it is entire. ∎

In the same way, we have

Proposition 3.6 *Let* σ *be a non self-dual cuspidal representation of* GL_n, *n even. Then the exterior square L-function* $L(s, \sigma, \wedge^2)$ *is entire.*

Remark 3.2 According to Langlands' functoriality, the self-dual cuspidal representations of GL_n, n even, are supposed to come from SO_n (resp. SO_{n+1}) if $L(s, \sigma, Sym^2)$ (resp. $L(s, \sigma, \wedge^2)$) has a pole at $s = 1$. See [Sh5].

3.2 $G = \mathrm{GL}_{n+m}$, $P = \mathrm{MN}$, $M = \mathrm{GL}_n \times \mathrm{GL}_m$, $n \neq m$

This is a special case of [M-W2, Appendix]. Let σ_1 (resp. σ_2) be a unitary cuspidal representation of GL_n (resp. GL_m). Moeglin-Waldspurger [M-W2, Appendix] proved that the

Rankin-Selberg L-function $L(s,\sigma_1\times\tilde\sigma_2)$ is holomorphic for $0<\mathrm{Re}\,s\le\frac12$ using a remarkable method. The functional equation then implies that it is entire if $m\neq n$. Here we want to give a different proof based on the fact that P is not self-conjugate.

Let $\sigma=\sigma_1\otimes\sigma_2$ be a cuspidal representation of $\mathrm{GL}_n\times\mathrm{GL}_m$. Then in (2.2), $r=r_1=\rho_n\otimes\tilde\rho_m$, where ρ_n and ρ_m are standard representations of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$, resp. Suppose σ_v is unramified. Then $\sigma_{1v}=\mathrm{Ind}_B^{\mathrm{GL}_n}\mu_1\otimes\cdots\otimes\mu_n$ and $\sigma_{2v}=\mathrm{Ind}_B^{\mathrm{GL}_m}\mu_1'\otimes\cdots\otimes\mu_m'$ for unramified quasi-characters $\mu_1,\dots,\mu_n,\mu_1',\dots,\mu_m'$ of F^*. Then the local Langlands' L-function for the representations $\rho_n\otimes\tilde\rho_m$ and σ_v is given by

$$L(s,\sigma_v,\rho_n\otimes\tilde\rho_m)=L(s,\sigma_{1v}\times\tilde\sigma_{2v})=\prod_{1\le i\le n,1\le j\le m}\left(1-\mu_i(\varpi)\mu_j'(\varpi)^{-1}q_v^{-s}\right)^{-1}.$$

Recall the following well-known facts.

Proposition 3.7

1. *[Sh1], [Sh4], [J-PS-S] For each v, the local Langlands' L-function $L(s,\sigma_{1v}\times\sigma_{2v})$ can be defined and the completed L-function $L(s,\sigma_1\times\tilde\sigma_2)=\prod_v L(s,\sigma_{1v}\times\tilde\sigma_{2v})$ can be continued meromorphically to all of $\mathbb{C}$ and satisfies the standard functional equation*

$$L(s,\sigma_1\times\tilde\sigma_2)=\epsilon(s,\sigma_1\times\tilde\sigma_2)L(1-s,\tilde\sigma_1\times\sigma_2).$$

2. *[J-S1] Let S be a finite set of places including archimedean places such that σ_v is unramified for $v\notin S$. The partial L-function $L_S(s,\sigma_1\times\tilde\sigma_2)=\prod_{v\notin S}L(s,\sigma_{1v}\times\tilde\sigma_{2v})$ is absolutely convergent for* $\mathrm{Re}\,s>1$ *and hence no zero there.*
3. *[J-S1], [Sh3] The completed L-function $L(s,\sigma_1\times\tilde\sigma_2)$ has no zeros and no poles on the line* $\mathrm{Re}\,s=1$.

Lemma 3.8 *For $\frac12\le\mathrm{Re}\,s<1$, $N(s,\sigma_v,w_0)$ is holomorphic and non-zero.*

Proof Since σ_{1v},σ_{2v} are generic, they can be written as follows:

$$\sigma_{1v}=\mathrm{Ind}_{M_1}^{\mathrm{GL}_n}\big(\pi_1(x_1)\otimes\cdots\otimes\pi_k(x_k)\otimes\tau_1\otimes\cdots\otimes\tau_q\otimes\pi_k(-x_k)\otimes\cdots\otimes\pi_1(-x_1)\big),$$
$$\sigma_{2v}=\mathrm{Ind}_{M_2}^{\mathrm{GL}_m}\big(\pi_1'(y_1)\otimes\cdots\otimes\pi_l'(y_l)\otimes\tau_1'\otimes\cdots\otimes\tau_p'\otimes\pi_l'(-y_l)\otimes\cdots\otimes\pi_1'(-y_1)\big),$$

where $\frac12>x_1\ge\cdots\ge x_k\ge0$, $\frac12>y_1\ge\cdots\ge y_l\ge0$ with $\pi_1,\dots,\pi_k,\pi_1',\dots,\pi_l'$, $\tau_1,\dots,\tau_q,\tau_1',\dots,\tau_p'$ discrete series representations. Therefore,

$$\begin{aligned}(3.2)\qquad I(s,\sigma_v)=I(s\tilde\alpha+\Lambda_0,\ &\pi_1\otimes\cdots\otimes\pi_k\otimes\tau_1\otimes\cdots\otimes\tau_q\otimes\pi_k\otimes\cdots\otimes\pi_1\\&\otimes\pi_1'\otimes\cdots\otimes\pi_l'\otimes\tau_1'\otimes\cdots\otimes\tau_p'\otimes\pi_l'\otimes\cdots\otimes\pi_1').\end{aligned}$$

where $s\tilde\alpha+\Lambda_0=(\frac s2+x_1,\dots,\frac s2+x_k,\frac s2,\dots,\frac s2,\frac s2-x_k,\dots,\frac s2-x_1,-\frac s2+y_1,\dots,-\frac s2+y_l,-\frac s2,\dots,-\frac s2,-\frac s2-y_l,\dots,-\frac s2-y_1)$. We identify $N(s,\sigma_v,w_0)$ with $N(\Lambda,\Sigma_v,w_0)$, where $\Lambda=s\tilde\alpha+\Lambda_0$, $\Sigma_v=\pi_1\otimes\cdots\otimes\pi_k\otimes\tau_1\otimes\cdots\otimes\tau_q\otimes\pi_k\otimes\cdots\otimes\pi_1\otimes\pi_1'\otimes\cdots\otimes\pi_l'\otimes\tau_1'\otimes\cdots\otimes\tau_p'\otimes\pi_l'\otimes\cdots\otimes\pi_1'$. We note that for $\frac12\le\mathrm{Re}\,s<1$, $\mathrm{Re}\big(\frac s2+x_i-(-\frac s2+y_j)\big)>0$ and $-\frac12<$

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$\mathrm{Re}\big(\frac{s}{2}-x_i-(-\frac{s}{2}+y_j)\big)<1$. Therefore by Lemma 3.3, the rank-one normalized intertwining operators attached to permutations among $\{\frac{s}{2}-x_1,\dots,\frac{s}{2}-x_k,-\frac{s}{2}+y_1,\dots,-\frac{s}{2}+y_l\}$ are holomorphic. So $N(\Lambda,\Sigma_v,w_0)$ is holomorphic. If Λ is in the closure of the positive Weyl chamber, it is non-zero. We argue as in [Zh, Theorem 3]. Suppose Λ is not in the closure of the positive Weyl chamber. Choose $w_1\in W$ so that $w_1\Lambda$ is in the closure of the positive Weyl chamber. Then

$$N\big(w_1\Lambda,w_1(\Sigma_v),w_0w_1^{-1}\big)=N(\Lambda,\Sigma_v,w_0)N\big(w_1\Lambda,w_1(\Sigma_v),w_1^{-1}\big).$$

By Proposition 1.2, $N\big(w_1\Lambda,w_1(\Sigma_v),w_0w_1^{-1}\big)$ and $N\big(w_1\Lambda,w_1(\Sigma_v),w_1^{-1}\big)$ are holomorphic and non-zero since $w_1\Lambda$ is in the closure of the positive Weyl chamber. Since $N(\Lambda,\Sigma_v,w_0)$ is holomorphic, it is non-zero. ∎

Remark 3.3 Moeglin-Waldspurger [M-W2, Appendix] proved much stronger result that $N(s,\sigma_v,w_0)$ is holomorphic for $\mathrm{Re}\, s>-e(\sigma_v)$, where $e(\sigma_v)$ is some positive number. The argument in [Zh, Theorem 3] proves that, for a tempered and generic representation σ_v, if $N(\nu,\sigma_v,w_0)$ is holomorphic at ν, then it is non-zero at ν under Conjecture 7.1 of [Sh1].

Therefore, we have

Theorem 3.9 **[M-W2, Appendix]** *Let σ_1 (σ_2) be a unitary cuspidal representation of GL_n (GL_m), $n\neq m$. Then the Rankin-Selberg L-function $L(s,\sigma_1\times\sigma_2)$ is entire.*

Proposition 3.10 *Let σ_1,σ_2 be unitary cuspidal representations of GL_n, where $\sigma_1\not\cong\sigma_2\otimes|\det(\ )|^t$ for all $t\in\mathbb{C}$. Then the Rankin-Selberg L-function $L(s,\sigma_1\times\sigma_2)$ is entire.*

3.3 The case G is a simply-connected split group of type E_6 and $P=\mathrm{MN}$, $M=\mathrm{GL}_1\cdot(\mathrm{SL}_2\times\mathrm{SL}_5)$ (almost direct product)

This is the case E_6-2 in [Sh4]. There is a canonical surjection $M\mapsto\mathrm{PGL}_2\times\mathrm{PGL}_5$. Let σ_1,σ_2 be cuspidal representations of $\mathrm{PGL}_2,\mathrm{PGL}_5$, resp. Then $\sigma_1\otimes\sigma_2$ can be considered as a cuspidal representation of M. Let S be a finite set of places, including all the archimedean places, such that for every $v\notin S$, σ_{1v}, σ_{2v}, are all unramified. For $v\notin S$, let $A(\sigma_{1v})=\{\mathrm{diag}(\alpha_{1v},\alpha_{2v})\}$ be the semisimple conjugacy classes attached to σ_{1v}. Let $A(\sigma_{2v})=\{\mathrm{diag}(\beta_{1v},\dots,\beta_{5v})\}$ be the one attached to σ_{2v}. Then the direct computation shows that

$$L(s,\sigma_{1v}\otimes\sigma_{2v},r_1)=L(s,\sigma_{1v}\otimes\tilde{\sigma}_{2v},\rho_2\otimes\wedge^2\rho_5)=\prod_{1\le i\le 2,1\le j<k\le 5}(1-\alpha_{iv}\beta_{jv}^{-1}\beta_{kv}^{-1}q_v^{-s})^{-1}$$

$$L(s,\sigma_{1v}\otimes\sigma_{2v},r_2)=L(s,\sigma_{2v})=\prod_{i=1}^{5}(1-\beta_{iv}q_v^{-s})^{-1},$$

where ρ_n is the standard representation of $\mathrm{GL}_n(\mathbb{C})$. In the same way as in Proposition 3.4, we can see that the normalized local intertwining operators satisfy Assumption (A), provided that Conjecture 7.1 of [Sh1] holds in this case. Unfortunately, the result of [C-Sh] does not apply to the exceptional group. Since the standard L-function $L(s,\sigma_2)$ has no zeros for Re $s \geq 1$, we have, by Corollary 2.2,

Theorem 3.11 *Let σ_1, σ_2 be cuspidal representations of* $\mathrm{PGL}_2, \mathrm{PGL}_5$, *resp. Suppose Conjecture 7.1 of [Sh1] holds for the exceptional group of type E_6. Then the completed L-function*

$$L(s, \sigma_1 \otimes \tilde{\sigma}_2, \rho_2 \otimes \wedge^2 \rho_5),$$

is entire.

3.4 The case G is a simply-connected split group of type E_6 and $P = \mathrm{MN}$, $M = \mathrm{GL}_1 \cdot D_5$ (almost direct product)

This is the case (xxiv) in [La1]. This case was suggested by Shahidi from the work [Mu-Sh]. Recall some facts from [Mu-Sh]. Let $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ be the set of simple roots of T with respect to the Borel subgroup B, which are labeled on Dynkin diagram in the standard way. Denote by $P = \mathrm{MN}$ ($P' = M'N'$, respectively) the maximal parabolic subgroup of G which corresponds to the set of simple roots $\theta = \Delta - \{\alpha_1\}$ ($\theta' = \Delta - \{\alpha_6\}$, respectively). Then M, M' are groups of type D_5. Let w_0 be the longest element of the Weyl group W modulo that of T in M. Then $w_0(\theta) = \theta'$ and $M' = w_0 M w_0^{-1}$. The adjoint representation of ${}^L M$ on ${}^L\mathfrak{n}$ is an irreducible representation of the lowest weight $\alpha_1^\vee$. Denote this representation by r. This is one of the two 16-dimensional irreducible half spin representations when restricted to the derived group of ${}^L M$ or the half-spin representation of ${}^L M = \mathrm{GSpin}(10, \mathbb{C})$ by abuse of terminology. Let σ be a generic cuspidal representation of $M(\mathbb{A})$. Then the completed L-function $L(s, \sigma, r)$ is defined.

Theorem 3.12 *Let σ be a generic cuspidal representation of* $\mathrm{SO}(10)$. *Then the completed $L(s, \sigma, r)$ is entire if Assumption (A) is satisfied.*

We can prove that Assumption (A) is satisfied for unramified places from Shahidi's result that $L(s, \sigma_v, r)$ is holomorphic for Re $s \geq 1$ [Sh4, Lemma 5.8]. However, we were not able to prove that the local normalized intertwining operators are holomorphic and non-zero for Re $s \geq \frac{1}{2}$ at ramified places. One serious obstacle is that we do not have the standard module conjecture for $\mathrm{SO}(2n)$. Nevertheless, in view of (2.2) and Corollary 2.2, we obtain the result that the partial L-function $L_S(s, \sigma, r)$ is holomorphic for Re $s > 0$: Let S be a finite set of places, including all the archimedean places, such that for every $v \notin S$, σ_v is unramified. Take $f = \otimes_v f_v$ such that for each $v \notin S$, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$ and let $\tilde{f}_v$ be the K_v-fixed function in the space of $I\big(-s, w_0(\sigma_v)\big)$, normalized the same way. Then (2.2) can be written as (see [Sh4, (2.7)])

$$M(s, \sigma, w_0) f = \frac{L_S(s, \sigma, r)}{L_S(1 + s, \sigma, r)} \otimes_{v \notin S} \tilde{f}_v \otimes \bigotimes_{v \in S} A(s, \sigma_v, w_0) f_v.$$

For each $v \in S$, $A(s, \sigma_v, w_0)$ is not a zero operator. By Corollary 2.2, $M(s, \sigma, w_0)$ is holomorphic for Re $s > 0$. Suppose $L_S(s, \sigma, r)$ has a pole for Re $s > 1$. Then for each $v \in S$, choose f_v such that $A(s, \sigma_v, w_0) f_v$ is not zero. From [Sh4, Theorem 5.1], $L_S(s, \sigma, r)$ has no

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poles for Re $s > 2$. We obtain a contradiction. In the same way, we see that $L_S(s, \sigma, r)$ is holomorphic for Re $s > 0$.

Again in the same way, we see that the partial L-function $L_S(s, \sigma_1 \otimes \tilde{\sigma}_2, \rho_2 \otimes \wedge^2 \rho_5)$ in Theorem 3.11 is holomorphic for Re $s > \frac{1}{2}$ without any assumption.

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