J. Austral. Math. Soc. (Series B) 24 (1982), 98–119

RAY ANALYSIS AND PUNCHING PROBLEMS FOR STRETCHED ELASTIC PLATES

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(Received 3 June 1981; revised 12 November 1981)

Abstract

The present paper presents a ray analysis for a problem of technical importance in fragmentation studies. The problem is that of suddenly punching a circular hole in either isotropic or transversely isotropic plates subjected to a uniaxial tension field. The ray method, which involves only differentiation, integration, and simple algebra, is shown to be particularly useful in clarifying the propagation process of the resulting unloading waves and obtaining the attendant discontinuities of the various quantities involved. Numerical results obtained from the ray analysis are presented in graphical form and compared with those obtained by more elaborate schemes.

1. Introduction

Here we consider the linear boundary value problem resulting from the normal impact of a flat-nosed cylindrical projectile on isotropic and transversely isotropic plates subjected to a uniform, uniaxial tensile stress. We assume that "plugging" occurs, that is, that a circular plug of material, of approximately the same diameter as the projectile is removed from the plate and unloading waves emanate from the boundary of the circular hole. We also assume that the plate is unbounded so that reflected waves are not considered. Miklowitz [6] studied the related simpler problem of an infinite plate stretched so that it is initially in a state of axially symmetric hydrostatic tension. A numerical solution to the present problem has been obtained by Haddow and Mioduchowski [2] employing the method of near characteristics.

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The ray method employed here is a formal procedure whereby problems involving partial differential equations may be transformed to consideration of ordinary differential equations for determining the various quantities (amplitude and phase) in a progressing wave expansion of the solution. The major developments in the theory and application of ray methods have been carried out by J. B. Keller and his co-workers [4].

2. Formulation of problem

We consider an unbounded isotropic or transversely isotropic thin elastic plate of thickness h which is in an initial state of uniform, uniaxial tensile stress $\sigma_y = S$, as shown in Figure 1. The axis of symmetry of the plate is chosen to coincide with

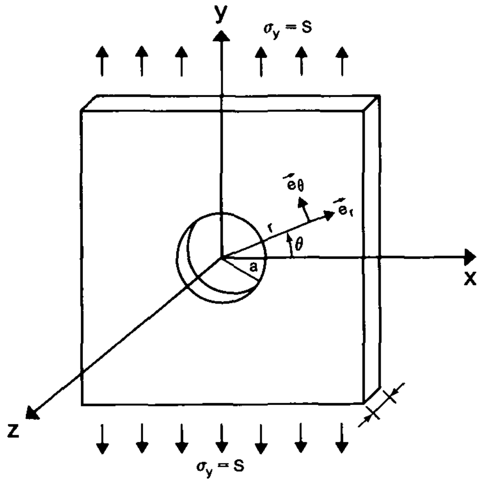

Figure 1. Infinite elastic plate in uniaxial tension field.

the z coordinate axis. In plane polar coordinates, the initial stress has non-zero components

$$\begin{aligned} \sigma_r &= (S/2)(1 - \cos 2\theta), \\ \sigma_\theta &= (S/2)(1 + \cos 2\theta), \\ \tau_{r\theta} &= (S/2)\sin 2\theta, \end{aligned} \tag{2.1}$$

where σ_r, σ_θ, and $\tau_{r\theta}$ are the radial, circumferential, and shear components, respectively, for generalized plane stress.

At $t = 0$, a flat-nosed cylindrical projectile of radius $a \gg h$, travelling with velocity w along the z-axis, strikes the plate and begins to punch out a hole of

https://doi.org/10.1017/S0334270000003349 Published online by Cambridge University Press

radius equal to its own. Under the assumptions of the so-called "plugging" process (see [6] for details) the boundary conditions at the hole may be written as

$$\begin{bmatrix} \sigma_r \\ \tau_{r\theta} \end{bmatrix}_{r=a} = \frac{S}{2}\begin{bmatrix} 1 - \cos 2\theta \\ \sin 2\theta \end{bmatrix}(1 - tH(t)/t^*)H(t^* - t), \tag{2.2}$$

where $H(\cdot)$ is the Heaviside step function and t^* is the punching time given by

$$t^* = 2h/w. \tag{2.3}$$

Equation (2.2) represents a linear change, with time, to zero magnitude, of the stress components at the edge of the hole. As in [6] and [2], results are obtained for values of t^* which are realistic for "plugging" of a metal plate.

For instantaneous plugging of the hole the boundary condition (2.2) becomes

$$\begin{bmatrix} \sigma_r \\ \tau_{r\theta} \end{bmatrix}_{r=a} = \frac{S}{2}\begin{bmatrix} 1 - \cos 2\theta \\ \sin 2\theta \end{bmatrix}(1 - H(t)). \tag{2.4}$$

The governing equations of motion for the plane stress problems being considered here are

$$\begin{aligned} \frac{\partial \sigma^r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} &= \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2}{r}\tau_{r\theta} &= \rho \frac{\partial^2 v}{\partial t^2}, \end{aligned} \tag{2.5}$$

where ρ is the density of the plate material, and u and v are the radial and circumferential components of displacement, respectively. The stress-strain relations pertinent to our situation are

$$\begin{aligned} \sigma_r &= \alpha\varepsilon_r + \beta\varepsilon_\theta, \\ \sigma_\theta &= \beta\varepsilon_r + \alpha\varepsilon_\theta, \\ \tau_{r\theta} &= (\alpha - \beta)\varepsilon_{r\theta}, \end{aligned} \tag{2.6}$$

where, for the case of transverse isotropy,

$$\alpha = (C_{11}C_{33} - C_{13}^2)/C_{33}, \quad \beta = (C_{12}C_{33} - C_{13}^2)/C_{33}, \tag{2.7}$$

where C_{ij} are the elastic parameters and for the isotropic case

$$\alpha = \frac{E}{1 - \nu^2}, \quad \beta = \frac{E\nu}{1 - \nu^2}, \tag{2.8}$$

where E is Young's modulus and ν is Poisson's ratio. For the isotropic case we have that

$$\alpha - \beta = E/(1 + \nu) = 2\mu,$$

where μ is the shear modulus. The components of strain, ε_r, ε_θ, and $\varepsilon_{r\theta}$, (in terms of the displacements u and v) when combined with (2.6) yield

$$\sigma_r = \alpha\frac{\partial u}{\partial r} + \frac{\beta}{r}\left(u + \frac{\partial v}{\partial \theta}\right),$$
$$\sigma_\theta = \beta\frac{\partial u}{\partial r} + \frac{\alpha}{r}\left(u + \frac{\partial v}{\partial \theta}\right),$$
$$\tau_{r\theta} = \frac{\alpha - \beta}{2}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right). \tag{2.9}$$

It is convenient to solve the problem with the initial state of stress removed and then superimpose the initial state on the solution obtained. Consequently, we replace (2.2) by

$$\begin{bmatrix}\sigma_r \\ \tau_{r\theta}\end{bmatrix}_{r=a} = \frac{S}{2}\begin{bmatrix}1 - \cos 2\theta \\ \sin 2\theta\end{bmatrix} f(t), \tag{2.10}$$

where

$$f(t) = \{[1 - (t/t^*)H(t)]H(t^* - t) - 1\} \tag{2.11}$$

(for instantaneous plugging $f(t) = -H(t)$) and we then have the quiescent initial conditions

$$\left.\begin{aligned} u(r,\theta,0) = v(r,\theta,0) = 0 \\ \frac{\partial u}{\partial t}(r,\theta,0) = \frac{\partial v}{\partial t}(r,\theta,0) = 0 \end{aligned}\right\}, \quad r > a, \quad 0 \leqslant \theta < 2\pi. \tag{2.12}$$

The boundary value problem to be solved consists of equations (2.5) together with (2.10) and (2.12).

3. A decomposition

We shall separate variables by expressing the stress and displacement components in the forms

$$\left.\begin{aligned} \sigma_r(r,\theta,t) &= \sigma_r^{(1)}(r,t) + \sigma_r^{(2)}(r,t)\cos 2\theta, \\ \sigma_\theta(r,\theta,t) &= \sigma_\theta^{(1)}(r,t) + \sigma_\theta^{(2)}(r,t)\cos 2\theta, \\ \tau_{r\theta}(r,\theta,t) &= \tau_{r\theta}^{(2)}(r,t)\sin 2\theta, \\ u(r,\theta,t) &= u^{(1)}(r,t) + u^{(2)}(r,t)\cos 2\theta, \\ v(r,\theta,t) &= v^{(2)}(r,t)\sin 2\theta. \end{aligned}\right\} \tag{3.1}$$

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Substituting from (3.1) into (2.9), (2.10), and (2.12) and introducing the potential functions Φ and Ψ defined by

$$\left.\begin{aligned} u^{(2)} &= \frac{\partial \Phi}{\partial r} + \frac{2}{r}\Psi, \\ v^{(2)} &= -\left(\frac{\partial \Psi}{\partial r} + \frac{2}{r}\Phi\right), \end{aligned}\right\} \tag{3.2}$$

our original problem then decomposes into the simultaneous solution of

$$\left.\begin{aligned} \frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}u^{(1)} &= \frac{\rho}{\alpha}\frac{\partial^2 u^{(1)}}{\partial t^2}, \\ L\Phi &= \frac{1}{C_1^2}\frac{\partial^2 \Phi}{\partial t^2}, \\ L\Psi &= \frac{1}{C_2^2}\frac{\partial^2 \Psi}{\partial t^2}, \end{aligned}\right\} \tag{3.3}$$

together with the initial conditions

$$\left.\begin{aligned} u^{(1)}(r,0) = \frac{\partial u^{(1)}}{\partial t}(r,0) = 0, \quad r > a, \\ \Phi(r,0) = \Psi(r,0) = \frac{\partial \Phi}{\partial t}(r,0) = \frac{\partial \Psi}{\partial t}(r,0) = 0, \quad r > a, \end{aligned}\right\} \tag{3.4}$$

and the boundary conditions

$$\left.\begin{aligned} \left\{\alpha\frac{\partial u^{(1)}}{\partial r} + \frac{\beta}{r}u^{(1)}\right\}_{r=a} &= \frac{S}{2}f(t), \\ \{M\Phi + N\Psi\}_{r=a} &= -\frac{S}{\alpha-\beta}f(t), \\ \{M\Psi + N\Phi\}_{r=a} &= -\frac{S}{\alpha-\beta}f(t), \end{aligned}\right\} \tag{3.5}$$

where

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}, \tag{3.6}$$

$$M \equiv \frac{1}{C_2^2}\frac{\partial^2}{\partial t^2} - \frac{2}{r} + \frac{8}{r^2}, \tag{3.7}$$

$$N \equiv \frac{4}{r}\frac{\partial}{\partial r} - \frac{4}{r^2}, \tag{3.8}$$

and C_1 and C_2 are the P- and S-type wave speeds defined by

$$C_1^2 = \frac{\alpha}{\rho}, \quad C_2^2 = \frac{\alpha - \beta}{2\rho}, \quad (C_1 > C_2). \tag{3.9}$$

Under our present decomposition there are two problems to be solved. Problem I consists of determining $u^{(1)}$ from the first equation in (3.3) together with the quiescent conditions of (3.4) and the first boundary condition in (3.5). Problem II consists of finding Φ and Ψ satisfying the second and third equations of (3.3) together with the initial and boundary conditions of (3.4) and (3.5). When these two problems have been solved we employ their solutions to find $\sigma_r^{(i)}, \sigma^{(i)}\theta$ $(i = 1, 2)$, and $\tau_{r\theta}^{(2)}$ which in turn (through (3.1)) leads us to expressions for σ_r, σ_θ, and $\tau_{r\theta}$ for the dynamic part of the problem. To these we then add the initial state of stress to obtain the complete solution.

It will be convenient in what follows to deal with nondimensional quantities. To this end we introduce the scheme

$$\begin{aligned}
&(\bar{\sigma}_r, \bar{\sigma}_\theta, \bar{\tau}_{r\theta}) = 2(\sigma_r, \sigma_\theta, \tau_{r\theta})/(\alpha - \beta),\\
&(\bar{u}, \bar{v}) = (u, v)/a, \quad (\bar{\Phi}, \bar{\Psi}) = (\Phi, \Psi)/a^2,\\
&\bar{r} = r/a, \quad (\bar{t}, \bar{t}^*) = C_2(t, t^*)/a,\\
&\bar{C}_1 = C_1/C_2 = (2\alpha/(\alpha - \beta))^{1/2}, \quad \bar{C}_2 = 1, \bar{S} = 2S/(\alpha - \beta),\\
&\bar{a} = 1, \quad \gamma = 2\beta/(\alpha - \beta).
\end{aligned} \tag{3.10}$$

Henceforth we use these nondimensional quantities but, for convenience, omit the bars. In nondimensional form we have

Problem I:

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{u^{(1)}}{r^2} = \frac{1}{C_1^2}\frac{\partial^2 u^{(1)}}{\partial t^2}, \quad r > 1, t > 0, \tag{3.11}$$

$$u^{(1)}(r, 0) = \frac{\partial u^{(1)}}{\partial t}(r, 0) = 0, \quad r > 1, \tag{3.12}$$

$$C_1^2 \frac{\partial u^{(1)}}{\partial r}(1, t) + \gamma u^{(1)}(1, t) = \frac{S}{2} f(t), \quad t > 0, \tag{3.13}$$

and

Problem II:

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} - \frac{4}{r^2}\Phi = \frac{1}{C_1^2}\frac{\partial^2 \Phi}{\partial t^2}, \quad r > 1, t > 0, \tag{3.14}$$

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{1}{r}\frac{\partial \Psi}{\partial r} - \frac{4}{r^2}\Psi = \frac{\partial^2 \Psi}{\partial t^2}, \quad r > 1, t > 0, \tag{3.15}$$

$$\Phi(r,0) = \Psi(r,0) = \frac{\partial\Phi}{\partial t}(r,0) = \frac{\partial\Psi}{\partial t}(r,0) = 0, \quad r > 1, \tag{3.16}$$

$$\begin{aligned}\frac{\partial^2\Phi}{\partial t^2}(1,t) - 2\frac{\partial\Phi}{\partial r}(1,t) + 8\Phi(1,t) + 4\frac{\partial\Psi}{\partial r}(1,t) - 4\Psi(1,t)\\ = -\frac{S}{2}f(t), \quad t > 0,\end{aligned} \tag{3.17}$$

$$\begin{aligned}\frac{\partial^2\Psi}{\partial t^2}(1,t) - 2\frac{\partial\Psi}{\partial r}(1,t) + 8\Psi(1,t) + 4\frac{\partial\Phi}{\partial r}(1,t) - 4\Phi(1,t)\\ = -\frac{S}{2}f(t), \quad t > 0.\end{aligned} \tag{3.18}$$

4. Solutions of Problems I and II

Problem I: We assume that the solution of I has the asymptotic form

$$u^{(1)}(r,t) \sim \sum_{n=0}^{\infty} U_n(r)F_n[t - P(r)], \tag{4.1}$$

where the F_n's are related by

$$F_n' = F_{n-1}, \quad n = 1, 2, \ldots . \tag{4.2}$$

The prime in (4.2) denotes differentiation with respect to the entire argument $t - P$ and (4.2) enables us to relate all of the F_n's to F_0 (the waveform) by successive integrations. The function $P(r)$ is called the phase function and $U_n(r)$ are amplitude functions. In (4.1), we assume that $U_n \equiv 0$ for $n < 0$.

We consider the wave given by (4.1) and whose propagation is governed by (3.11). Substituting (4.1) into (3.11) and employing (4.2) in the resulting expression, we obtain

$$\begin{aligned}U_{n-2}'' - 2P'U_{n-1}' + (P')^2U_n - P''U_{n-1} + \frac{1}{r}U_{n-2}'\\ - \frac{1}{r}P'U_{n-1} - \frac{1}{r^2}U_{n-2} - \frac{1}{C_1^2}U_n = 0.\end{aligned} \tag{4.3}$$

where the primes refer to differentiation with respect to r. Setting $n = 0$ in (4.3) and assuming, without loss of generality, that $U_0 \neq 0$, yields

$$(P')^2 = \frac{1}{C_1^2}. \tag{4.4}$$

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Equation (4.4) is the well-known eikonal equation of geometrical optics. Integrating this ordinary differential equation along a ray associated with P-type waves gives

$$P(r)=\bar{P}\pm\int_1^r\frac{d\tau}{C_1}=\bar{P}\pm\frac{r-1}{C_1},\quad \text{where } \bar{P}=P(1), \tag{4.5}$$

where the $\pm$ signs are associated with outgoing and incoming waves, respectively. This equation enables us to determine the phase at any point on a ray in terms of its value at $r=1$. The value at $r=1$ will be determined from the boundary condition.

The transport equation for U_n, derived from (4.3) with the aid of (4.4), is

$$2P'U_n'+\frac{P'}{r}U_n=U''_{n-1}+\left(\frac{1}{r}U_{n-1}\right)',\quad n\geqslant 0. \tag{4.6}$$

The general solution of this first-order, linear ordinary differential equation is

$$\left.\begin{aligned}U_n(r)=\bar{U}_n r^{-1/2}\pm\frac{C_1}{2}\int_1^r\left(\frac{\tau}{r}\right)^{1/2}\left\{U''_{n-1}(\tau)+\left(\frac{1}{r}U_{n-1}(\tau)\right)'\right\}d\tau,\\ \text{where } \bar{U}_n=U_n(1).\end{aligned}\right\} \tag{4.7}$$

From (4.7), it may be shown by induction that

$$U_n(r)=r^{-1/2}\sum_{j=0}^{n}u_{jn}r^{-j},\quad n\geqslant 0, \tag{4.8}$$

where the coefficients u_{jn} will be determined for outgoing waves. Substituting (4.8) into (4.7), taking the plus sign for outgoing waves, and simplying, gives

$$u_{jn}=\begin{cases}K(j)u_{j-1,n-1} & \text{if } 1\leqslant j\leqslant n,\\ \bar{U}_n-\sum_{j=1}^{n}K(j)u_{j-1,n-1} & \text{if } j=0,\, n>0,\\ \bar{U}_0 & \text{if } j=n=0,\\ 0 & \text{if } j<0 \text{ or } j>n,\end{cases} \tag{4.9}$$

where

$$K(j)=(C_1/2j)\left[1-(j-1/2)^2\right]. \tag{4.10}$$

The unknown quantities in our solution, that is, $\bar{P}, \bar{U}_n$, and F_n are now determined from the boundary conditions (3.13). Inserting (4.1) into (3.13) and employing (4.2) in the resulting expression, we obtain

$$\sum_{n=0}^{\infty}\left[C_1^2\bar{U}'_{n-1}-C_1\bar{U}_n+\gamma\bar{U}_{n-1}\right]F_{n-1}(t-\bar{P})\sim\frac{S}{2}f(t). \tag{4.11}$$

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We thus choose

$$\begin{aligned}&\bar{U}_0 = 0, \quad F_0 = f(t),\\ &\bar{U}_1 = -S/2C_1, \quad \bar{P} = 0,\\ &C_1^2\bar{U}'_{n-1} - C_1\bar{U}_n + \gamma\bar{U}_{n-1} = 0, \quad n \geqslant 2.\end{aligned} \tag{4.12}$$

Combining (4.9) and (4.12) we have explicitly that

$$u_{jn} = \begin{cases} K(j)u_{j-1,n-1} & \text{if } 1 \leqslant j \leqslant n,\\ 0 & \text{if } j = n = 0,\\ -S/2C_1 & \text{if } j = 0, n = 1,\\ \sum_{j=1}^{n} [\gamma/C_1 - C_1(j-1/2) - K(j)]u_{j-1,n-1} & \text{if } j = 0, n \geqslant 2,\\ 0 & \text{if } j < 0 \text{ or } j > n. \end{cases} \tag{4.13}$$

From (4.12) and (4.2) we find from Duhamel's theorem that

$$F_n = \frac{H(t)}{n!}\frac{\partial}{\partial t}\int_0^t (t-\tau)^n f(\tau)\,d\tau. \tag{4.14}$$

For instantaneous plugging of the hole, $f(t) = -H(t)$, and

$$F_n = -H(t)t^n/n!. \tag{4.15}$$

For a punching time $t^* > 0$, we employ $f(t)$ from (2.11) in (4.14) to obtain

$$F_n = \begin{cases} \dfrac{1}{t^*(n+1)!}\left[(t-t^*)^{n+1} - t^{n+1}\right], & t > t^*,\\ -\dfrac{t^{n+1}}{t^*(n+1)!}, \quad 0 < t < t^*,\\ 0, \quad t < 0. \end{cases} \tag{4.16}$$

Writing (4.16) with the aid of Heaviside functions gives

$$F_n = \frac{1}{t^*(n+1)!}\left[(t-t^*)^{n+1}H(t-t^*) - t^{n+1}H(t)\right]. \tag{4.17}$$

The complete expansion of the solution to Problem I is then given by

$$u^{(1)}(r,t) \sim r^{-1/2}\sum_{n=0}^{\infty} F_n[t-(r-1)/C_1]\sum_{j=0}^{\infty} u_{jn}r^{-j}, \tag{4.18}$$

where the u_{jn} are given recursively by (4.13) and the F_n are given by (4.15) for instantaneous plugging and by (4.16) or (4.17) for plugging time $t^* > 0$.

Problem II: Again, as in Problem I, we seek progressing wave solutions to this initial/boundary value problem. These solutions are assumed to have the asymptotic forms

$$\Phi(r,t) \sim \sum_{n=0}^{\infty} \phi_n(r) F_n\left[t - P_\phi(r)\right], \tag{4.19}$$

$$\Psi(r,t) \sim \sum_{n=0}^{\infty} \psi_n(r) F_n\left[t - P_\psi(r)\right], \tag{4.20}$$

where the F_n's are related by (4.2) and we assume that

$$\phi_n \equiv 0, \quad \psi_n \equiv 0, \quad n < 0.$$

Substituting from (4.19) and (4.20) into (3.14) and (3.15) and employing arguments similar to those in the treatment of Problem I we obtain the eikonal and transport equations, that is,

$$(P_\phi')^2 = 1/C_1^2, \quad (P_\psi')^2 = 1, \tag{4.21}$$

$$2P_\phi'\phi_n' + \frac{1}{r}P_\phi'\phi_n = \phi_{n-1}'' + \frac{1}{r}\phi_{n-1}' - \frac{4}{r^2}\phi_{n-1}, \quad n \geqslant 0, \tag{4.22}$$

$$2P_\psi'\psi_n' + \frac{1}{r}P_\psi'\psi_n = \psi_{n-1}'' + \frac{1}{r}\psi_{n-1}' - \frac{4}{r^2}\psi_{n-1}, \quad n \geqslant 0. \tag{4.23}$$

The solutions of (4.21)–(4.23) are

$$\left.\begin{aligned}
P_\phi(r) &= \bar{P}_\phi \pm \int_1^r \frac{d\tau}{C_1} = \bar{P}_\phi \pm \frac{r-1}{C_1} \quad \text{where } \bar{P}_\phi = P_\phi(1),\\
P_\psi(r) &= \bar{P}_\psi \pm \int_1^r d\tau = \bar{P}_\psi \pm (r-1) \quad \text{where } \bar{P}_\psi = P_\psi(1),\\
\phi_n(r) &= \bar{\phi}_n r^{-1/2} \pm \frac{C_1}{2}\int_1^r \left(\frac{\tau}{r}\right)^{1/2}\left\{\phi_{n-1}''(\tau) + \frac{1}{\tau}\phi_{n-1}'(\tau) - \frac{4}{\tau^2}\phi_{n-1}(\tau)\right\} d\tau,\\
\psi_n(\tau) &= \bar{\psi}_n r^{-1/2} \pm \frac{1}{2}\int_1^r \left(\frac{\tau}{r}\right)^{1/2}\left\{\psi_{n-1}''(\tau) + \frac{1}{\tau}\psi_{n-1}'(\tau) - \frac{4}{\tau^2}\psi_{n-1}(\tau)\right\} d\tau,\\
\bar{\phi}_n &= \phi_n(1), \quad \bar{\psi}_n = \psi_n(1).
\end{aligned}\right\} \tag{4.24}$$

From the expressions for ϕ_n, ψ_n in (4.24), it can be shown by induction that

$$\left.\begin{aligned}
\phi_n(r) &= r^{-1/2}\sum_{j=0}^{\infty} \phi_{jn} r^{-j}, \quad n \geqslant 0,\\
\psi_n(r) &= r^{-1/2}\sum_{j=0}^{\infty} \psi_{jn} r^{-j}, \quad n \geqslant 0,
\end{aligned}\right\} \tag{4.25}$$

where the coefficients ϕ_{jn} and ψ_{jn} will be determined for outgoing waves. Subsituting ϕ_n and ψ_n from (4.25) into the integrated forms of the transport equations in (4.24) and simplying gives

$$\phi_{jn} = \begin{cases} H(j)\phi_{j-1,n-1} & \text{if } 1 \leqslant j \leqslant n, \\ \bar{\phi}_n - \sum_{j=0}^{n} H(j)\phi_{j-1,n-1} & \text{if } j = 0, n > 0, \\ \bar{\phi}_0 & \text{if } j = n = 0, \\ 0 & \text{if } j < 0 \text{ or } j > n, \end{cases} \tag{4.26}$$

$$\psi_{jn} = \begin{cases} G(j)\psi_{j-1,n-1} & \text{if } 1 \leqslant j \leqslant n, \\ \bar{\psi}_n - \sum_{j=1}^{n} G(j)\psi_{j-1,n-1} & \text{if } j = 0, n > 0, \\ \bar{\psi}_0 & \text{if } j = n = 0, \\ 0 & \text{if } j < 0 \text{ or } j > n, \end{cases} \tag{4.27}$$

where

$$H(j) = \frac{C_1}{2j}\left[4 - (j - 1/2)^2\right], \tag{4.28}$$

$$G(j) = \frac{1}{2j}\left[4 - (j - 1/2)^2\right]. \tag{4.29}$$

The unknown quantities in our solution, namely, $\bar{P}_\phi$, $\bar{P}_\psi$, $\bar{\phi}_n$, $\bar{\psi}_n$, and F_n must now be determined from the boundary conditions (3.17) and (3.18). Inserting the expansions for Φ and Ψ from (4.19) and (4.20) into (3.17) and (3.18) gives

$$\begin{aligned} \sum_{n=0}^{\infty} \Big[&\bar{\phi}_n F_{n-2}(t - \bar{P}_\phi) - 2\bar{\phi}_n' F_n(t - \bar{P}_\phi) + 2\bar{\phi}_n F_{n-1}(t - \bar{P}_\phi)/C_1 \\ &+ 8\bar{\phi}_n F_n(t - \bar{P}_\phi) + 4\bar{\psi}_n' F_n(t - \bar{P}_\psi) \\ &- 4\bar{\psi}_n F_{n-1}(t - \bar{P}_\psi) - 4\bar{\psi}_n F_n(t - \bar{P}_\psi)\Big] \sim -\frac{S}{2} f(t) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \Big[&\bar{\psi}_n F_{n-2}(t - \bar{P}_\psi) - 2\bar{\psi}_n' F_n(t - \bar{P}_\psi) + 2\bar{\psi}_n F_{n-1}(t - \bar{P}_\psi) \\ &+ 8\bar{\psi}_n F_n(t - \bar{P}_\psi) + 4\bar{\phi}_n' F_n(t - \bar{P}_\phi) \\ &- 4\bar{\phi}_n F_{n-1}(t - \bar{P}_\phi)/C_1 - 4\bar{\phi}_n F_n(t - \bar{P}_\phi)\Big] \sim -\frac{S}{2} f(t). \end{aligned} \tag{4.31}$$

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The conditions (4.30) and (4.31) are simultaneously satisfied by taking

$$\left.\begin{array}{l}\bar{\phi}_0 = 0, \quad \bar{\psi}_0 = 0, \quad \bar{P}_\phi = \bar{P}_\psi = 0,\\ \bar{\phi}_1 = 0, \quad \bar{\psi}_1 = 0, \quad F_0 = f(t),\\ \bar{\phi}_2 = -S/2, \quad \bar{\psi}_2 = -S/2,\\ \bar{\phi}_n - 2\bar{\phi}'_{n-2} + 2\bar{\phi}_{n-1}/C_1 + 8\bar{\phi}_{n-2} + 4\bar{\psi}'_{n-2} - 4\bar{\psi}_{n-1} - 4\bar{\psi}_{n-2} = 0, \quad n \geqslant 3,\\ \bar{\psi}_n - 2\bar{\psi}'_{n-2} + 2\bar{\psi}_{n-1} + 8\bar{\psi}_{n-2} + 4\bar{\phi}'_{n-2} - 4\bar{\phi}_{n-1}/C_1 - 4\bar{\phi}_{n-2} = 0, \quad n \geqslant 3.\end{array}\right\} \tag{4.32}$$

Combining these initial conditions with the first two equations in (4.24) gives (for outgoing waves)

$$\left.\begin{array}{l}P_\phi(r) = (r-1)/C_1,\\ P_\psi(r) = r - 1.\end{array}\right\} \tag{4.33}$$

The complete expansions for Φ and Ψ are given by

$$\Phi(r,t) \sim r^{-1/2} \sum_{n=0}^{\infty} F_n[t-(r-1)/C_1] \sum_{j=1}^{n} \phi_{jn} r^{-j}, \tag{4.34}$$

$$\Psi(r,t) \sim r^{-1/2} \sum_{n=0}^{\infty} F_n[t-(r-1)] \sum_{j=1}^{n} \psi_{jn} r^{-j}, \tag{4.35}$$

where ϕ_{jn} and ψ_{jn} are given recursively by (4.26) and (4.27) in combination with the boundary conditions (4.32). For instantaneous plugging the F_n are given by (4.15) whereas for punching time $t^* > 0$ they are given by (4.16) or (4.17).

5. Composing the complete solution

In this section we describe how to construct the complete solution for the stress fields. As well, we point out certain information which the formal series reveal about the wave character of the solution.

Employing (3.2) in the relationships among the stress and displacement components, we obtain

$$\sigma_r^{(1)} = C_1^2 \frac{\partial u^{(1)}}{\partial r} + \frac{\gamma}{r} u^{(1)}, \tag{5.1}$$

$$\sigma_\theta^{(1)} = \gamma \frac{\partial u^{(1)}}{\partial r} + C_1^2 u^{(1)}, \tag{5.2}$$

$$\sigma_r^{(2)} = \frac{\partial^2 \Phi}{\partial t^2} + \frac{2}{r}\left(\frac{4}{r}\Phi - \frac{\partial \Phi}{\partial r} + 2\frac{\partial \Psi}{\partial r} - \frac{2}{r}\Psi\right), \tag{5.3}$$

$$\sigma_\theta^{(2)} = \frac{\beta}{\alpha}\frac{\partial^2\Phi}{\partial t^2} - \frac{2}{r}\left(\frac{4}{r}\Phi - \frac{\partial\Phi}{\partial r} + 2\frac{\partial\Psi}{\partial r} - \frac{2}{r}\Psi\right), \tag{5.4}$$

$$\tau_{r\theta}^{(2)} = -\frac{\partial^2\Psi}{\partial t^2} - \frac{2}{r}\left(\frac{4}{r}\Psi - \frac{\partial\Psi}{\partial r} + 2\frac{\partial\Phi}{\partial r} - \frac{2}{r}\Phi\right). \tag{5.5}$$

In obtaining the last three formulas we have made use of (3.14) and (3.15), the differential equations for Φ and Ψ. Combining (5.1)–(5.5) with the formal series solutions to Problems I and II using the relation (4.2), we obtain series expansions for $\sigma_r^{(1)}, \sigma_\theta^{(1)}, \sigma_r^{(2)}, \sigma_\theta^{(2)}$, and $\tau_{r\theta}^{(2)}$. Then using these results in (3.1) we find the solutions for the stesses $\sigma_r(r,\theta,t)$, $\sigma_\theta(r,\theta,t)$ and $\tau_{r\theta}(r,\theta,t)$. Finally, to these we must add the initial stresses (2.1).

On examining the formal solutions obtained by the ray method we see that $u^{(1)}$ and Φ are zero for $t < (r-1)/C_1$ while Ψ is zero for $t < r-1$. Thus a wave-front leaves the opening at $t = 0$ ahead of which the stresses have their initial static values. This leading wavefront propagates radially and, since $C_1 > C_2 = 1$, has speed C_1.

In addition the form of the series solutions facilitate locating and determining the magnitude of jumps in the stresses and their derivatives. In particular, we find that σ_r, σ_θ and $\tau_{r\theta}$ are continuous, but have finite discontinuities in their time derivatives and r-derivatives. Graphically the discontinuities in the first derivatives appear as creases in the solution surfaces. There are four distinct lines in the rt-plane along which these creases occur. Hence the present problem has four recognizable wavefronts propagating in the radial direction. Two of these leave the opening with speeds C_1 and C_2 at $t = 0$, one of which is the leading wavefront mentioned above. The remaining two leave the opening at $t = t^*$ when the punching is complete, and also propagate with speeds C_1 and C_2.

The location and magnitude of the discontinuities in the time derivative of the stress components are given in Table 1. The first column in the table gives the equation of the wavefront on which a discontinuity occurs while the remaining columns give the magnitude of each of the discontinuities. The square brackets have the significance

$$[A] = A_2 - A_1$$

where A_2 and A_1 are the values of A ahead of and behind the corresponding wavefront. The jump in the r-derivative of a component of stress differs from the corresponding jumps in the time derivative by a multiplicative factor. For a wavefront moving with speed C_1 the factor is $-C_1$ and for a front moving with speed $C_2 = 1$, the factor is -1.

TABLE 1
Location and magnitude of jumps in time derivative of stress components

Wavefronts	$\left[\frac{\partial \sigma_r}{\partial t}\right]$	$\left[\frac{\partial \sigma_\theta}{\partial t}\right]$	$\left[\frac{\partial \tau_{r\theta}}{\partial t}\right]$
$t = \frac{(r-1)}{C_1}$	$\frac{S}{2t^* r^{1/2}}(1 - \cos 2\theta)$	$\frac{S\beta}{2t^* r^{1/2}\alpha}(1 - \cos 2\theta)$	0
$t = r - 1$	0	0	$\frac{S}{2t^* r^{1/2}}\sin 2\theta$
$t = t^* + \frac{(r-1)}{C_1}$	$-\frac{S}{2t^* r^{1/2}}(1 - \cos 2\theta)$	$-\frac{S\beta}{2t^* r^{1/2}\alpha}(1 - \cos 2\theta)$	0
$t = t^* + r - 1$	0	0	$-\frac{S}{2t^* r^{1/2}}\sin 2\theta$

6. Numerical results

Numerical results are obtained from the formal series solutions. The various recurrence formulas generated by the ray method, although complicated, are in a form suitable for coding in a programming language. Since the computer program which summed the series required very little computing time, we made several runs, each time increasing the number of terms in the series. The output of each run, for a particular component of stress, was in agreement for a certain clearly defined range of the independent variables r and t. This procedure indicated that the series solutions are convergent and it was clear for what values of r and t our numerical results are valid.

All results, presented graphically in Figures 2 to 7, are for the isotropic case with Poisson's ratio $\nu = 0.3$. The curves in Figures 2 and 3 show the variation of σ_θ with nondimensional time for $t^* = 0.25$, 0.5 and 1.0. The curves shown in the remaining figures are for $t^* = 1.0$, the realistic value for plugging of an aluminium plate, 0.5 mm thick, by an 11 mm diameter projectile, with impact speed 700 ms^{-1} (see [2]).

The discontinuities in the first derivative of the various components of stress appear as corners on our graphs. According to Table 1, all our graphs will have corners except those for $\theta = 0$. Certain corners are not visible in some of the figures due to the magnitude of the slopes involved and the scale of the graphs. Thus for example, the curves in Figure 3 have corners at $t = t^*$, but only those for $t^* = 0.25$, 0.5 are clearly visible, while the one for $t^* = 1.0$ is not.

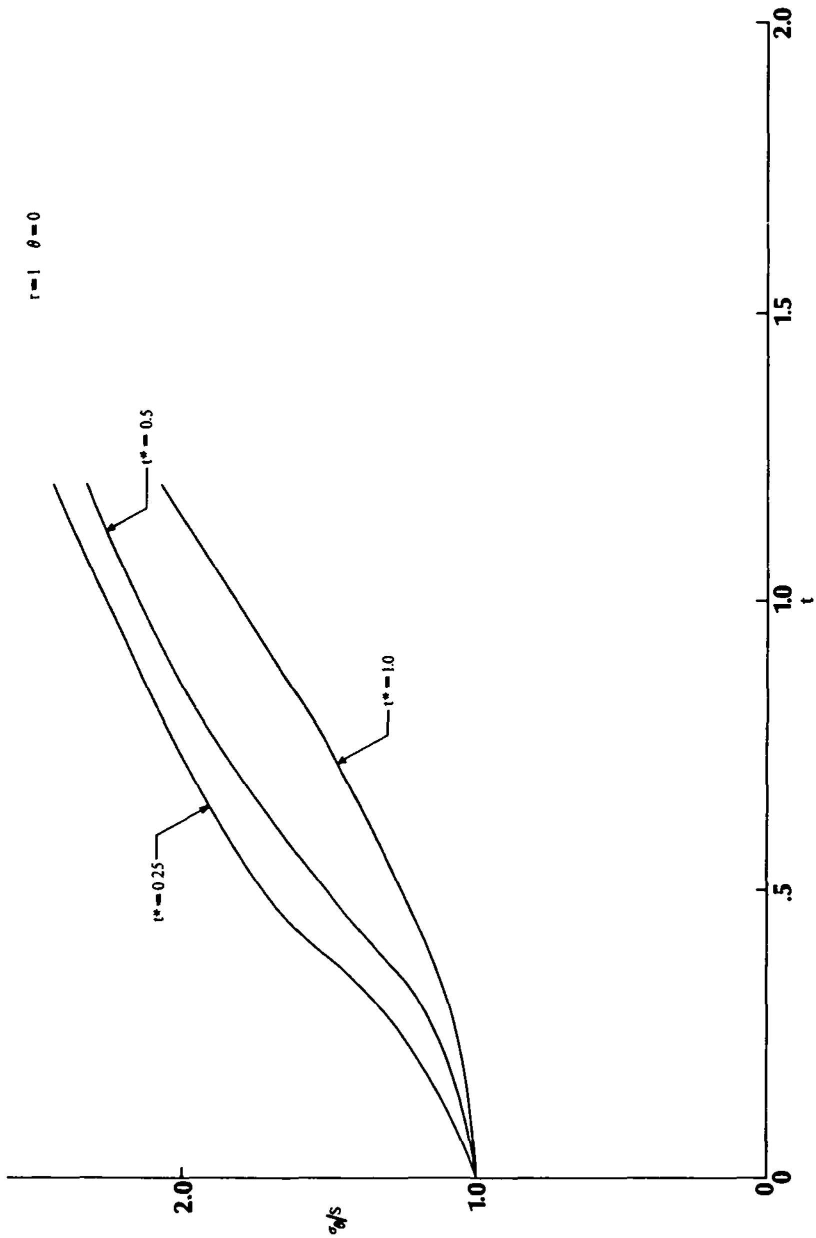

Figure 2. Variation of σ_θ/S with nondimensional time at $r = 1$, $\theta = 0$ for $t^* = 0.25$, 0.5 and 1.0.

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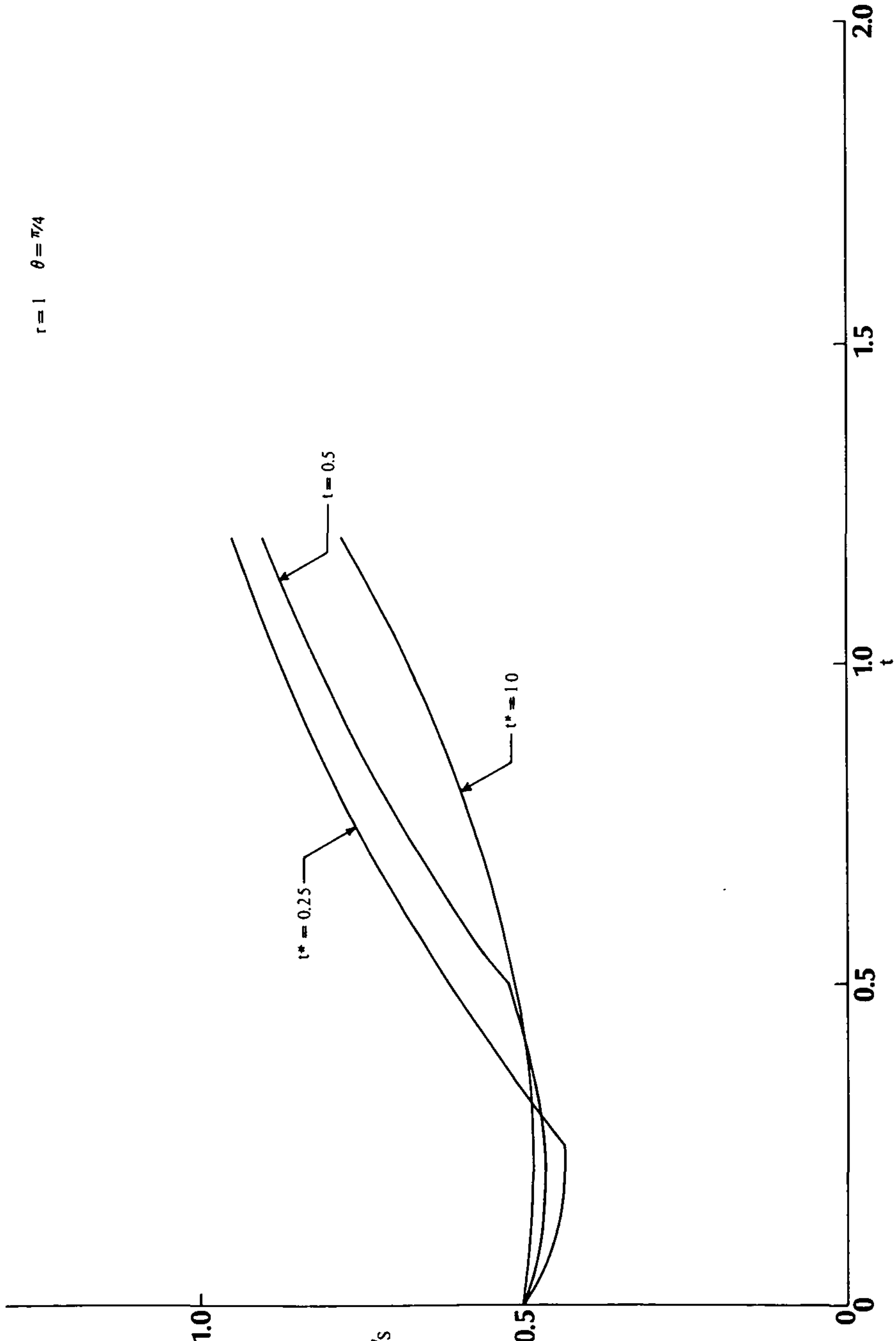

Figure 3. Variation of σ_θ/S with nondimensional time at $r = 1$, $\theta = \pi/4$ for $t^* = 0.25$, 0.5 and 1.0.

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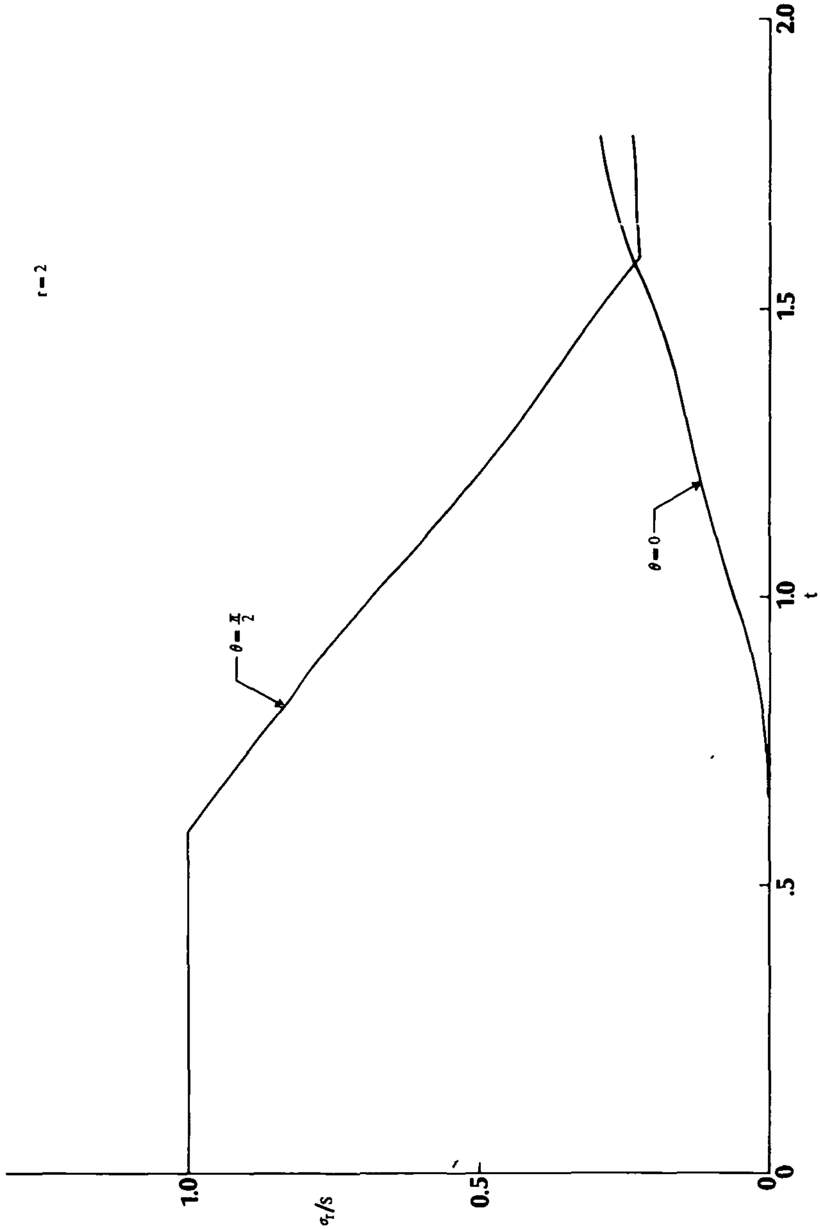

Figure 4. Variation of σ_r/S with nondimensional time at $r = 2$, $\theta = 0$ and $\theta = \pi/2$ for $t^* = 1.0$.

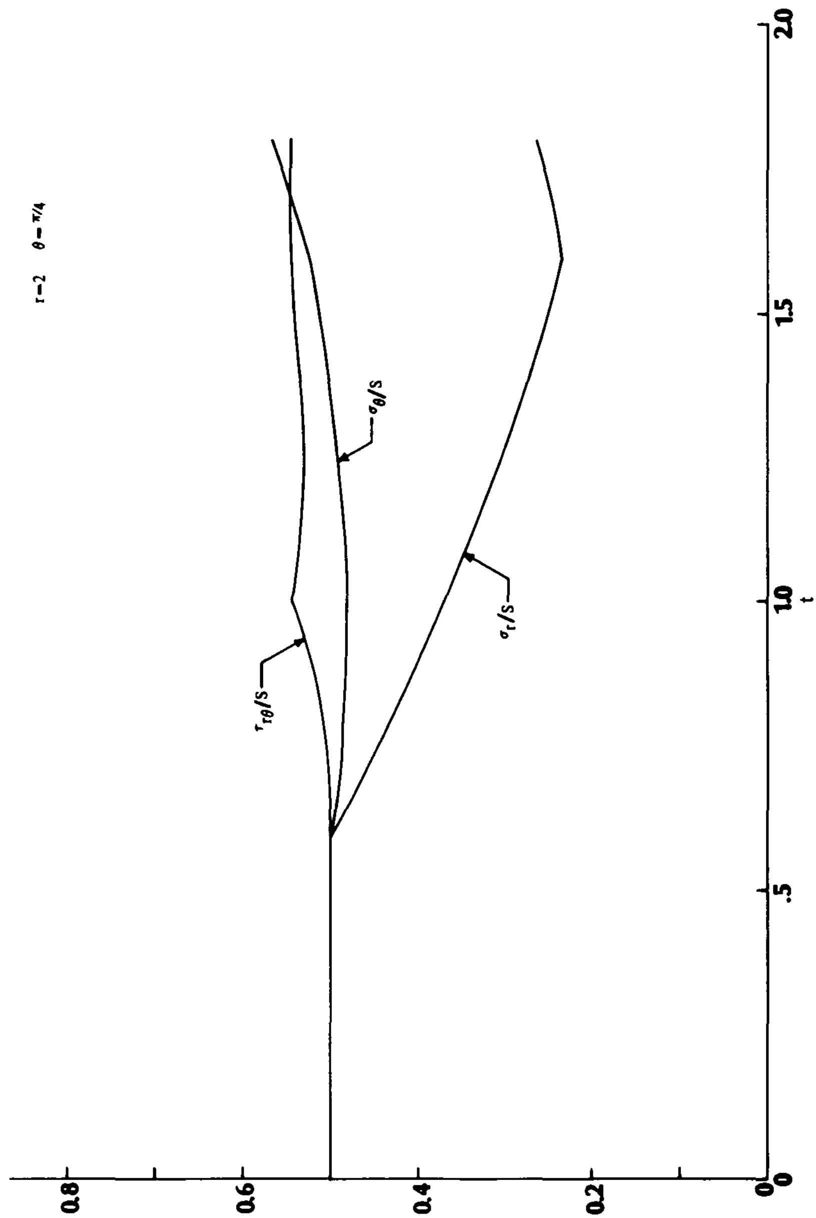

Figure 5. Variation of σ_r/S, σ_θ/S, and $\tau_{r\theta}/S$ with nondimensional time at $r = 2$, $\theta = \pi/4$ for $t^* = 1.0$.

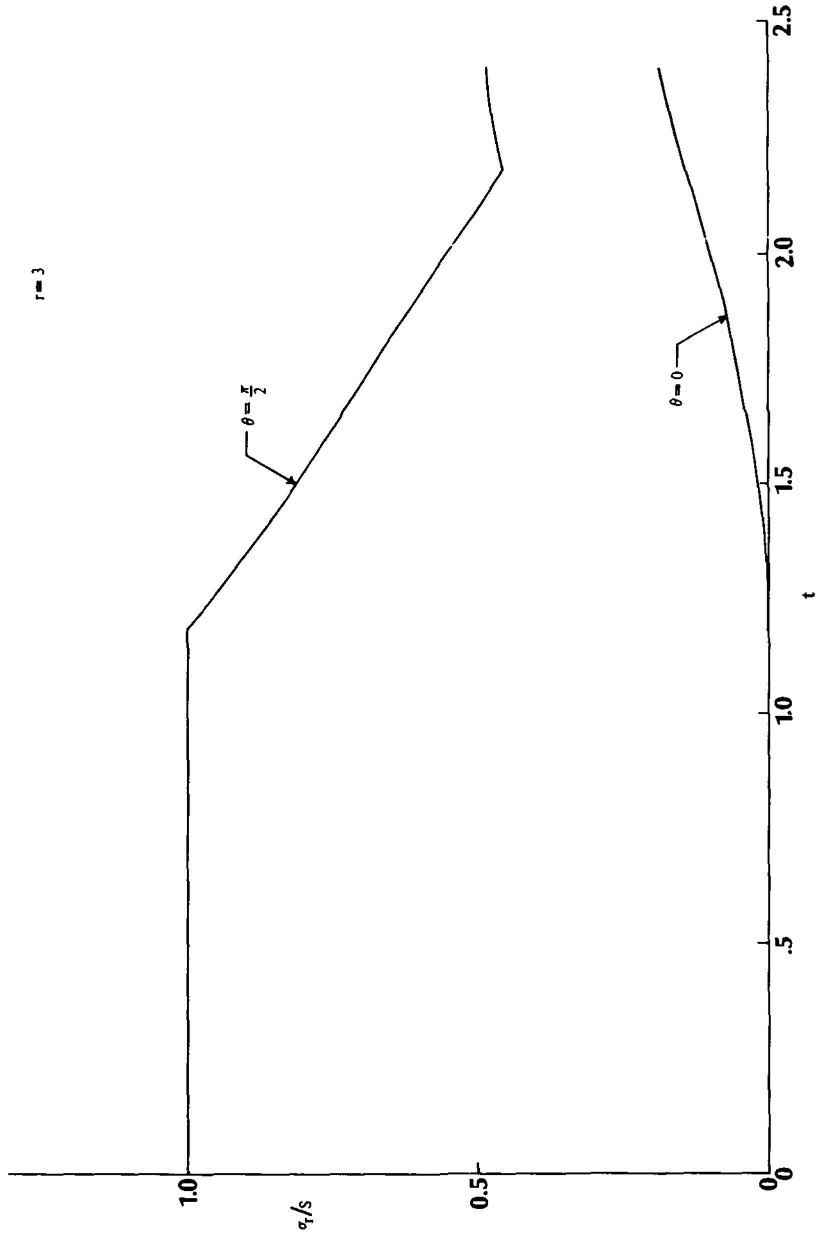

Figure 6. Variation of σ_r/S with nondimensional time at $r = 3$, $\theta = 0$ and $\theta = \pi/2$ for $t^* = 1.0$.

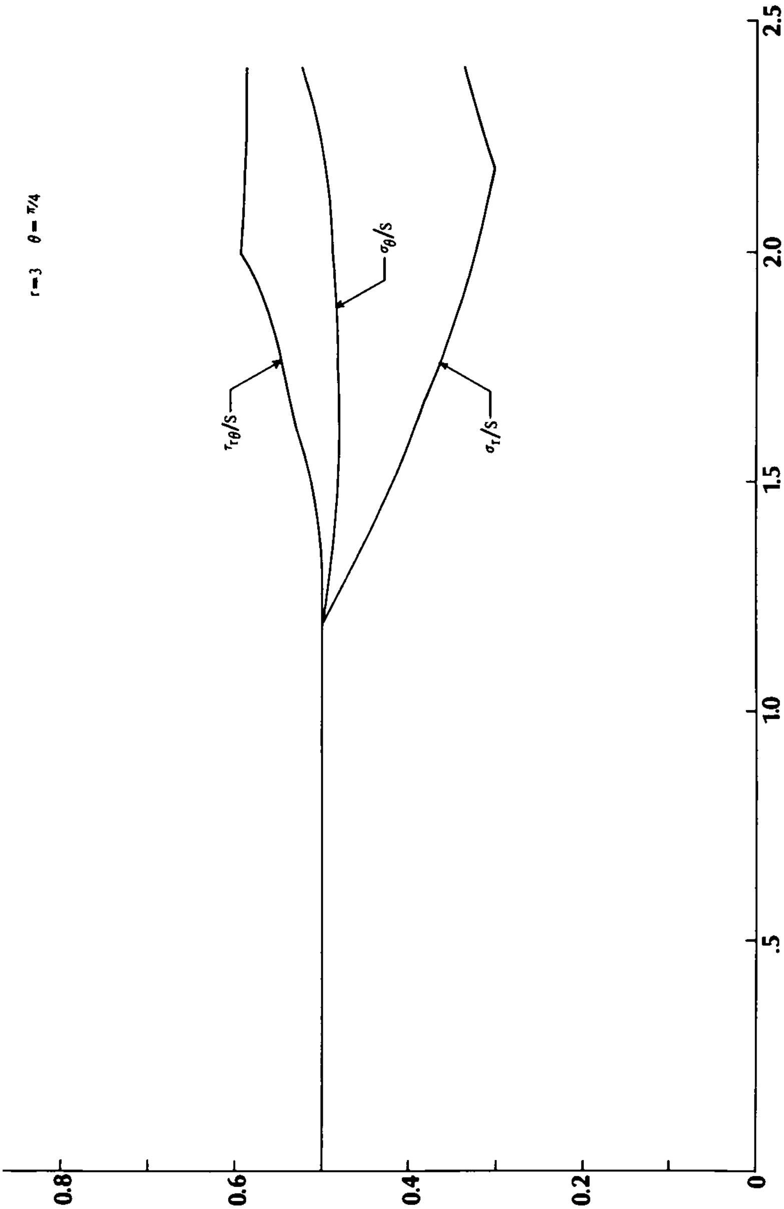

Figure 7. Variation of σ_r/S and $\tau_{r\theta}/S$ with nondimensional time at $r = 3$, $\theta = \pi/4$ for $t^* = 1.0$.

7. Summary

In this paper we have obtained plane stress solutions to the two-dimensional problem of elastic waves propagating from a suddenly punched hole in a stretched elastic plate. These solutions were obtained by a ray analysis which is direct and involves only ordinary differentiation, integration and algebra. A clear picture of the propagation process was obtained and we were able to identify four wavefronts which were not discovered by the more elaborate numerical scheme used in [2].

The ray method has been applied to problems of wave propagation in solids by many authors [1], [3], [5], [7], (see [4] for a more extensive list of references). As was pointed out by Karal and Keller [3], the formal series solutions obtained by this method are not necessarily convergent. The present paper appears to be one of the first to use the full series to obtain numerical results and in the process shed some light on the convergence of the series. Because of the complicated nature of the recurrence formulas (4.13), (4.26) and (4.27), a rigorous analysis of the convergence of (4.1), (4.19) and (4.20) is difficult and has not been carried out. For practical purposes this is not necessary. By using an increasing number of terms, it is apparent that each series is convergent for a certain range of the independent variable. The numerical results obtained from these convergent series are valid for important time intervals after the leading wavefront has passed a station in the plate. These results are in excellent agreement with those presented in [2].

Acknowledgements

The research reported in this paper was supported in part by NSERC Grants A 4039 (T.B.M.) A 5157 (R.J.T.) and A 3958 (D. W.B.). This work was carried out while T.B.M. was a recipient of a McCalla Research Professorship.

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