Nagoya Math. J. **208** (2012), 201–263
DOI 10.1215/00277630-1815249

MATRIX COEFFICIENTS OF THE LARGE DISCRETE SERIES REPRESENTATIONS OF $\mathrm{Sp}(2;\mathbf{R})$ AS HYPERGEOMETRIC SERIES OF TWO VARIABLES

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In Memoriam: Hiroshi Saito

Abstract. We investigate the radial part of the matrix coefficients with minimal K-types of the large discrete series representations of $\mathrm{Sp}(2;\mathbf{R})$. They satisfy certain difference-differential equations derived from Schmid operators. This system is reduced to a holonomic system of rank 4, which is finally found to be equivalent to higher-order hypergeometric series in the sense of Appell and Kampé de Fériet.

Contents

Introduction

We investigate the radial part of the matrix coefficients of the *large* discrete series representations with minimal K-type on the real symplectic group $G = \mathrm{Sp}(2,\mathbf{R})$ of rank 2.

Given a Hilbert representation (π, H) of G, for some vectors $v, w \in H$ the function $c_{v,w} : g \in G \to (\pi(g)v, w)_H$ is called a *matrix element* or a *matrix*

Received January 31, 2011. Revised February 29, 2012. Accepted May 2, 2012.
2010 Mathematics Subject Classification. Primary 22E30; Secondary 22E45, 43A90, 33C65.

coefficient of the representations π. Among the set of equivalence classes of the discrete series representations of G (or the irreducible square integrable representations of G), there are $4 = 8/2 = |W_G|/|W_K|$ different classes with the same infinitesimal characters. Here, W_G is the Weyl group of G, and W_K is the Weyl group of a maximal compact subgroup K, whose orders are 8 and 2, respectively.

In these four discrete series representations with the same infinitesimal character, one belongs to the holomorphic discrete series and another to the antiholomorphic discrete series. The Gelfand-Kirillov dimension of these two representations is 3, and that of the remaining two discrete series representations is 4, which is equal to the dimension of the maximal unipotent subgroup. We call the latter two representations the *large* discrete series representations, following Kostant and Vogan. This kind of representation is rather different from the holomorphic discrete series; in particular, they have the Whittaker model. But these representations are also very important. For example, they also contribute to the relative Lie algebra cohomology.

If we realize a holomorphic discrete series representation in $L^2(G)$ and see its double K-finite elements, then the restriction of these functions to the split Cartan subgroup A in view of the Cartan decomposition $G = KAK$ is a (Laurent) rational function in the matrix entries of elements in A. This follows the explicit formula by Hua of the Bergmann kernel.

If we consider the same problem for the large discrete series representations, we can guess that more difficult transcendental functions appear in the matrix entries of the elements of A. However, as we have already shown in [10] in the case of Whittaker functions, these functions are not as "transcendental" as in the case when π is a principal series representation, where they should appear as BC_2-type hypergeometric functions (see [1], [2], [5], [6], [7], [14], [15]).

Our target in this paper is to grasp these functions for the large discrete series, which are *mid*-transcendental between rationals and BC_2-type transcendentals. The answer is the hypergeometric functions of two variables in the sense of Appell and Kampé de Fériet. It is essentially F_2 or, more generally speaking, F_D of Lauricella and a kind of *half system* of BC_2-type hypergeometric functions.

First, we want to have the holonomic system for the A-radial part of the matrix coefficients with minimal K-type of the large discrete series representations.

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In the $G \times G$ birepresentation on $L^2(G)$ via the right and the left regular representations, any discrete series representation π is realized as a closed subspace in $L^2(G)$ isomorphic to the outer tensor product $\pi^* \boxtimes \pi$ (with multiplicity 1, a conclusion of the Plancherel theorem). Here π^* is the contragredient representation of G.

A deep result, but now rather well known, is that there exists the minimal K-type (τ_μ, W_μ) in π. Then $\tau_\mu^* \boxtimes \tau_\mu$ is a finite-dimensional $(K \times K)$-bimodule, consisting of real analytic functions on G.

There is a method to characterize those functions with these special double K-types in $\pi^* \boxtimes \pi$, by utilizing Schmid operators (see [16]). And there is a way to compute the A-radial parts of these Schmid operators, though it is rather tedious and long. Arranging these functions in a canonical way, we have a (unique) vector-valued function on G, which is also the reproducing kernel of this discrete series.

The purpose of this paper is to go through this procedure until we can write the function in terms of the classical object, Appell's hypergeometric functions.

The organization of this paper is the following. Section 1 presents generalities and basic symbols on our group G and its Lie subgroups and corresponding Lie algebras; Section 2, basic results on the discrete series representations of G and the Schmid operators; Section 3, the explicit expression of the Schmid operators with respect to the standard basis of K-modules, and we detail a system of differential-difference equations in two variables. In Section 4 the system of equations is rewritten in a new system of variables, so that we have a more tractable system. In Section 5 we show the extremal components of the radial part of the matrix coefficients satisfy Appell's differential equations of type 2, and we present a power series solution and an integrable expression for these extremal components. In Section 6, power series and integral expressions are extended for all components of the radial parts of the matrix coefficients. Section 7 presents a short postscript for further research.

Chronologically speaking, our first result on the special functions on $\mathrm{Sp}(2, \mathbf{R})$ is [11], that is, the Whittaker functions belonging to the large discrete series. Somewhat later, Miyazaki [10] pointed out that the "shape" of K-types of the generalized principal series representations obtained by the parabolic induction with respect to the Jacobi parabolic subgroup (corresponding to the long root) from a discrete series of $\mathrm{SL}(2, \mathbf{R})$, is the same as that of the large discrete series. We can prove that the Whittaker function with the *corner* K-type is quite similar to that of the large discrete series (see [10]). On the other hand, Iida [9] investigated the matrix coefficients with minimal K-types of the principal series representations induced via the minimal parabolic subgroup and those with the corner K-types of the generalized principal series induced via the Jacobi parabolic subgroup. He and others showed that the radial part of the matrix coefficients of the latter representations satisfied a holonomic system in two variables of rank 4. We note here that this 4 is the degree of the associate variety of our discrete series.

Since the K-type of these representations is the same as that of the large discrete series up to *translation*, it is reasonable to believe that the radial parts of the matrix coefficients of the large discrete series are written by similar functions. And we have confirmed that in this paper.

The holonomic system for c_I in Section 2 itself was already obtained in the fall of 1994. The solutions at peripheral entries were obtained relatively early because the reduced holonomic system for them is similar to that of Iida [9]. The whole system of solutions took until 1998. An informal report on the result was contributed to RIMS, Kokyuroku (see [12], [13]). I apologize to the mathematics community for taking so long to prepare the final version of the paper.

Since my original intention was to apply this kernel function for trace formulas, this opportunity seems to be adequate for me to publish my results.

Also I have to add that the ideas here originated from a talk with my former teacher Ihara in the early 1980s, while we were sitting beside the Yasuda auditorium at Hongo campus, about the article by Hirai [8] on the character formula of the discrete series of $\mathrm{Sp}(2,\mathbf{R})$.

§1. Generalities on the real symplectic group of rank 2

1.1. The group and related subgroups and the corresponding Lie algebras

Let $E_{i,j}$ be the matrix unit in the space $M_4(\mathbf{R})$ of real square matrices of size 4, with 1 as its (i,j)-component and 0 at the other entries. We set $J = E_{1,3} + E_{2,4} - E_{3,1} - E_{4,2}$. Then the real symplectic group $G = \mathrm{Sp}(2,\mathbf{R})$ of rank 2 is a subgroup of the real special linear $\mathrm{SL}_4(\mathbf{R})$ of size 4, defined by

$$\mathrm{Sp}(2,\mathbf{R}) = \left\{ g \in \mathrm{SL}_4(\mathbf{R}) \,\middle|\, {}^t gJg = J \right\}.$$

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A maximal compact subgroup K of G is given by

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}(2,\mathbf{R}) \;\middle|\; A, B \in M_2(\mathbf{R}) \right\},$$

which is the fixed part of the associated Cartan involution $\theta : g \mapsto {}^t g^{-1}$. The Cartan symmetric decomposition for the corresponding Lie algebras is denoted as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Here we use the standard convention to denote the Lie algebra of a Lie group X by the corresponding German lowercase letter $\mathfrak{x}$.

A maximal abelian subalgebra $\mathfrak{a}$ in $\mathfrak{p}$ is chosen as

$$\mathfrak{a} = \left\{ \mathrm{diag}(x_1, x_2, -x_1, -x_2) \;\middle|\; x_1, x_2 \in \mathbf{R} \right\},$$

with the standard basis $H_1 = \mathrm{diag}(1,0,-1,0)$ and $H_2 = \mathrm{diag}(0,1,0,-1)$, and we put $A = \exp(\mathfrak{a})$. It defines a split Cartan subgroup of G. The root system for $(\mathfrak{g}, \mathfrak{a})$ is realized as

$$\Phi(\mathfrak{g}, \mathfrak{a}) = \{\pm 2e_1, \pm 2e_2, \pm e_1 \pm e_2\}$$

with respect to the standard basis $\{e_1 = (1,0), e_2 = (0,1)\}$ in the 2-dimensional Euclidean space, with $e_1(x_1, x_2) = x_1$ and $e_2(x_1, x_2) = x_2$, or equivalently, $e_i(H_j) = \delta_{ij}$ (δ_{ij} being the Kronecker delta).

We fix a positive system by the simple roots $\{e_1 - e_2, 2e_2\}$; then the unipotent radical $\mathfrak{n}$ of the corresponding Borel subalgebra is given by

$$\mathfrak{n} = \mathbf{R}E_{e_1-e_2} \oplus \mathbf{R}E_{2e_1} \oplus \mathbf{R}E_{e_1+e_2} \oplus \mathbf{R}E_{2e_2},$$

with

$$\begin{aligned} E_{e_1-e_2} &= E_{1,2} - E_{4,3}, & E_{2e_1} &= E_{1,3}, \\ E_{e_1+e_2} &= E_{2,3} + E_{1,4}, & E_{2e_2} &= E_{2,4}. \end{aligned}$$

Set $N = \exp(\mathfrak{n})$. Then we have Iwasawa decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad \text{and} \qquad G = KAN.$$

The homogeneous space G/K is a Hermitian symmetric space of C_{II}-type, and its complex structure on the tangent space $\mathfrak{p}$ at the origin $[K] \in G/K$

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is given by the adjoint action of the central element $\iota = (1/\sqrt{2})\left(\begin{smallmatrix} 1_2 & 1_2 \\ -1_2 & 1_2 \end{smallmatrix}\right)$ in K. The eigenspaces of $\mathrm{Ad}_{\mathfrak{p}}(\iota)$ with the eigenvalues $\pm\sqrt{-1}$ in the complexification $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p} \otimes_{\mathbf{R}} \mathbf{C}$ are given by

$$\mathfrak{p}_{\pm} = \left\{ \begin{pmatrix} C & \pm\sqrt{-1}C \\ \pm\sqrt{-1}C & -C \end{pmatrix} \in \mathfrak{g} \,\middle|\, C \in M_2(\mathbf{C}) \right\}.$$

1.2. Cartan decomposition of the basis of $\mathfrak{p}_{\pm}$

Let $a = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A$ be a regular element of A; that is,

$$a_i \neq 1 \quad (i = 1, 2), \qquad a_1 \neq a_2, \qquad \text{and} \qquad a_1 a_2 \neq 1.$$

Then we have Cartan double coset decompositions

$$\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k} \qquad \text{and} \qquad \mathfrak{g} = \mathrm{Ad}(a)\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}.$$

Here we collect the formulas of Cartan decompositions of standard basis of $\mathfrak{p}_{\pm}$. First, we consider the case $\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$. Before stating the formulas, we describe the standard basis of $\mathfrak{p}_{\pm}$.

Notation 1.1. For a square matrix x of size 2, we set

$$p_+(x) := \begin{pmatrix} x & \sqrt{-1}x \\ \sqrt{-1}x & -x \end{pmatrix}, \qquad p_-(x) := \begin{pmatrix} x & -\sqrt{-1}x \\ -\sqrt{-1}x & -x \end{pmatrix}.$$

Then we define a basis in each of $\mathfrak{p}_{\pm}$ by

$$X_{(2,0)} = p_+(e_{11}), \qquad X_{(1,1)} = p_+(e_{12} + e_{21}), \qquad X_{(0,2)} = p_+(e_{22}) \in \mathfrak{p}_+;$$
$$X_{(-2,0)} = p_-(e_{11}), \qquad X_{(-1,-1)} = p_-(e_{12} + e_{21}), \qquad X_{(0,-2)} = p_-(e_{22}) \in \mathfrak{p}_-$$

with the matrix units in $M_2(\mathbf{C})$:

$$e_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad e_{21} = {}^t e_{12}, \qquad e_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The inverse of the mapping

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathfrak{k} \mapsto A + iB \in \mathfrak{u}(2)$$

is denoted by κ, and its complexification $M_2(\mathbf{C}) \cong \mathfrak{k}_{\mathbf{C}}$ is denoted by the same symbol. We put

$$H_i' := \kappa(e_{ii}) \quad (i = 1, 2), \qquad X := \kappa(e_{12}), \qquad \bar{X} := \kappa(e_{21}).$$

Moreover, we set

$$E_{-2e_i} = {}^t E_{2e_i} \quad (i = 1, 2), \qquad E_{-e_1 \pm e_2} = {}^t E_{e_1 \mp e_2}.$$

For our later convenience, we introduce some notation for hyperbolic functions in $\log a_i$.

NOTATION 1.2. For real positive numbers a, a_1, a_2, we set

$$sh(a) = \frac{1}{2}(a - a^{-1}), \qquad ch(a) = \frac{1}{2}(a + a^{-1}),$$
$$th(a) = sh(a)/ch(a), \qquad cth(a) = ch(a)/sh(a), \qquad \text{and}$$
$$D = D(a_1, a_2) = sh^2(a_1) - sh^2(a_2) = ch^2(a_1) - ch^2(a_2).$$

Note here that the last function is antisymmetric with respect to the permutation of the variables $(a_1, a_2) \leftrightarrow (a_2, a_1)$. Moreover, for $a = (a_1, a_2)$, we introduce the other four functions:

$$e_{pq}(a) = (-1)^{p-1} D^{-1} sh(a_p) ch(a_q) \quad (p, q \in \{1, 2\}).$$

LEMMA 1.1. *With respect to the decomposition* $\mathfrak{g} = \mathrm{Ad}(a^{-1})\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$, *we have*

$$X_{(-2,0)} = -sh(a_1^2)^{-1}\mathrm{Ad}(a^{-1})H_1' + H_1 + ct(a_1^2)H_1',$$
$$\frac{1}{2}X_{(-1,-1)} = e_{11}(a)X + e_{22}(a)\bar{X} - e_{12}(a)\mathrm{Ad}(a^{-1})X - e_{21}(a)\mathrm{Ad}(a^{-1})\bar{X},$$
$$X_{(0,-2)} = -sh(a_2^2)^{-1}\mathrm{Ad}(a^{-1})H_2' + H_2 + ct(a_2^2)H_2',$$
$$X_{(2,0)} = sh(a_1^2)^{-1}\mathrm{Ad}(a^{-1})H_1' + H_1 - ct(a_1^2)H_1',$$
$$\frac{1}{2}X_{(1,1)} = -e_{22}(a)X - e_{11}(a)\bar{X} + e_{21}(a)\mathrm{Ad}(a^{-1})X + e_{12}(a)\mathrm{Ad}(a^{-1})\bar{X},$$
$$X_{(0,2)} = sh(a_2^2)^{-1}\mathrm{Ad}(a^{-1})H_2' + H_2 - ct(a_2^2)H_2'.$$

Proof. Recall that

$$X_{(\pm 2,0)} = H_1 \pm \sqrt{-1}(E_{2e_1} + E_{-2e_1}),$$
$$X_{(0,\pm 2)} = H_2 \pm \sqrt{-1}(E_{2e_2} + E_{-2e_2}),$$
$$X_{(-1,-1)} = -2X + 2E_{e_1-e_2} - 2\sqrt{-1}E_{e_1+e_2},$$

and

$$X_{(1,1)} = 2\bar{X} + 2E_{e_1-e_2} + 2\sqrt{-1}E_{e_1+e_2}.$$

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It is enough to have Cartan decompositions for elements in $\mathfrak{p}$. For example, for $\kappa(\sqrt{-1}e_{11}) = E_{1,3} - E_{3,1} = E_{2e_1} - E_{-2e_1}$, we have $\mathrm{Ad}(a^{-1}) \cdot \kappa(\sqrt{-1}e_{11}) = a_1^{-2}E_{1,3} - a_1^2 E_{3,1}$. Apply the Cartan involution $\theta : W \in \mathfrak{g} \mapsto -{}^tW$. Then we have

$$\mathrm{Ad}(a^{-1})\kappa(\sqrt{-1}e_{11}) = -sh(a_1^2)(E_{1,3} + E_{3,1}) + ch(a_1^2)(E_{1,3} - E_{3,1})$$

or, equivalently,

$$i(E_{13} + E_{31}) = sh^{-1}(a_1)\mathrm{Ad}(a^{-1})\kappa(e_{11}) - ct(a_1^2)\kappa(e_{11}).$$

This gives the decomposition for $X_{(\pm 2,0)}$. The remaining cases are treated similarly. □

The other decomposition, $\mathfrak{g} = \mathrm{Ad}(a)\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{k}$, is discussed similarly.

Lemma 1.2. *We have*

$$\begin{aligned}
X_{(-2,0)} &= sh(a_1^2)^{-1}\mathrm{Ad}(a)H_1' + H_1 - ct(a_1^2)H_1', \\
\frac{1}{2}X_{(-1,-1)} &= -e_{11}(a)X - e_{22}(a)\bar{X} + e_{12}(a)\mathrm{Ad}(a)X + e_{21}(a)\mathrm{Ad}(a)\bar{X}, \\
X_{(0,-2)} &= sh(a_2^2)^{-1}\mathrm{Ad}(a)H_2' + H_2 - ct(a_2^2)H_2', \\
X_{(2,0)} &= -sh(a_1^2)^{-1}\mathrm{Ad}(a)H_1' + H_1 + ct(a_1^2)H_1', \\
\frac{1}{2}X_{(1,1)} &= e_{22}(a)X + e_{11}(a)\bar{X} - e_{21}(a)\mathrm{Ad}(a)X - e_{12}\mathrm{Ad}(a)\bar{X}, \\
X_{(0,2)} &= -sh(a_2^2)^{-1}\mathrm{Ad}(a)H_2' + H_2 + ct(a_2^2)H_2'.
\end{aligned}$$

Proof. The proof is similar to that of Lemma 1.1. □

1.3. Representations of the maximal compact subgroup

We recall some basic facts on the irreducible representations of the maximal compact subgroup K, because our explicit computations in the later sections frequently use the K-types of the representations of G. The group K is isomorphic to the compact unitary group $U(2)$ of degree 2. The complexification of the Lie algebra of $U(2)$ has four generators:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad H' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \bar{X} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The irreducible finite-dimensional representation τ_λ of K with highest weight $\lambda = (l_1, l_2) \in \mathbf{Z}^{\oplus 2}$ has the standard basis $\{v_i\}_{0 \le i \le d_\lambda}$ $(d_\lambda = l_1 - l_2)$

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in the representation space W_λ, so that the associated action of the Lie algebra is given by

$$Zv_i = (l_1 + l_2)v_i, \qquad H'v_i = (2i - d_\lambda)v_i,$$
$$Xv_i = (i+1)v_{i+1}, \qquad \bar{X}v_i = (d_\lambda + 1 - i)v_{i-1}.$$

Now let us consider the irreducible decomposition of the tensor product

$$W_\lambda \otimes W_{(k+1,k-1)} \cong W_{\mu_1} \oplus W_{\mu_2} \oplus W_{\mu_3},$$

with $\mu_1 = (l_1 + k + 1, l_2 + k - 1)$, $\mu_2 = (l_1 + k, l_2 + k)$, and $\mu_3 = (l_1 + k - 1, l_2 + k + 1)$.

We need explicit computation of Clebsch-Gordan coefficients, which is well known for the case $K_{\mathbf{C}} \cong \mathrm{GL}(2, \mathbf{C})$.

LEMMA 1.3. *Let $\{v'_i\}$ be the standard basis of each target space W_{μ_k} $(k = 1,\ 2,\ 3)$.*

(i) *Let P^{down} be the projector from $W_\lambda \otimes W_{k+1,k-1}$ to W_{μ_3}. Then in terms of standard basis, up to scalar multiple, it is given by*

$$P^{\mathrm{down}}(v_i \otimes w_2) = v'_i \quad (0 \le i \le d-2),$$
$$P^{\mathrm{down}}(v_i \otimes w_1) = -2v'_{i-1} \quad (1 \le i \le d-1),$$
$$P^{\mathrm{down}}(v_i \otimes w_0) = v'_{i-2} \quad (2 \le i \le d).$$

For other standard generators, we have $P^{\mathrm{down}}(v_i \otimes w_j) = 0$, if either $i + j > d$ or $i + j < 2$.

(ii) *Let P^{even} be the projector $W_\lambda \otimes W_{(k+1,k-1)} \to W_{\mu_2}$. Then up to scalar multiple, it is given by*

$$P^{\mathrm{even}}(v_i \otimes w_2) = (i+1)v'_{i+1} \quad (0 \le i \le d-1),$$
$$P^{\mathrm{even}}(v_i \otimes w_1) = (d - 2i)v'_i \quad (0 \le i \le d),$$
$$P^{\mathrm{even}}(v_i \otimes w_0) = -(d + 1 - i)v'_{i-1} \quad (1 \le i \le d),$$

and $P^{\mathrm{even}}(v_d \otimes w_2) = P^{\mathrm{even}}(v_0 \otimes w_0) = 0$.

(iii) *Let P^{up} be the projector from $W_\lambda \otimes W_{k+1,k-1}$ to W_{μ_1}. Then it is given by*

$$P^{\mathrm{up}}(v_i \otimes w_2) = \frac{(i+1)(i+2)}{2} v'_{i+2},$$

$$P^{\mathrm{up}}(v_i \otimes w_1) = (i+1)(d+1-i)v'_{i+1},$$
$$P^{\mathrm{up}}(v_i \otimes w_0) = \frac{(d+1-i)(d+2-i)}{2}v'_{i-1},$$

for $0 \le i \le d$.

Proof. This is an elementary fact. See Lemmas 2.1–2.3 of [11]. □

§2. The discrete series representations of $\mathrm{Sp}(2;\mathbf{R})$

Let $\hat{G}$ be the unitary dual of a real semisimple Lie group G with finite center, that is, the set of unitary equivalence classes $[\pi]$ of irreducible unitary representation π of G, equipped with the dual topology. Then π is called a *discrete series* representation if $[\pi]$ is isolated in $\hat{G}$.

The group G has discrete series if and only if $\operatorname{rank} G = \operatorname{rank} K$, that is, if and only if G has a compact Cartan subgroup T. Harish-Chandra gave a description of the subset $\hat{G}_{DS}$ of $\hat{G}$ consisting of the discrete series representations in terms of the unitary characters of T.

For $G = \mathrm{Sp}(2,\mathbf{R})$, we have the following description.

2.1. Parameterization of the discrete series

The root system of G with respect to a compact Cartan subgroup

$$T = \exp\bigl(\mathbf{R}(E_{1,3} - E_{3,1}) + \mathbf{R}(E_{2,4} - E_{4,2})\bigr)$$

is also given by a set of vectors in the Euclidean plane:

$$\{\pm 2\varepsilon_1, \pm 2\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}.$$

Here

$$\varepsilon_1\bigl(r_1(E_{1,3} - E_{3,1}) + r_2(E_{2,4} - E_{4,2})\bigr) = \sqrt{-1}r_1,$$
$$\varepsilon_2\bigl(r_1(E_{1,3} - E_{3,1}) + r_2(E_{2,4} - E_{4,2})\bigr) = \sqrt{-1}r_2.$$

We fix a subset of simple roots and the associated positive roots by

$$\{\varepsilon_2, \varepsilon_1 - \varepsilon_2\}, \qquad \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\varepsilon_2\},$$

respectively.

Let T be a compact Cartan subgroup; then the set of the unitary characters (or their derivations) is identified naturally with $\mathbf{Z} \oplus \mathbf{Z}$, and the subset consisting of dominant integral weights is

$$\Xi = \bigl\{(n_1, n_2) \bigm| n_i \in \mathbf{Z}, n_1 \ge n_2\bigr\}.$$

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Because an isomorphism class of finite-dimensional irreducible representations of K is determined by its highest weight, there is a bijection between $\hat{K}$ and Ξ.

Because the half-sum of the positive roots is integral, the discrete series representations of $\mathrm{Sp}(2;\mathbf{R})$ are parameterized by the subset of regular elements in Ξ:

$$\Xi' = \big\{\nu = (n_1, n_2) \,\big|\, n_i \in \mathbf{Z}, n_1 > n_2, n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0\big\}.$$

The subsets $\Xi_I = \{(n_1, n_2) \mid n_1 > n_2 > 0\}$ and $\Xi_{IV} = \{(n_1, n_2) \mid 0 > n_1 > n_2\}$ parameterize the holomorphic discrete series and the antiholomorphic discrete series, respectively. Set

$$\Xi_{II} = \big\{(n_1, n_2) \,\big|\, n_1 > 0 > n_2, n_1 + n_2 > 0\big\},$$

and set

$$\Xi_{III} = \big\{(n_1, n_2) \,\big|\, n_1 > 0 > n_2, 0 > n_1 + n_2\big\}.$$

Then the union $\Xi_{II} \cup \Xi_{III}$ parameterizes the large discrete series.

2.2. Formulation of the main problem

In general, given a discrete series representation (π, H) of a semisimple Lie group G, the matrix coefficients

$$c_{v,w}(x) := \big(\pi(x)v, w\big) \quad (v, w \in H)$$

are square-integrable, that is, they belong to $L^2(G)$. Because of the intertwining property

$$c_{v,w}(xg) = c_{\pi(g)v,w}(x) \quad (x, g \in G)$$

and the orthogonality relation of Godement for square-integrable representations of G, for a fixed nonzero vector $w \in H$, the map

$$v \in H \mapsto c_{v,w} \in L^2(G)$$

defines an injective intertwining isometry between unitary G-modules, if one regards $L^2(G)$ as a G-module via the right regular action. Similarly, for a fixed $v \in H$, the map

$$w \in H \mapsto c_{v,w} \in L^2(G)$$

defines an intertwining isometry from the contragredient representation π^* of π to the left regular G-module $L^2(G)$. Move both v and w in H; then the

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coefficients $c_{v,w}$ generate a $G\times G$ birepresentation $\pi^*\hat{\boxtimes}\pi$ which occurs with multiplicity 1 inside the $(G\times G)$-module $L^2(G)$ by the Plancherel theorem.

Let (τ,W) be the K-module, contragredient to the minimal K-type τ^* of π with representation space $W^*\hookrightarrow H$, and let us choose a basis $\{v_i\}_{1\le i\le m}$ in W and the dual basis $\{v_j^*\}_{1\le j\le m}$ in the dual space W^* of W. Then the contragredient representation τ on W is the minimal K-type of π^*. We choose another basis $\{w_j\}_{1\le j\le m}$, which is possibly different from $\{v_j^*\}_{1\le j\le m}$.

Put

$$\Phi(g)=\sum_i\sum_j c_{v_i^*,w_j^*}(g)v_i\otimes w_j,$$

which is a $(W\otimes_{\mathbf{C}}W^*)$-valued function on G. Clearly, this does not depend on the choice of basis.

Let us consider the discrete series representation π_Λ with Harish-Chandra parameter $\Lambda\in\Xi_{III}$. Let $\lambda=(l_1,l_2)$ be the highest weight of (τ,W); that is, its dual $(-l_2,-l_1)$ is the Blattner parameter, which is the parameter of the minimal K-type τ_λ^* of π_Λ.

Notation 2.1. Let us choose a standard basis $\{v_i\}_{0\le i\le d}$ in W, and let $\{v_i^*\}_{0\le i\le d}$ be its dual basis in W^*. Let us also choose a standard basis $\{w_j\}_{0\le j\le d}$ in W^*, and let $\{w_j^*\}_{0\le j\le d}$ be its dual in W.

Then we put

$$\Phi(g)=\sum_i\sum_j c_{v_i^*,w_j^*}(g)v_i\otimes w_j,$$

which is a $(W\otimes_{\mathbf{C}}W^*)$-valued function on G.

The main problem of this paper is to determine the A-radial part of this Φ.

2.3. Realization of the discrete series via Schmid equations

First, we recall the definition of the gradient operators.

Notation 2.2. For two continuous finite-dimensional representations (τ_i,W_i) $(i=1,2)$ of K, we set

$$C^\infty_{\tau_1,\tau_2}(K\backslash G/K):=\{f:G\to W_1\otimes W_2, C^\infty\text{-function} \mid f(k_1xk_2)=\tau_1(k_1)\tau_2(k_2)^{-1}f(x)\ \forall k_1,k_2\in K, x\in G\}.$$

Let a basis $\{X_i\}$ of $\mathfrak{p}$ and its dual basis $\{X_j^*\}$ in the dual $\mathfrak{p}^*$ be such that $\langle X_i,X_j^*\rangle=\delta_{ij}$. Then we can define the *right gradient operator* as

$$\nabla^R_{\tau_1,\tau_2}:C^\infty_{\tau_1,\tau_2}(K\backslash G/K)\to C^\infty_{\tau_1,\tau_2\otimes\mathrm{Ad}_{\mathfrak{p}^*}}(K\backslash G/K).$$

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Here for $f \in C^\infty_{\tau_1,\tau_2}(K\backslash G/K)$, we set

$$\nabla^R_{\tau_1,\tau_2}(f) := \sum_i R_{X_i}(f) \otimes X_i^*.$$

Here $R_X(f)$ is the right derivation:

$$R_X(f)(g) := \lim_{t\to 0} \frac{1}{t}\bigl\{f\bigl(g\cdot \exp(tX)\bigr) - f(g)\bigr\}.$$

Similarly, we can define the *left gradient operator*

$$\nabla^L_{\tau_1,\tau_2} : C^\infty_{\tau_1,\tau_2}(K\backslash G/K) \to C^\infty_{\tau_1\otimes \mathrm{Ad}_{\mathfrak{p}},\tau_2}(K\backslash G/K)$$

by

$$\nabla^L_{\tau_1,\tau_2}(f) := \sum_i L_{X_i}(f) \otimes X_i^*.$$

Here $L_X(f)$ is the left derivation:

$$L_X(f)(g) := \lim_{t\to 0} \frac{1}{t}\bigl\{f\bigl(\exp(-tX)\cdot g\bigr) - f(g)\bigr\}.$$

If we decompose the complexification $\mathfrak{p}_{\mathbf{C}}$ into a direct sum $\mathfrak{p}_+ \oplus \mathfrak{p}_-$ of holomorphic part and antiholomorphic part, then $\nabla^\heartsuit$ ($\heartsuit \in \{R, L\}$) is a sum $\nabla^\heartsuit_+ + \nabla^\heartsuit_-$. Let $\{Y_i\}$ be a basis of $\mathfrak{p}_-$, and let $\{Y_j^*\}$ be its dual basis in $\mathfrak{p}_-^*$ which is identified with $\mathfrak{p}_+$ via the Killing form. Then we put

$$\nabla^R_{\tau_1,\tau_2,+}\bigl(f(g)\bigr) := \sum_i R_{Y_i}(f) \otimes Y_i^*,$$

and we define the other three operators $\nabla^R_{\tau_1,\tau_2,-}, \nabla^L_{\tau_1,\tau_2,+}, \nabla^L_{\tau_1,\tau_2,-}$ similarly.

Now let τ_μ^* be the minimal K-type of the discrete series representation. Then by the minimality property of the K-type τ_μ^*, we have

$$P^{\mathrm{even}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,+}\Phi = 0,$$

$$P^{\mathrm{down}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,+}\Phi = 0,$$

$$P^{\mathrm{down}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,-}\Phi = 0.$$

Here $P^{\mathrm{even}}, P^{\mathrm{odd}}$ are the linear operators

$$C^\infty_{\tau_\lambda,\tau_\mu^*\otimes \mathrm{Ad}_{\mathfrak{p}_\pm^*}}(K\backslash G/K) \to C^\infty_{\tau_\lambda,\tau_{\mu_i}}(K\backslash G/K)$$

corresponding to the projectors to the simple K-modules τ_{μ_i} in the Clebsch-Gordan decomposition $\tau_\mu^* \otimes \mathfrak{p}_\pm^* \cong \bigoplus_{i=1}^3 \tau_{\mu_i}$, with P^{even} for $\mu_i = \mu^* \pm (1,1)$ and P^{down} for $\mu_i = \mu^* \pm (0,2)$ (see also [11, Lemmas 3.1–3.3]).

Schmid [16] proved the converse. Namely, we have the following.

PROPOSITION 2.1. *The discrete series representation π_Λ^*, contragredient to π_Λ, is realized in the closure of the solution of the above three equations in $C^\infty_{\tau_\mu}(G/K)$, if Λ is sufficiently regular.*

REMARK. The condition "sufficiently regular" was necessary to assure the ellipticity of certain operators, and so forth. This is not necessary at least for $\mathrm{Sp}(2,\mathbf{R})$. Anyway, we do not use this result on the sufficiency.

2.4. Radial part of the gradient operators

Our function ϕ belongs to the space

$$C^\infty_{\tau_L,\tau_R}(K\backslash G/K) := \{f : G \to W_{\tau_R} \otimes W_{\tau_L} \mid f(k_1 g k_2) = \tau_L(k_1)\tau_R(k_2)^{-1} f(g), k_1, k_2 \in K, g \in G\}$$

with some K-types (τ_L, τ_R).

NOTATION 2.3. In the following, we denote by $\rho_A(\nabla^*_*)$ the A-radial part of ∇^*_*s. We suppress the subscripts τ_L, τ_R in ∇ to simplify the notation. We also denote by ∂_1 and ∂_2 the Euler operators $a_i \frac{\partial}{\partial a_i}$ $(i = 1,2)$, respectively.

COMPUTATION. By the Cartan double K decomposition, we have

$$\rho_A(R_{X_{(-2,0)}})\phi = \{\partial_1 - sh^{-1}(a_1^2)\cdot\tau_L(H_1') - cth(a_1^2)\cdot\tau_R(H_1')\}\phi.$$

Then we have

$$\begin{aligned}
&\{\rho_A(R_{X_{(-2,0)}})\phi\} \otimes X_{(2,0)} \\
&\quad = \{\partial_1 - sh^{-1}(a_1^2)\cdot\tau_L(H_1') - cth(a_1^2)\cdot(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_1')\}(\phi \otimes X_{(2,0)}) \\
&\qquad + cth(a_1^2)\phi \otimes [H_1', X_{(2,0)}] \\
&\quad = \{\partial_1 - sh^{-1}(a_1^2)\cdot\tau_L(H_1') - cth(a_1^2)\cdot(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_1') + 2cth(a_1^2)\} \\
&\qquad \times (\phi \otimes X_{(2,0)}).
\end{aligned}$$

Here we note that $[H_1', X_{(2,0)}] = 2X_{(2,0)}$.

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The Cartan decomposition implies also that

$$\frac{1}{2}\rho_A(R_{X_{(-1,-1)}})\phi = \{-e_{12}(a)\tau_L(X) - e_{21}(a)\tau_L(\bar{X}) - e_{11}(a)\tau_R(X) - e_{22}(a)\tau_R(\bar{X})\}\phi.$$

Hence, we have

$$\begin{aligned}\left\{\frac{1}{2}\rho_A(R_{X_{(-1,-1)}})\phi\right\} \otimes X_{(1,1)} &= -\{e_{12}(a)\tau_L(X) + e_{21}(a)\tau_L(\bar{X})\}(\phi \otimes X_{(1,1)}) \\ &\quad - e_{11}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X)(\phi \otimes X_{(1,1)}) + e_{11}(a)\phi \otimes [X, X_{(1,1)}] \\ &\quad - e_{22}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(\bar{X})(\phi \otimes X_{(1,1)}) + e_{22}(a)\phi \otimes [\bar{X}, X_{(1,1)}] \\ &= \{-e_{12}(a)\tau_L(X) - e_{21}(a)\tau_L(\bar{X}) \\ &\quad - e_{11}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X) - e_{22}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(\bar{X})\}(\phi \otimes X_{(1,1)}) \\ &\quad + 2e_{11}(a)(\phi \otimes X_{(2,0)}) + 2e_{22}(a)(\phi \otimes X_{(0,2)}).\end{aligned}$$

Finally, noting that $[H_2', X_{(0,2)}] = 2X_{(0,2)}$, we have

$$\begin{aligned}\{\rho_A(R_{X_{(0,-2)}})\phi\} \otimes X_{(0,2)} &= \{\partial_2 - sh^{-1}(a_2^2) \cdot \tau_L(H_2') - cth(a_2^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2')\}(\phi \otimes X_{(0,2)}) \\ &\quad + cth(a_2^2)(\phi \otimes [H_2', X_{(0,2)}]) \\ &= \{\partial_2 - sh^{-1}(a_2^2) \cdot \tau_L(H_2') - cth(a_2^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2') + 2cth(a_2^2)\} \\ &\quad \times (\phi \otimes X_{(0,2)}).\end{aligned}$$

Summing up these computations, we have

PROPOSITION 2.2.

$$\begin{aligned}\rho_A(\nabla^R_{\tau_L,\tau_R,+})\phi &= \{\partial_1 - sh^{-1}(a_1^2)\tau_L(H_1') - cth(a_1^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_1') \\ &\quad + 2cth(a_1^2) + 2e_{11}(a)\}(\phi \otimes X_{(2,0)}) \\ &\quad + \{-e_{12}(a)\tau_L(X) - e_{21}(a)\tau_L(\bar{X}) - e_{11}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X) \\ &\quad - e_{22}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(\bar{X})\}(\phi \otimes X_{(1,1)})\end{aligned}$$

$$+\{\partial_2 - sh^{-1}(a_2^2)\tau_L(H_2') - cth(a_2^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2')$$
$$+ 2cth(a_2^2) + 2e_{22}(a)\}(\phi \otimes X_{(0,2)}).$$

Similarly we have

$$\rho_A(\nabla^R_{\tau_L,\tau_R,-})\phi$$
$$= \{\partial_1 + sh^{-1}(a_1^2)\tau_L(H_1') + cth(a_1^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_-})s(H_1')$$
$$+ 2cth(a_1^2) + 2e_{11}(a)\}(\phi \otimes X_{(2,0)})$$
$$+ \{e_{21}(a)\tau_L(X) + e_{12}(a)\tau_L(\bar{X})$$
$$+ e_{22}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_-}) + e_{11}(a)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_-})(\bar{X})\}(\phi \otimes X_{(1,1)})$$
$$+ \{\partial_2 + sh^{-1}(a_2^2)\tau_L(H_2') + cth(a_2^2)(\tau_R \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2')$$
$$+ 2cth(a_2^2) + 2e_{22}(a)\}(\phi \otimes X_{(0,2)}).$$

For the left gradient operators, we have

PROPOSITION 2.3.

$$\rho_A(\nabla^L_{\tau_L,\tau_R,-})\phi$$
$$= \{-\partial_1 - sh^{-1}(a_1^2)\tau_R(H_1') - cth(a_1^2)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_-})(H_1')$$
$$- 2cth(a_1^2) + 2e_{11}(a)\}(\phi \otimes X_{(-2,0)})$$
$$+ \{-e_{21}(a)\tau_R(X) - e_{12}(a)\tau_R(\bar{X}) + e_{11}(a)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_-})(X)$$
$$+ e_{22}(a)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_-})(\bar{X})\}(\phi \otimes X_{(-1,-1)})$$
$$+ \{-\partial_2 - sh^{-1}(a_2^2)\tau_R(H_2') - cth(a_2^2)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_-})(H_2')$$
$$- 2cth(a_2^2) + 2e_{22}(a)\}(\phi \otimes X_{(0,-2)}).$$

$$\rho_A(\nabla^L_{\tau_L,\tau_R,+})\phi$$
$$= \{-\partial_1 + sh^{-1}(a_1^2)\tau_R(H_1') + cth(a_1^2)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_+})s(H_1')$$
$$- 2cth(a_1^2) + 2_{11}(a)\}(\phi \otimes X_{(2,0)})$$
$$+ \{e_{12}(a)\tau_R(X) + e_{21}(a)\tau_L(\bar{X}) - e_{11}(a)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_+})(X)$$
$$- e_{22}(a)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_+})(\bar{X})\}(\phi \otimes X_{(1,1)})$$
$$+ \{-\partial_2 + sh^{-1}(a_2^2)\tau_R(H_2') + cth(a_2^2)(\tau_L \otimes \mathrm{Ad}_{\mathfrak{p}_+})(H_2')$$
$$- 2cth(a_2^2) + 2e_{22}(a)\}(\phi \otimes X_{(0,2)}).$$

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§3. Explicit formulas of Schmid operators

We write Schmid operators in terms of coefficients explicitly.

We write $\lambda=(l_1,l_2)$, $\mu=(m_1,m_2)$, and λ',μ' the target K-types

$$\Psi(a):=P^{\heartsuit}\nabla^{\spadesuit}_{\tau_\lambda,\tau_\mu,\varepsilon}(\Phi)_{|A}(a)=\sum_{i=1}^{d_{\mu'}}\sum_{j=1}^{d_{\mu'}}d_{i,j}(a)(w'_i\otimes v'_j),$$

where $\heartsuit\in\{\text{even},\text{down}\}$, $\spadesuit\in\{R,L\}$, and $\varepsilon\in\{+,-\}$.

3.1. Schmid operators

3.1.1. Right actions.

LEMMA 3.1 (Chirality operators).

(i) *Let* $\Psi=P^{\text{even}}\cdot\rho_A(\nabla^R_{\tau_\lambda,\tau_\mu,+})\Phi$. *Then* $\mu'=(m_1-1,m_2-1)$, *and*

$$\begin{aligned}
&d^{R;\text{even}}_{i,j;+}\\
&\quad=i\big\{\partial_1-(l_2+j)sh(a_1^2)^{-1}-(m_2+i-1)ct(a_1^2)-(d_\mu-2i)e_{11}(a)\big\}c_{i-1,j}\\
&\qquad+(d_\mu-2i)\big\{-je_{12}(a)c_{i,j-1}-(d_\lambda-j)e_{21}(a)c_{i,j+1}\big\}\\
&\qquad-(d_\mu-i)\big\{\partial_2-(l_1-j)sh(a_2^2)^{-1}-(m_1-i-1)ct(a_2^2)\\
&\qquad+(d_\mu-2i)e_{22}(a)\big\}c_{i+1,j}.
\end{aligned}$$

(ii) *Let* $\Psi=P^{\text{down}}\cdot\rho_A(\nabla^R_{\tau_\lambda,\tau_\mu,+})\Phi$. *Then* $\mu'=(m_1-2,m_2)$, *and*

$$\begin{aligned}
&d^{R;\text{down}}_{i,j;+}\\
&\quad=\big\{\partial_1-(l_2+j)sh(a_1^2)^{-1}-(m_2+i)ct(a_1^2)+2(i+1)e_{11}(a)\big\}c_{i,j}\\
&\qquad+2je_{12}(a)c_{i+1,j-1}(a)+2(d_\lambda-j)e_{21}(a)c_{i+1,j+1}\\
&\qquad+\big\{\partial_2-(l_1-j)sh(a_2^2)^{-1}-(m_1-i-2)ct(a_2^2)\\
&\qquad+2(d_\mu-i-1)e_{22}(a)\big\}c_{i+2,j}.
\end{aligned}$$

Here we assume that $0\le i\le d_{\mu'}=d_\mu-2$.

(iii) *(Adjoint operator). Let* $\Psi=P^{\text{down}}\cdot\rho_A(\nabla^R_{\tau_\lambda,\tau_\mu,-})\Phi$. *Then* $\mu'=(m_1,m_2+2)$, *and for* $0\le i\le d_\mu-2=d_{\mu'},0\le j\le d_\lambda$, *we have*

$$\begin{aligned}
&d^{R;\text{down}}_{i,j;-}\\
&\quad=\big\{\partial_1+(l_2+j)sh(a_1^2)^{-1}+(m_2+i+2)ct(a_1^2)
\end{aligned}$$

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$$+ 2(d_\mu - i - 1)e_{11}(a)\big\}c_{i+2,j}$$
$$+ 2je_{21}(a)c_{i+1,j-1} + 2(d_\lambda - j)e_{11}(a)c_{i+1,j+1}$$
$$+ \big\{\partial_2 + (l_1 - j)sh(a_1^2)^{-1} + (m_1 - i)ct(a_2^2) + 2(i+1)e_{22}(a)\big\}c_{i,j}.$$

Proof. This proposition and the next are shown by direct computation employing Propositions 2.2 and 2.3 and Lemma 1.3. We indicate here only the key points of the computation.

To prove (i), we start from the first formula in Proposition 2.2 with $\tau_R = \tau_\mu$ and $\tau_L = \tau_\lambda$. Note that

$$P^{\text{even}}\big\{(\tau_\mu \otimes \mathrm{Ad}_{\mathfrak{p}_+})(Y)(\Phi \otimes X_{(a,b)}) = \tau_{\mu'}(Y)P^{\text{even}}(\Phi \otimes X_{(a,b)})\big\}$$

for any $Y \in \mathfrak{k}_{\mathbf{C}}$ and $X_{(a,b)} \in \mathfrak{p}_+$. Next we identify $X_{(2,0)}, X_{(1,1)}, X_{(0,2)}$ with w_2, w_1, w_0 of Lemma 2.3. Now note that the operators before $P^{\text{even}}(\Phi \otimes X_{(2,0)})$ and $P^{\text{even}}(\Phi \otimes X_{(0,2)})$ are diagonal operators. Therefore, their contributions to the coefficients of $v_i' \otimes w_j$ come up from the coefficients $c_{i-1,j}$ and $c_{i+1,j}$, respectively, with corresponding operators. The factors before the brackets $\{*\}$ come up by Lemma 2.3.

The remaining are four terms involving the functions $e_{pq}(a)$. They come from the operator before $\Phi \otimes X_{(1,1)}$. Among them, two involving τ_λ have no relation with the gradient and the projector. The actions of $\tau_\lambda(X)$, $\tau_\lambda(\bar{X})$ appear as the coefficients $-j$ and $d_\lambda - j$. Finally, there remain two terms, which should be handled most carefully. □

3.1.2. Left actions.

LEMMA 3.2 (*Chirality operators*).

(i) *Let* $\Psi = P^{\text{even}} \cdot \rho_A(\nabla^L_{\tau_\lambda,\tau_\mu,-})\Phi$, *and let* $\lambda' = (l_1, l_2)$. *Then for* $0 \le j \le d_\lambda$,

$$\begin{aligned} d^{L;\text{even}}_{i,j;-} &= (d_\lambda - j)\big\{\partial_1 + (m_2 + i)sh(a_1^2)^{-1} + (l_2 + j + 1)ct(a_1^2) \\ &\quad + (d_\lambda - 2j)e_{11}(a)\big\}c_{i,j+1} \\ &\quad + (d_\lambda - 2j)\big\{+ie_{21}(a)c_{i-1,j} + (d_\mu - i)e_{12}(e)c_{i+1,j}\big\} \\ &\quad - j\big\{\partial_2 + (m_1 - i)sh(a_2^2)^{-1} + (l_1 - j + 1)ct(a_2^2) \\ &\quad - (d_\lambda - 2j)e_{22}(a)\big\}c_{i,j-1}. \end{aligned}$$

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(ii) *Let* $\Psi = P^{\mathrm{down}} \cdot \rho_A(\nabla^L_{\tau_\lambda,\tau_\mu,-})\Phi$, *and let* $\lambda' = (l_1, l_2+2)$. *Then*

$$\begin{aligned} d^{L;\mathrm{down}}_{i,j;-} & \\ &= \{-\partial_1 - (m_2+i)sh(a_1^2)^{-1} - (l_2+j+2))ct(a_1^2) \\ &\quad + 2(d_\lambda - j - 1)e_{11}(a)\}c_{i,j+2} \\ &\quad + \{-\partial_2 - (m_1 - i)sh(a_2^2)^{-1} - (l_1 - j)ct(a_2^2) + 2(j+1)e_{22}(a)\}c_{i,j} \\ &\quad - 2ie_{21}(a)c_{i-1,j+1} - 2(d_\mu - i)e_{12}(a)c_{i+1,j+1}. \end{aligned}$$

Here we assume that $0 \le j \le d_{\lambda'} = d_\lambda - 2$.

(iii) *(Adjoint operator). Assume that* $\lambda' = (l_1 - 2, l_2)$ *and that* $\Psi = P^{\mathrm{down}} \cdot \rho_A(\nabla^L_{\tau_\lambda,\tau\mu,+})\Phi$. *Then for* $0 \le i \le d_\mu, 0 \le j \le d_{\lambda'} = d_\lambda - 2$, *we have*

$$\begin{aligned} d^{L;\mathrm{down}}_{i,j;+} & \\ &= -\{\partial_1 - (m_2+i)sh(a_1^2)^{-1} - (l_2+j)ct(a_1^2) + 2(j+1)e_{11}(a)\}c_{i,j} \\ &\quad - 2ie_{12}(a)c_{i-1,j+1} - 2(d_\mu - i)e_{21}(a)c_{i+1,j+1} \\ &\quad - \{\partial_2 - (m_1 - i)sh(a_2^2)^{-1} - (l_1 - j - 2)ct(a_2^2) \\ &\quad + 2(d_\lambda - j - 1)e_{22}(a)\}c_{i,j+2}. \end{aligned}$$

3.2. The Schmid equations

The Schmid operators are obtained by setting $\Psi = 0$, that is, $d_{i,j} = 0$, in the formulas of the previous section. By adding some equations, we have simpler equations.

3.2.1. The right equations.

PROPOSITION 3.1. *The system of equations*

$$\begin{aligned} P^{\mathrm{even}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,+}\Phi &= 0, \\ P^{\mathrm{down}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,+}\Phi &= 0, \\ P^{\mathrm{down}} \cdot \nabla^R_{\tau_\lambda,\tau_\mu,-}\Phi &= 0 \end{aligned}$$

is represented by the following system in terms of the coefficients $\{c_{i,j}\}$ *of* Φ.

Right chirality equations:

$$(R_+.1)\quad \begin{aligned} &\{\partial_1-(l_2+j)sh(a_1^2)^{-1}-(m_2+i)ct(a_1^2)+(i+1)e_{11}(a)\}c_{i,j}\\ &\quad+(d_\mu-i-1)e_{22}(a)c_{i+2,j}+je_{12}(a)c_{i+1,j-1}\\ &\quad+(d_\lambda-j)e_{12}(a)c_{i+1,j+1}=0,\end{aligned}$$

$$(R_+.2)\quad \begin{aligned} &\{\partial_2-(l_1-j)-(m_1-i-2)ct(a_2^2)+(d_\mu-i-1)e_{22}(a)\}c_{i+2,j}\\ &\quad+(i+1)e_{11}(a)c_{i,j}+je_{12}(a)c_{i+1,j-1}\\ &\quad+(d_\lambda-j)e_{21}(a)c_{i+1,j+1}=0.\end{aligned}$$

Right adjoint equations:

$$(R_-.3)\quad \begin{aligned} &\{\partial_1+(l_2+j)sh(a_1^2)^{-1}+(m_2+i+2)ct(a_1^2)+2(d_\mu-i-1)e_{11}(a)\}c_{i+2,j}\\ &+2je_{21}(a)c_{i+1,j-1}+2(d_\lambda-j)e_{12}(a)c_{i+1,j+1}\\ &+\{\partial_2+(l_1-j)sh(a_2^2)^{-1}+(m_1-i)ct(a_2^2)\\ &+2(i+1)e_{22}(a)\}c_{i,j}=0.\end{aligned}$$

3.2.2. The left equations. Similarly to the right Schmid equations, we can consider the left system.

PROPOSITION 3.2. *The explicit formulas of the left Schmid equations*

$$P^{\text{even}}\cdot\nabla^L_{\tau_\lambda,\tau_\mu,-}\Phi=0,$$
$$P^{\text{down}}\cdot\nabla^L_{\tau_\lambda,\tau_\mu,-}\Phi=0,$$
$$P^{\text{down}}\cdot\nabla^L_{\tau_\lambda,\tau_\mu,+}\Phi=0$$

in terms of the coefficients of Φ *are given as follows.*

Left chirality equations:

$$(L_-.1)\quad \begin{aligned} &\{\partial_1+(m_2+i)sh(a_1^2)^{-1}+(l_2+j+2)ct(a_1^2)\\ &+(d_\lambda-j-1)e_{11}(a)\}c_{i,j+2}\\ &+(d_\mu-i)e_{12}(a)c_{i+1,j+1}+ie_{21}(a)c_{i-1,j+1}+(j+1)e_{22}(a)c_{i,j}=0,\end{aligned}$$

$$(L_-.2)\quad \begin{aligned} &\{\partial_2+(m_1-i)sh(a_2^2)^{-1}+(l_1-j)ct(a_2^2)+(j+1)e_{22}(a)\}c_{i,j}\\ &\quad+ie_{21}(a)c_{i-1,j+1}+(d_\mu-i)e_{12}c_{i+1,j+1}\\ &\quad+(d_\lambda-j-1)e_{11}(a)c_{i,j+2}=0.\end{aligned}$$

Left adjoint equations:

$$(L_+.3)\quad \begin{aligned}&\{\partial_1-(m_2+i)sh(a_1^2)^{-1}-(l_2+j)ct(a_1^2)+2(j+1)e_{11}(a)\}c_{i,j}\\&\quad+2ie_{12}(a)c_{i-1,j+1}+2(d_\mu-i)e_{21}(a)c_{i+1,j+1}\\&\quad+\{\partial_2-(m_1-i)sh(a_1^2)^{-1}-(l_1-j-2)ct(a_2^2)\\&\quad+2(d_\lambda-j-1)e_{22}(a)\}c_{i,j+2}=0.\end{aligned}$$

3.2.3. Derivation of a holonomic system from the Schmid operators. We prepare some macro symbols to denote our difference-differential equations. With the help of these symbols, some symmetries of the system become apparent. These are some symbols concerning the indices $I=(i,j)=(i_1,i_2)$ which belong to the product set $\{0,d\}^2$, $\{0,d\}$ being the set of numbers $\{0,1,\ldots,d-1,d\}$.

NOTATION 3.1. For $I=(i,j)$, we define integers

$$s(I)=\frac{1}{2}(d-i-j),\qquad w(I)=|s(I)|,$$
$$c_1(I)=\frac{1}{2}(i-j-L),\qquad c_2(I)=\frac{1}{2}(j-i-L).$$

Here we recall that we set $L=l_1+l_2$. The last number $w(I)$ is called the *weight* of the index I. Moreover, for $p\in\{1,2\}$, we set

$$\partial_p^{\pm}(I):=\partial_p\mp s(I)ct(a_p)-c_p(I)th(a_p).$$

Finally, for the indices $I=(i,j)$, we set $I_{(a,b)}=(i+a,j+b)$ for $a,b\in\mathbf{Z}$. When we use this notation in the subscript, say, $c_{I(+1,-1)}$, the use of double subscripts is avoided.

PROPOSITION 3.3. *Here are the differential-difference equations given by the Schmid operators.*

(a) *(Chirality equations):*

$$(C1^+)\quad \begin{aligned}&\{\partial_1^+(I_{(0,2)})+(d-j-1)e_{11}(a)\}c_{I_{(0,2)}}+(d-i)e_{12}(a)c_{I_{(1,1)}}\\&\quad+(j+1)e_{22}(a)c_I+ie_{21}(a)c_{I_{(-1,1)}}=0,\end{aligned}$$

$$(C1^-)\quad \begin{aligned}&\{\partial_1^-(I)+(i+1)e_{11}(a)\}c_I+je_{12}(a)c_{I_{(1,-1)}}\\&\quad+(d-j)e_{21}(a)c_{I_{(1,1)}}+(d-i-1)e_{22}(a)c_{I_{(2,0)}}=0,\end{aligned}$$

$$(C2^+)\quad \begin{aligned}&\{\partial_2^+(I_{(2,0)})+(d-i-1)e_{22}(a)\}c_{I_{(2,0)}}+(d-j)e_{21}(a)c_{I_{(1,1)}}\\&\quad+je_{12}(a)c_{I_{(1,-1)}}+(i+1)e_{11}(a)c_I=0,\end{aligned}$$

$$(C2^-)\quad \begin{aligned}&\{\partial_2^-(I)+(j+1)e_{22}(a)\}c_I+ie_{21}(a)c_{I_{(-1,1)}}\\&\quad+(d-i)e_{12}(a)c_{I_{(1,1)}}+(d-j-1)e_{11}(a)c_{I_{(0,2)}}=0.\end{aligned}$$

(b) *(Adjoint equations):*

$$(C3^r)\quad \begin{aligned}&\{\partial_2^-(I)+2c_2(I)th(a_2)+2(i+1)e_{22}(a)\}c_I+2je_{21}(a)c_{I_{(1,-1)}}\\&\quad+\{\partial_1^+(I_{(2,0)})+2c_1(I_{(2,0)})th(a_1)+2(d-i-1)e_{11}(a)\}c_{I_{(2,0)}}\\&\quad+2(d-j)e_{12}(a)c_{I_{(1,1)}}=0,\end{aligned}$$

$$(C3^l)\quad \begin{aligned}&\{\partial_1^-(I)+2c_1(I)th(a_1)+2(j+1)e_{11}(a)\}c_I+2ie_{12}(a)c_{I_{(-1,1)}}\\&\quad+\{\partial_2^+(I_{(0,2)})+2c_2(I_{(0,2)})th(a_2)+2(d-j-1)e_{22}(a)\}c_{I_{(0,2)}}\\&\quad+2(d-i)e_{21}(a)c_{I_{(1,1)}}=0.\end{aligned}$$

Proof. This is just a paraphrase of Propositions 3.1 and 3.2. □

3.2.4. Symmetry with respect to the indices I. We define an involutive automorphism on the set $\{0,d\}^2$ of indices by

$$I=(i,j)\to I'=(d-j,d-i).$$

Obviously, we have

$$s(I')=-s(I),\qquad c_k(I')=c_k(I)\quad (k=1,2).$$

Hence,

$$\partial_k^{\pm}(I')=\partial_k^{\mp}(I)\quad (k=1,2).$$

Also for $\varepsilon_1,\varepsilon_2\in\{0,\pm1,\pm2\}$,

$$\{I_{(\varepsilon_1,\varepsilon_2)}\}'=I'_{(-\varepsilon_2,-\varepsilon_1)}.$$

Apply the involution $'$ to the equations of our holonomic system, say, to $(1.\pm)$. Then it is transformed to similar equations $(1.\mp)$. Therefore, we have the following.

Corollary 3.1. *If we replace the functions $\{c_I\}$ by another system $\{\tilde c_I\}$ defined*

$$\tilde c_I=c_{I'}=c_{d-j,d-i}\quad \textit{for each } I=(i,j),$$

then they satisfy the same holonomic system as for $\{c_I\}$.

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3.3. Linear relations among coefficients c_I

To proceed further, we need some simple linear relations among the coefficients $\{c_I\}$.

LEMMA 3.3. *We have the six-term relations*

$$(F6A)_I \quad \begin{aligned} &(j-1)c_{I_{(0,-2)}} - (d-i-1)c_{I_{(+2,0)}} \\ &\quad + \frac{ch(a_1)}{ch(a_2)}\{ic_{I_{(-1,-1)}} - (d-j)c_{I_{(+1,+1)}}\} \\ &\quad - 2s(I)ct(a_1)th(a_2)c_I - 2s(I)\frac{sh(a_1)}{sh(a_2)}c_{I_{(+1,-1)}} = 0 \\ &(0 \le i \le d-1, 1 \le j \le d), \end{aligned}$$

$$(F6B)_I \quad \begin{aligned} &(i-1)c_{I(-2,0)} - (d-j-1)c_{I(0,+2)} \\ &\quad + \frac{ch(a_2)}{ch(a_1)}\{jc_{I(-1,-1)} - (d-i)c_{I(+1,+1)}\} \\ &\quad - 2s(I)ct(a_2)th(a_1)c_I - 2s(I)\frac{sh(a_2)}{sh(a_1)}c_{I(-1,+1)} = 0 \\ &(1 \le i \le d, 0 \le j \le d-1). \end{aligned}$$

Proof. To get (F6A), apply the shift $I \mapsto I_{(0,-2)}$ to $(C1^+)$. Then subtract $(C1^-)$ from that. Note that $d-i-j = 2s(I)$ and $\partial_1^+(I) - \partial_1^+(I) = -2s(I)ct(a_1)$. Then the result is

$$\begin{aligned} &-2s(I)\{ct(a_1) - e_{11}(a)\}c_I + 2s(I)e_{12}(a)c_{I(+1,-1)} \\ &\quad + ie_{21}(a)c_{I(-1,-1)} - (d-j)e_{21}(a)c_{I(+1,+1)} \\ &\quad + (j-1)e_{22}(a)c_{I(0,-2)} - (d-i-1)e_{22}(a)c_{I(2,0)} = 0. \end{aligned}$$

The coefficient of c_I is equal to

$$2s(I)ct(a_1)\{-1 + sh^2(a_1)/D\} = 2s(I)ct(a_1)sh^2(a_2)/D.$$

Therefore, dividing the whole formula by $e_{22}(a)$, we have (F6A).

To get (F6B), apply the shift $I \mapsto I_{(-2,0)}$ to $(C2^+)$ and subtract $(C2^-)$. After that, divide by $e_{11}(a)$. ∎

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LEMMA 3.4. *We have the following two kinds of nine-term relations:*

$$
(F9A)_I \qquad
\begin{aligned}
&2(i-1)c_{I(-2,0)} - (j-1)c_{I(0,-2)} \\
&\quad - 2(d-j-1)c_{I(0,+2)} + (d-i-1)c_{I(+2,0)} \\
&\quad + \Big\{-(l_1+l_2-i+j)\frac{D}{ch(a_1)ch(a_2)} + (2j-i)\frac{ch(a_1)}{ch(a_2)}\Big\}c_{I(-1,-1)} \\
&\quad - 2s(I)\Big\{\frac{D}{sh(a_1)sh(a_2)} + 2\frac{sh(a_2)}{sh(a_1)}\Big\}c_{I(-1,+1)} \\
&\quad - 2s(I)ct(a_1)th(a_2)c_I + 2s(I)\frac{sh(a_2)}{sh(a_1)}c_{I(+1,-1)} \\
&\quad + \Big\{(l_1+l_2-i+j)\frac{D}{ch(a_1)ch(a_2)} - (d+j-2i)\frac{sh(a_1)}{sh(a_2)}\Big\} \\
&\quad \times c_{I(+1,+1)} = 0,
\end{aligned}
$$

and (F9B), obtained similarly.

REMARK 3.1. We do not give the relation (F9B) explicitly, which is obtained by symmetry. It is deduced from (F9A) and the symmetrized nine-term relation (F6A) + (F6B), because the latter is equivalent to (F9A) + (F9B). So we omit it here; however, we have its equivalent form in Lemma 4.5(ii).

Proof of Lemma 3.4. Compute

$$(C1^+)_{I(+1,-1)} - (C3^r)_{I(-1,+1)} - (C1^-)_{I(-1,-1)} + (C3^l)_{I(-1,-1)}.$$

Here $(C1^+)_{I(+1,-1)}$ means the formula obtained from $(C1^+)_I$ after application of the shift $I \mapsto I_{(1,-1)}$ to the index I, and the other terms have similar meaning. Then we have

$$
\begin{aligned}
&2(i-1)e_{12}(a)c_{I(-2,0)} - (j-1)e_{12}(a)c_{I(0,-2)} \\
&\quad + (d-i-1)e_{12}(a)c_{I(+2,0)} - 2(d-j-1)e_{12}(a)c_{I(0,+2)} \\
&\quad + \big\{2c_1(I_{(-1,-1)})th(a_1) + (2j-i)e_{11}(a)\big\}c_{I(-1,-1)} \\
&\quad - \big\{2c_1(I_{(+1,+1)})th(a_1) + (d-2i+j)e_{11}(a)\big\}c_{(I(+1,+1)} \\
&\quad + 2s(I)e_{21}(a)c_I - 2s(I_{(+1,+1)}e_{22}(a)c_{I(+1,-1)} \\
&\quad - 2s(I_{(+1,-1)})\big\{ct(a_2) - 2e_{22}(a)\big\}c_{I(-1,+1)} = 0.
\end{aligned}
$$

Divide this by $e_{12}(a)$ to get the formula (F9A). ∎

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Lemmas 3.3 and 3.4 imply the following linear relations among five adjacent coefficients.

LEMMA 3.5. *We have*

$$(F5)_I \qquad \begin{aligned} &-(L-i-j)sh(a_1)sh(a_2)c_{I(-1,-1)} \\ &\qquad -2s(I)ch(a_1)ch(a_2)c_{I(-1,+1)}+4s(I)c_I \\ &\qquad -2s(I)ch(a_1)ch(a_2)c_{I(+1,-1)} \\ &\qquad +(L-2d+i+j)sh(a_1)sh(a_2)c_{I(+1,+1)}=0 \\ &(1\le i,j\le d-1). \end{aligned}$$

REMARK 3.2. The above five-term relations decay for I around the corner $(0,0)$ or $(1,0),(0,1)$. This means that the systems of transcendentals $\{c_I\}$ which are solutions of our holonomic system in Proposition 3.3 are essentially of dimension 4 over the rational function field $\mathbf{C}[th(a_1)^{\pm 1}, th(a_2)]$. The number 4 is found to be the rank of our holonomic system and also the degree of the associated variety of the discrete series π_Λ.

§4. Reduction of the holonomic system

4.1. Change of functions and variables

Up to the previous section, we had an explicit system of differential equations in terms of the coefficients $\{c_I\}$. In order to make this system simpler to handle, we introduce new functions $\{h_I\}$, which differ from $\{c_I\}$ by simple multipliers. Also, we change the variables a_i by $-sh^2(a_i)$.

DEFINITION 4.1 (Multipliers). We set

$$\begin{aligned} \mu_I^{\pm}(a) &= \{sh(a_1)sh(a_2)\}^{\pm s(I)}\prod_{i=1}^{2} ch(a_i)^{c_i(I)} \\ &= \{sh(a_1)sh(a_2)\}^{\pm(d-i-j)/2}\cdot \\ &\qquad \{ch(a_1)ch(a_2)\}^{-L/2}\{ch(a_1)/ch(a_2)\}^{(i-j)/2}. \end{aligned}$$

DEFINITION 4.2 (New functions). We set

$$c_I(a)=\mu_I^+(a)h_I^+(a)=\mu_I^-(a)h_I^-(a).$$

We consider mainly only h_I^+ in the later sections for the reason of symmetry between the systems $\{h_I^{\pm}\}$. Therefore, we drop the superscript $+$ in the symbol from now on.

Remark 4.1 (Symmetry for the multipliers). We have $\mu^-_{I'}(a) = \mu^+_I(a)$.

Definition 4.3 (Change of variables). We set

$$x_i = -sh^2(a_i) \quad (i = 1, 2).$$

Obviously, $ch^2(a_i) = 1 - x_i = -(x_i - 1)$. Note that

$$\frac{\partial}{\partial x_i} = -\frac{1}{sh(a_i^2)}\partial_i = -\frac{1}{2sh(a_i)ch(a_i)}\partial_i \quad (i = 1, 2).$$

Now we rewrite the equations for $\{c_I\}$ in Proposition 3.3 in terms of new functions $\{h_I\}$ and *new variables* x_i, which are found to be a holonomic system for $\{h_I\}$.

Proposition 4.1.

(a) *(Chirality equations) We have*

$$(H1^+) \quad \left\{\frac{\partial}{\partial x_1} + \frac{d-j-1}{2(x_1-x_2)}\right\}h_{I(0,2)} + \frac{d-i}{2(x_1-x_2)}h_{I(1,1)} + \frac{(j+1)x_2}{2(x_1-x_2)}h_I + \frac{ix_2}{2(x_1-x_2)}h_{I(-1,1)} = 0,$$

$$(H1^-) \quad \left\{x_1\frac{\partial}{\partial x_1} + s(I) + \frac{(i+1)x_1}{2(x_1-x_2)}\right\}h_I + \frac{jx_1}{2(x_1-x_2)}h_{I(1,-1)} + \frac{d-j}{2(x_1-x_2)}h_{I(1,1)} + \frac{d-i-1}{2(x_1-x_2)}h_{I(2,0)} = 0,$$

$$(H2^+) \quad \left\{\frac{\partial}{\partial x_2} - \frac{d-i-1}{2(x_1-x_2)}\right\}h_{I(2,0)} - \frac{d-j}{2(x_1-x_2)}h_{I(1,1)} - \frac{jx_1}{2(x_1-x_2)}h_{I(1,-1)} - \frac{(i+1)x_1}{2(x_1-x_2)}h_I = 0,$$

$$(H2^-) \quad \left\{x_2\frac{\partial}{\partial x_2} + s(I) - \frac{(j+1)x_2}{2(x_1-x_2)}\right\}h_I - \frac{ix_2}{2(x_1-x_2)}h_{I(-1,1)} - \frac{d-i}{2(x_1-x_2)}h_{I(1,1)} - \frac{d-j-1}{2(x_1-x_2)}h_{I(0,2)} = 0.$$

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(b) *(Adjoint equations) We have*

$$
(H3r)\qquad
\begin{aligned}
&(x_2-1)\Big\{x_2\frac{\partial}{\partial x_2}+s(I)+c_2(I)\frac{x_2}{x_2-1}-\frac{(i+1)x_2}{x_1-x_2}\Big\}h_I\\
&\quad -j\frac{(x_1-1)x_2}{x_1-x_2}h_{I(1,-1)}-(d-j)\frac{x_2-1}{x_1-x_2}h_{I(1,1)}\\
&\quad -(x_1-1)\Big\{\frac{\partial}{\partial x_1}+\frac{c_1(I(2,0))}{x_1-1}+\frac{d-i-1}{x_1-x_2}\Big\}h_{I(2,0)}=0,
\end{aligned}
$$

$$
(H3l)\qquad
\begin{aligned}
&(x_1-1)\Big\{x_1\frac{\partial}{\partial x_1}+s(I)+c_1(I)\frac{x_1}{x_1-1}+\frac{(j+1)x_1}{x_1-x_2}\Big\}h_I\\
&\quad +i\frac{x_1(x_2-1)}{x_1-x_2}h_{I(-1,1)}+(d-i)\frac{x_1-1}{x_1-x_2}h_{I(1,1)}\\
&\quad -(x_2-1)\Big\{\frac{\partial}{\partial x_2}+\frac{c_2(I(0,2))}{x_2-1}-\frac{d-j-1}{x_1-x_2}\Big\}h_{I(0,2)}=0.
\end{aligned}
$$

Proof. These are immediately derived from the system for $\{c_I\}$. We should note the identity

$$
\mu_I^+(a)^{-1}\cdot\partial_i\{\mu_I^+(a)f(a)\}=\{\partial_i+s(I)ct(a_i)+c_i(I)th(a_i)\}f. \qquad \square
$$

REMARK 4.2 (Symmetry). By the involution $I\to I'=(d-j,d-i)$, the system of equations for $\{h_I\}$ $(s(I)\geq 0)$ can be regarded as a system of equations for $\{h_I^-\}$ $(s(I')\leq 0)$.

4.2. Inductive equations

We derive from the basic system of the previous section for $\{h_I\}$, some simpler equations in this section. We call them *inductive equations* because they will be used to find solutions for general I from peripheral entries, such as $h_{0,0}$, $h_{0,1}$.

LEMMA 4.1. *We have*

$$
\text{(i)}\qquad
\begin{gathered}
\frac{\partial}{\partial x_1}h_{I(0,2)}+\frac{\partial}{\partial x_2}h_{I(1,1)}-\frac{i}{2}h_{I(-1,1)}-\frac{j+1}{2}h_I=0\\
(I=(i,\ j)\in\{0,d-1\}\times\{-1,d-2\}),
\end{gathered}
$$

$$
\text{(ii)}\qquad
\begin{gathered}
\frac{\partial}{\partial x_1}h_{I(1,1)}+\frac{\partial}{\partial x_2}h_{I(2,0)}-\frac{i+1}{2}h_I-\frac{j}{2}h_{I(1,-1)}=0\\
(I=(i,j)\in\{-1,d-2\}\times\{0,d-1\}).
\end{gathered}
$$

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Proof. Shift the index I in the formula $(H2^+)$ by $I \to I(-1,1)$, and add the obtained formula to $(H1^+)$ in Proposition 4.1. The equation (ii) is just the index shift $I \to I(1,-1)$ in (i). ∎

LEMMA 4.2 (Inductive equations of type I). *We have*

$$\text{(i)} \qquad \frac{\partial}{\partial x_1} h_{I(0,2)} = -\Big(x_2 \frac{\partial}{\partial x_2} + s(I)\Big) h_I \quad (j \in \{0, d-2\}),$$

$$\text{(ii)} \qquad \frac{\partial}{\partial x_2} h_{I(2,0)} = -\Big(x_1 \frac{\partial}{\partial x_1} + s(I)\Big) h_I \quad (i \in \{0, d-2\}),$$

$$\text{(iii)} \qquad \begin{aligned} &\frac{\partial}{\partial x_1} h_{I(1,1)} - \frac{j}{2} h_{I(1,-1)} - \Big(x_1 \frac{\partial}{\partial x_1} + s(I) + \frac{i+1}{2}\Big) h_I = 0 \\ &\qquad \big(I = (i,j) \in \{0, d-2\} \times \{0, d-1\}\big), \end{aligned}$$

$$\text{(iv)} \qquad \begin{aligned} &\frac{\partial}{\partial x_2} h_{I(1,1)} - \frac{i}{2} h_{I(-1,1)} - \Big(x_2 \frac{\partial}{\partial x_2} + s(I) + \frac{j+1}{2}\Big) h_I = 0 \\ &\qquad \big(I = (i,j) \in \{0, d-1\} \times \{0, d-2\}\big). \end{aligned}$$

Proof. Add $(H1^+)$ and $(H2^-)$ to get (i), and add $(H1^-)$ and $(H2^+)$ to get (ii). To show (iii) (resp., (iv)), subtract (ii) (resp., (i)) of the present lemma from (ii) (resp., (i)) of Lemma 4.1. ∎

REMARK 4.3. Note that up to this point, we use only the chirality equations.

We have more complicated inductive equations derived from the right and left adjoint equations.

LEMMA 4.3 (Inductive equations of type II). *We have*

$$\text{(v)} \qquad \begin{aligned} &(x_2 - 1)\Big\{\frac{\partial}{\partial x_2} + \frac{d-j-1+c_2(I_{(0,2)})}{x_2 - 1}\Big\} h_{I(0,2)} + i h_{I(-1,1)} \\ &\quad - (x_1 - 1)\Big\{x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2} + 3s(I) + j + 1 + \frac{c_1(I) x_1}{x_1 - 1}\Big\} h_I = 0, \end{aligned}$$

$$\text{(vi)} \qquad \begin{aligned} &(x_1 - 1)\Big\{\frac{\partial}{\partial x_1} + \frac{d-i-1+c_1(I_{(2,0)})}{x_1 - 1}\Big\} h_{I(2,0)} + j h_{I(1,-1)} \\ &\quad - (x_2 - 1)\Big\{2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + 3s(I) + i + 1 + \frac{c_2(I) x_2}{x_2 - 1}\Big\} h_I = 0. \end{aligned}$$

Proof. Subtract $(H3l)$ (resp., $(H3r)$) from $(H2^-) \times 2(x_1 - 1)$ (resp., $(1^-) \times 2(x_2 - 1)$) to get (v) (resp., (iv)). ∎

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4.3. Linear relations in terms of $\{h_I\}$

By eliminating the derivative terms in the equations by addition and subtraction among them, we have various linear relations of h_I.

LEMMA 4.4 (Six-term relations). *We have*

$$\begin{aligned}&s(I)x_2h_I+s(I)x_1h_{I(1,-1)}+\frac{j-1}{2}x_1x_2h_{I(0,-2)}\\ \text{(F6-i)}\qquad&\qquad+\frac{i}{2}x_1x_2h_{I(-1,-1)}-\frac{d-j}{2}h_{I(1,1)}-\frac{d-i-1}{2}h_{I(2,0)}=0\\ &(I=(i,j)\in\{0,d-1\}\times\{1,d\}).\end{aligned}$$

Proof. This is just a paraphrase of Lemma 3.3. The direct way to get this is to apply the shift $I\to I(0,-2)$ to $(H1^+)$; then we have

$$\begin{aligned}&\Big\{\frac{\partial}{\partial x_1}+\frac{d+1-j}{2(x_1-x_2)}\Big\}h_I+\frac{d-i}{2(x_1-x_2)}h_{I(1,-1)}+\frac{(j-1)x_2}{2(x_1-x_2)}h_{I(0,-2)}\\ &\quad+\frac{ix_2}{2(x_1-x_2)}h_{I(-1,-1)}=0\quad(j\in\{1,d\});\end{aligned}$$

multiply this by x_1, and subtract $(H1^-)$. Note that $d-i-j=2s(I)$. ∎

REMARK 4.4. By eliminating ∂_2 in place of ∂_1 of the proof of Lemma 4.4, we have a similar six-term linear relation. It is identical to Lemma 4.4 up to the shift. For the convenience of reference, we write it explicitly here:

$$\begin{aligned}&s(I)x_2h_{I(-1,1)}+s(I)x_1h_I+\frac{j}{2}x_1x_2h_{I(-1,-1)}+\frac{i-1}{2}x_1x_2h_{I(-2,0)}\\ \text{(F6-ii)}\qquad&-\frac{d-j-1}{2}h_{I(0,2)}-\frac{d-i}{2}h_{I(1,1)}=0\quad(i\in\{1,d\},j\in\{0,d-1\}).\end{aligned}$$

LEMMA 4.5 (Nine-term linear relations). *We have*

$$\begin{aligned}&x_1x_2\big\{2(i-1)(x_2-1)h_{I(-2,0)}-(j-1)(x_1-1)h_{I(0,-2)}\\ &\qquad+\big\{(2j-i)(x_1-1)+(i-j-L)(x_1-x_2)\big\}h_{I(-1,-1)}\\ &\qquad+2s(I)(x_1+x_2)(x_2-1)h_{I(-1,1)}+2s(I)(x_1-1)x_2h_I\\ \text{(i)}\qquad&\qquad-2s(I)(x_1-1)x_2h_{I(1,-1)}\\ &\qquad+(d-i-1)(x_1-1)h_{I(2,0)}-2(d-j-1)(x_2-1)h_{I(0,2)}\\ &\qquad-\big\{(d-2i+j)(x_1-1)+(i-j-L)(x_1-x_2)\big\}h_{I(1,1)}=0\\ &(i\in\{1,d-1\},j\in\{1,d-1\}),\end{aligned}$$

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$$
\text{(ii)}\quad
\begin{aligned}
&x_1x_2\{2(j-1)(x_1-1)h_{I(0,-2)}-(i-1)(x_2-1)h_{I(-2,0)}\\
&\qquad+\{(j-2i)(x_2-1)+(j-i-L)(x_1-x_2)\}h_{I(-1,-1)}\\
&\qquad+2s(I)x_1(x_2-1)h_{I(-1,1)}-2s(I)x_1(x_2-1)h_I\\
&\qquad-s(I)(x_1-1)(x_1+x_2)h_{I(1,-1)}\\
&\qquad+2(d-i-1)(x_1-1)h_{I(2,0)}-(d-j-1)(x_2-1)h_{I(0,2)}\\
&\qquad+\{(d-2j+i)(x_2-1)-(j-i-L)(x_1-x_2)\}h_{I(1,1)}=0\\
&\quad(i,j\in\{1,d-1\}).
\end{aligned}
$$

Proof. We have the following.

(i) Set

$$A=(H1^+)_{I(1,-1)}\cdot(x_1-1)+(H3^r)_{I(-1,1)}.$$

Then

$$
\begin{aligned}
e(x)^{-1}A=&\{-(d+j-2i)\cdot(x_1-1)-c_1(I_{(1,1)})\cdot 2(x_1-x_2)\}h_{I(1,1)}\\
&+(d-i-1)(x_1-1)h_{I(2,0)}-2(d-j-1)(x_2-1)h_{I(0,2)}\\
&+j(x_1-1)x_2h_{I(1,-1)}+(i-2j-1)(x_1-1)x_2h_I\\
&+(x_2-1)\cdot 2(x_1-x_2)\Big\{x_2\frac{\partial}{\partial x_2}+s(I_{(-1,1)})\\
&+c_2(I_{(-1,1)})\frac{x_2}{x_2-1}-2ix_2e(x)\Big\}h_{I(-1,1)}\\
&(i\in\{1,d-1\},j\in\{0,d-1\}).
\end{aligned}
$$

Set

$$B=\{(H1^-)_{I(-1,-1)}\cdot(x_1-1)+(H3^l)_{I(-1,-1)}\}x_2.$$

Then

$$
\begin{aligned}
e(x)^{-1}B=&(x_2-1)\cdot 2(x_1-x_2)\Big\{x_2\frac{\partial}{\partial x_2}+c_2\big(I(-1,1)\big)\frac{x_2}{x_2-1}\\
&-2(d-j)x_2e(x)\Big\}h_{I(-1,1)}\\
&+(-d+2i-j-1)(x_1-1)x_2h_I+(d-i)(x_1-1)x_2h_{I(1,-1)}\\
&+(j-1)(x_1-1)x_1x_2h_{I(0,-2)}-2(i-1)x_1x_2(x_2-1)h_{I(-2,0)}
\end{aligned}
$$

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$$+(x_1-1)\Big\{(i-2j)+c_1\big(I(-1,-1)\big)\frac{1}{x_1-1}\cdot 2(x_1-x_2)\Big\}$$
$$\times x_1x_2h_{I(-1,-1)}=0 \quad (i\in\{1,d\}, j\in\{1,d-1\}).$$

The subtraction $(A-B)e(x)^{-1}$ yields formula (i).

(ii) By symmetry, compute

$$(H2^+)_{I(-1,1)}\cdot(x_2-1)+(H3^l)_{I(1,-1)}$$
$$-x_2\big\{(x_2-1)\cdot(H2^-)_{I(-1,-1)}-(H3^r)_{I(-1,-1)}\big\}. \qquad \square$$

LEMMA 4.6 (Five-term relations). *For $i,j\in\{1,d-1\}$, we have*

$$(d-i-j)\big\{2h_I+(x_1-1)h_{I(1,-1)}+(x_2-1)h_{I(-1,1)}\big\}$$
$$+(i+j-L)x_1x_2h_{I(-1,-1)}+(i+j+L-2d)h_{I(1,1)}=0.$$

REMARK 4.5. We may rewrite the above relation as

$$s(I)\big\{2h_I+(x_1-1)h_{I(1,-1)}+(x_2-1)h_{I(-1,1)}\big\}$$
$$+\big\{(d-L)/2-s(I)\big\}x_1x_2h_{I(-1,-1)}-\big\{(d-L)/2+s(I)\big\}h_{I(1,1)}=0.$$

Proof. Compute first a combination of the six-term relations:

$$2(x_1-1)\times(\text{F6-i})+(-4)(x_2-1)\times(\text{F6-ii}),$$

which equals

$$2s(I)\big\{(x_1-1)x_2-2_1(x_2-1)\big\}h_I$$
$$2s(I)x_1(x_1-1)h_{I(1,-1)}-4s(I)x_2(x_2-1)h_{I(-1,1)}$$
$$\big\{ix_1x_2(x_1-1)-2jx_1x_2(x_2-1)\big\}h_{I(-1,-1)}$$
$$+(j-1)x_1x_2(x_1-1)h_{I(0,-2)}-2(i-1)x_1x_2(x_2-1)h_{I(-2,0)}$$
$$\big\{-(d-j)(x_1-1)+2(d-i)(x_2-1)\big\}h_{I(1,1)}$$
$$-(d-i-1)(x_1-1)h_{I(2,0)}+2(d-j-1)(x_2-1)h_{I(0,2)}=0$$
$$(i,j\in\{1,d\}).$$

After that, add the nine-term relation (i) of Lemma 4.5. Then we have

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$$\begin{aligned}
4s(I)(x_1-x_2)h_I + 2s(I)(x_1-x_2)(x_1-1)& \\
\times h_{I(1,-1)} + 2s(I)(x_1-x_2)(x_2-1)h_{I(-1,1)}& \\
x_1x_2\cdot(i+j-L)(x_1-x_2)h_{I(-1,-1)}& \\
+(2d-i-j-L)(x_1-x_2)h_{I(1,1)} = 0.&
\end{aligned}$$

Divide the last equality by (x_1-x_2) to get Lemma 4.6. ∎

LEMMA 4.7 (Initial values). *For $i,j\in\{1,d-1\}$, at $(x_1,x_2)=(0,0)$, we have*

$$s(I)\{2h_I(0)-h_{I(1,-1)}(0)-h_{I(-1,1)}(0)\} = \Big\{s(I)+\frac{d-L}{2}\Big\}h_{I(1,1)}(0).$$

Proof. Set $x_1=x_2=0$ in Lemma 4.6. ∎

4.4. The initial values $h_I(0,0)$

We determine the values of $h_I(x_1,x_2)$ at the origin $(0,0)$. We start with the case of diagonal entries h_I with $s(I)=0$.

LEMMA 4.8. *If $s(I)=0$ (i.e., $j=d-i$), then*

$$h_I(0,0)=h_{i,d-i}(0)=c_0(-1)^i\binom{d}{i}^{-1}=c_0(-1)^{(i-j+d)/2}\frac{i!j!}{d!},$$

where c_0 is a constant independent of i.

Proof. The normalization condition for the matrix coefficient $\Phi(a_1,a_2)$ should be $\Phi(1,1)=1_{d+1}$, with 1_{d+1} being the unit matrix of size $d+1$, if Φ is written in terms of some basis in the representation space of the minimal K-type of the left side and its dual basis in the minimal K-type of the right side. However, our formulation uses the standard basis for the K-types of both sides. In view of the relation between the standard basis and the dual basis for them, this condition for $\Phi(1,1)$ is equivalent to

$$c_{i,j}=\begin{cases} c_0(-1)^i\binom{d}{i}^{-1} & \text{if } i+j=d,\\ 0, & \text{if } i+j\neq d.\end{cases}$$

Note finally that $h_{i,d-i}(0)=c_{i,d-i}(0)$ if $s(I)=0$ by definition. ∎

REMARK 4.6. The constant c_0 depends on the choice of the identification of the standard basis in W and the dual basis in the same W of the standard

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basis in the contragredient space W^*. There is no canonical way to specify it completely. Even if one fixes the length of the standard basis and the dual standard basis, c_0 still has ambiguity up to a complex number with modulus 1.

Next we consider the relation between $h_I(0)$ with the same $s(I)$.

LEMMA 4.9.

$$(d-j)h_{i,j+1}(0)+(d-i)h_{i+1,j}(0)=0.$$

Proof. Input $x_1=0, x_2=0$ in Lemma 4.4 (the six-term relations). Then

$$(d-j)h_{i+1,j+1}(0)+(d-i-1)h_{i+2,j}(0)=0.$$

Replace i by $i-1$ to get our lemma. ∎

Now we determine the value $h_I(0)$ for general I with $s(I)\geq 0$. By Lemma 4.6 (the five-term relation) or Lemma 4.7, we have

$$s(I)\big\{2h_I(0)-h_{I(-1,1)}(0)-h_{I(1,-1)}(0)\big\}=\Big(\frac{1}{2}(d-L)+s(I)\Big)h_{I(1,1)}(0).$$

On the other hand, Lemma 4.8 shows that

$$\begin{aligned}
&2h_I(0)-h_{I(-1,1)}-h_{I(1,-1)}\\
&\quad=\frac{1}{(d-i)(d-j)}\big\{2(d-i)(d-j)+(d-i)(d-i+1)\\
&\qquad+(d-j+1)(d-j)\big\}h_I(0)\\
&\quad=\frac{(2d+1-i-j)(2d-i-j)}{(d-i)(d-j)}h_I(0).
\end{aligned}$$

Thus, we have the following.

LEMMA 4.10. *For $I=(i,j)$,*

$$h_I(0)=\frac{\big(s(I)+\frac{1}{2}(d-L)\big)(d-i)(d-j)}{s(I)(2d+1-i-j)(2d-i-j)}h_{I(1,1)}(0).$$

Summing up these equalities, we have the following.

PROPOSITION 4.2. *If $s(I)\geq 0$, then*

$$\frac{1}{d+1}h_I(0)=c_0(-1)^{(i-j+d)/2}\binom{\frac{1}{2}(d-L)+s(I)}{s(I)}\frac{(d-i)!(d-j)!}{(2d+1-i-j)!}.$$

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Proof. When $s(I)=0$, the right-hand side RHS(I) of our statement, which is a function in $I=(i,j)=(i,d-i)$, equals

$$c_0(-1)^i\frac{(d-i)!i!}{(d+1)!}=\frac{1}{d+1}h_{i,d-i}(0)$$

by Lemma 4.8. So this case is settled.

For the case $s(I)>0$, by definition we have the first equality for the ratio

$$\begin{aligned}&\mathrm{RHS}\big(I(-1,-1)\big)/\,\mathrm{RHS}(I)\\ &\quad=\frac{s(I)+\frac{1}{2}(d-L)+1}{s(I)+1}\frac{(d+1-i)(d+1-j)}{(2d+3-i-j)(2d+2-i-j)}\\ &\quad=h_{I(-1,-1)}(0)/h_I(0),\end{aligned}$$

and the second equality was shown in Lemma 4.10. Hence, our proposition follows immediately by induction with respect to $s(I)$. □

§5. Modified system of F_2 for extremal entries

In order to solve this involved system of difference-differential equations for h_I, we have to borrow an idea from chess: "checkmate by two knights," that is, force the opponent's king to a corner first, and checkmate!

We have to find the holonomic system for the corner entries of the functions h_I. We show that the extremal entries h_I with $s(I)=\pm[\frac{d}{2}]$ are solutions of certain holonomic systems of rank 4 with singularities along the divisor $x_1x_2(x_1-1)(x_2-1)(x_1-x_2)=0$ and at infinity, which is called the *modified system of (Appell's)* F_2 in Takayama [17, Section 2], that consists of an Euler-Darboux equation (Proposition 5.1) and a Poisson equation (Proposition 5.2) These two equations are deduced from part of the inductive equations in Section 4.2.

We treat the Euler-Darboux equation first.

5.1. The Euler-Darboux equation for extremal entries

We want to determine h_I when the index I attains the highest possible weight $w(I)=|s(I)|=[d/2]$, that is, when

$$\begin{cases}I=(0,0)\text{ or }I=(d,d) & \text{for } d \text{ even},\\ I\in\{(0,1),(1,0),(d,d-1),(d-1,d)\} & \text{for } d \text{ odd}.\end{cases}$$

By symmetry, it suffices to consider only positive $s(I)$, that is, $I=(0,1), I=(1,0)$ if d is odd. When d is even, it is convenient to discuss also $h_{1,1}$, not only

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$h_{0,0}$. To simplify the notation, we suppress the parentheses of the subscript in $h_{(i,j)}$, to write $h_{i,j}$.

PROPOSITION 5.1 (Euler-Darboux equations).

(a) *If d is even, both $h_{0,0}$ and $h_{1,1}$ satisfy the equation*

$$\left\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+1}{2}\frac{1}{x_1 - x_2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\right\} h_{i,i} = 0 \quad (i = 0, 1).$$

(b) *If d is odd, $h_{0,1}$ and $h_{1,0}$ satisfy the equations*

$$\left\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d+2}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{d}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_2}\right\} h_{0,1} = 0$$

and

$$\left\{\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{d}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{d+2}{2}\frac{1}{x_1 - x_2}\frac{\partial}{\partial x_2}\right\} h_{1,0} = 0,$$

respectively.

Proof. If d is even, we have

$$\frac{\partial}{\partial x_i} h_{1,1} = \left(x_i \frac{\partial}{\partial x_i} + \frac{d+1}{2}\right) h_{0,0} \quad (i = 1, 2),$$

by setting $I = (0, 0)$ in Lemma 4.2(iii),(iv) among the inductive equations of type I.

Now apply the operator $\frac{\partial}{\partial x_2}$ to one of the last formulas, and utilize the remaining one, to get

$$\begin{aligned}\frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} &= \left(x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2}\right)\frac{\partial}{\partial x_2} h_{0,0} \\ &= x_2^{-1}\left(x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2}\right)\left(\frac{\partial}{\partial x_2} h_{1,1} - \frac{d+1}{2} h_{0,0}\right).\end{aligned}$$

Multiply x_2 to the above formula to have

$$\begin{aligned}x_2 \frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} &= \left(x_1 \frac{\partial}{\partial x_1} + \frac{d+1}{2}\right)\frac{\partial}{\partial x_2} h_{1,1} - \frac{d+1}{2}\frac{\partial}{\partial x_1} h_{1,1} \\ &= x_1 \frac{\partial}{\partial x_1 \partial x_2} h_{1,1} - \frac{d+1}{2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) h_{1,1},\end{aligned}$$

which is the equation in question for $h_{1,1}$.

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For $h_{0,0}$, we have

$$\frac{\partial^2}{\partial x_1 \partial x_2} h_{1,1} = \frac{\partial}{\partial x_1}\Big(x_2 \frac{\partial}{\partial x_2} + \frac{d+1}{2}\Big) h_{0,0} = \Big(x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_1}\Big) h_{0,0}.$$

Change the role of the variables x_1, x_2 to get the other formula, and subtract it from the original one. Then we have the asymmetric equality

$$\Big\{\Big(x_1 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_2}\Big) - \Big(x_2 \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{d+1}{2}\frac{\partial}{\partial x_1}\Big)\Big\} h_{0,0} = 0,$$

which is the desired equation for $h_{0,0}$. Thus, (a) is proved.

Now we settle the case of d odd. Set $I = (0,-1)$ in $(H1^+)$, and set $I = (-1,0)$ in $(H2^+)$. Then

$$\Big(\frac{\partial}{\partial x_1} + \frac{d}{2}\frac{1}{x_1 - x_2}\Big) h_{0,1} + \frac{d}{2}\frac{1}{x_1 - x_2} h_{1,0} = 0$$

and

$$\Big(\frac{\partial}{\partial x_2} - \frac{d}{2}\frac{1}{x_1 - x_2}\Big) h_{1,0} - \frac{d}{2}\frac{1}{x_1 - x_2} h_{0,1} = 0.$$

Eliminate $h_{1,0}$ in the second formula by using the first one. Then we have the equation for $h_{0,1}$. Eliminate $h_{0,1}$ in the first formula by the second formula. Then we have the other equation for $h_{1,0}$. □

5.2. The Poisson equations for the peripheral and extremal entries

We deduce the other partial differential equations for the extremal h_I. We start with an equation valid for each of the peripheral entries, that is, for h_I with $i = 0$ (or $j = 0$ by symmetry).

Lemma 5.1. *If $I = (0,j)$, that is, $i = 0$, we have an equation*

$$(\#)_j \qquad \begin{aligned} &\Big\{\sum_{k=1}^{2} x_i(x_i - 1)\frac{\partial^2}{\partial x_i^2} + 2(x_1 - 1)x_2 \frac{\partial^2}{\partial x_1 \partial x_2}\Big\} h_I \\ &\quad + \sum_{k=1}^{2}\Big\{\Big(\frac{d-L}{2} + d - j + 3\Big)x_i - \big(s(I) + 1\big)\Big\}\frac{\partial}{\partial x_i} h_I \\ &\quad - (d+1)\frac{\partial}{\partial x_1} h_I + \Big(\frac{d-j}{2} + 1\Big)\Big(\frac{d-L}{2} + \frac{d-j}{2} + 1\Big) h_I = 0. \end{aligned}$$

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Proof. Set $i=0$ in Lemma 4.3(v) of the inductive equation. Then we have

$$(x_2-1)\Big(\frac{\partial}{\partial x_2}+\frac{d-j-1+c_2(I_{(0,2)})}{x_2-1}\Big)h_{I(0,2)}$$
$$-(x_1-1)\Big(x_1\frac{\partial}{\partial x_1}2x_2\frac{\partial}{\partial x_2}+3s(I)+j+1+\frac{c_1(I)x_1}{x_1-x_2}\Big)h_I=0.$$

Apply the operator $\frac{\partial}{\partial x_1}$ to the above formula. Then, since the operator on the function $h_{I(0,2)}$ depends only on the variable x_2, it commutes with $\frac{\partial}{\partial x_1}$. Recalling the inductive equation

$$\frac{\partial}{\partial x_1}h_{I(0,2)}=-\Big(x_2\frac{\partial}{\partial x_2}+s(I)\Big)h_I,$$

we have an equation

$$\frac{\partial}{\partial x_1}\Big\{(x_1-1)\Big(x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+3s(I)+j+1+\frac{c_1(I)x_1}{x_1-1}\Big)h_I\Big\}$$
$$+\Big\{(x_2-1)\frac{\partial}{\partial x_2}+d-j-1+c_2(I_{(0,2)})\Big\}\Big(x_2\frac{\partial}{\partial x_2}+s(I)\Big)h_I=0.$$

Note here that

$$\begin{aligned}3s(I)+j+1+c_1(I)&=2s(I)+j+1+\big\{s(I)+c_1(I)\big\}\\&=2s(I)+\frac{d-L}{2}+1=\frac{d-L}{2}+d-j+1.\end{aligned}$$

Then the last equation is found to be the desired equation by direct computation. ∎

The following Poisson equations are obtained from the above lemma.

Proposition 5.2.

(a) *If d is even,*

$$\Big\{\sum_{k=1}^{2}x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+(d+1)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$
$$+\sum_{k=1}^{2}\Big\{\Big(\frac{d-L}{2}+2\Big)x_k-\Big(\frac{d}{2}+1\Big)\Big\}\frac{\partial}{\partial x_k}$$
$$+\Big(\frac{d}{2}+1\Big)\Big(\frac{d-L}{2}+\frac{d}{2}+1\Big)\Big\}h_{0,0}=0.$$

(b) *If d is odd,*

$$\Big\{\sum_{k=1}^{2} x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+(d+2)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-d\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$
$$+\sum_{k=1}^{2}\Big\{\Big(\frac{d-L}{2}+2k-2\Big)x_k-\Big(\frac{d+2k-3}{2}\Big)\Big\}\frac{\partial}{\partial x_k}$$
$$+\Big(\frac{d-1}{2}+1\Big)\Big(\frac{d-L}{2}+\frac{d-1}{2}+1\Big)\Big\}h_{0,1}=0$$

and

$$\Big\{\sum_{k=1}^{2} x_k(x_k-1)\frac{\partial^2}{\partial x_i^2}+d\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+2)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_2}$$
$$+\sum_{k=1}^{2}\Big\{\Big(\frac{d-L}{2}+4-2k\Big)x_k-\Big(\frac{d+3-2k}{2}\Big)\Big\}\frac{\partial}{\partial x_k}$$
$$+\Big(\frac{d-1}{2}+1\Big)\Big(\frac{d-L}{2}+\frac{d-1}{2}+1\Big)\Big\}h_{1,0}=0.$$

Proof. When d is even (resp., odd), set $j=0$ (resp., $j=1$) in formula $(\#)_j$ of Lemma 5.1, and applying the Euler-Darboux equation, replace the term $2(x_1-1)x_2\frac{\partial^2}{\partial x_1\partial x_2}$ by

$$(d+1+j)\frac{(x_1-1)x_2}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{(x_1-1)x_2}{x_1-x_2}\frac{\partial}{\partial x_1}$$
$$=(d+1+j)\frac{x_1(x_1-1)}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{x_2(x_2-1)}{x_1-x_2}\frac{\partial}{\partial x_1}$$
$$-(d+1+j)\frac{x_1-1}{x_1-x_2}\frac{\partial}{\partial x_1}-(d+1-j)\frac{x_2}{x_1-x_2}\frac{\partial}{\partial x_2}$$

with $j=0$ (resp., $j=1$). □

5.3. The solutions for the modified system of F_2

We recall some basic facts on the regular solutions of the modified system of F_2 (see [1], [2], [9, Section 8.2]).

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NOTATION 5.1. We define an operator Q_{B_1,B_2}, called the *Euler-Darboux operator*, by

$$Q_{B_1,B_2} = \frac{\partial^2}{\partial x_1 \partial x_2} - \frac{B_2}{x_1 - x_2}\frac{\partial}{\partial x_1} + \frac{B_1}{x_1 - x_2}\frac{\partial}{\partial x_2}.$$

Here B_1, B_2 are constants in x_1 and x_2.

LEMMA 5.2. *If a function $h(x_1, x_2)$, analytic around the origin, satisfies the equation*

$$Q_{B_1,B_2} h = 0 \quad (B_1 > 0, B_2 > 0),$$

it has a power series expansion of the form

$$h(x_1, x_2) = \sum_{m_1, m_2 \geq 0} \frac{(B_1)_{m_1}(B_2)_{m_2}\xi(m_1 + m_2)}{m_1! m_2!} x_1^{m_1} x_2^{m_2},$$

with some series $\{\xi(k)\}_{k \in \mathbf{N}}$. Here we set

$$(B)_k = \frac{\Gamma(B + k)}{\Gamma(B)}.$$

Moreover, let $F_h(z)$ be a function in one variable z, regular around $z = 0$, defined by the restriction of h to the diagonal:

$$h(z, z) = \frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1 + B_2)} F_h(z) = \sum_{k \geq 0} \frac{(B_1 + B_2)_k}{k!} \xi(k) z^k.$$

Then the Eulerian integral formula for the β-function implies an integral expression

$$h(x_1, x_2) = \int_0^1 F_h\big(tx_1 + (1 - t)x_2\big) t^{B_1 - 1}(1 - t)^{B_2 - 1}\, dt,$$

if the integral of the right-hand side converges.

The modified F_2 system consists of the above Q_{B_1,B_2} and a Poisson operator $P = P_{A,B_1,B_1,C;\lambda}$. The key property of the Poisson operator is the following "intertwining property" for the series of the above type.

LEMMA 5.3. *Assume that h satisfies the condition in Lemma 5.2,*

$$Q_{B_1,B_2} h = 0,$$

and let $F_h(z)$ be the associated power series of one variable defined there. Let

$$P = \sum_{i=1}^{i} x_i(x_i - 1)\frac{\partial^2}{\partial x_i^2}$$
$$+ \Big\{(A + B_1 - B_2 + 1)x_1 + B_2 - C + 2B_2\frac{x_1(x_1-1)}{x_1 - x_2}\Big\}\frac{\partial}{\partial x_1}$$
$$+ \Big\{(A - B_1 + B_2 + 1)x_2 + B_1 - C - 2B_1\frac{x_2(x_2-1)}{x_1 - x_2}\Big\}\frac{\partial}{\partial x_2} - \lambda.$$

Then $h_1 = Ph$ also satisfies $Q_{B_1,B_2}h_1 = 0$, and for the operator

$$L = z(z-1)\frac{d^2}{dz^2} - \big\{C - (A + B_1 + B_2 + 1)z\big\}\frac{d}{dz} - \lambda,$$

we have an intertwining property:

$$F_{Ph}(z) = L \cdot F_h(z).$$

Remark 5.1. This is an analogy of [9, Lemma 8.6, p. 720]. Though Iida formulated this for the integral expression of h, strictly speaking, to justify it one needs extra work to verify its convergence in the definite interval $[0, 1]$. So we prefer to formulate in terms of (formal) power series.

Proof. Since $Q_{B_1,B_2}h = 0$, we have an equality

$$Ph = \big\{P + 2(x_1 - 1)x_2Q_{B_1,B_2} - (x_1 - x_2)Q_{B_1,B_2}\big\}h.$$

Set

$$P_0h = \sum_{i=1}^{2}\sum_{j=1}^{2} x_ix_j\frac{\partial^2}{\partial x_i\partial x_j} + (A + B_1 + B_2 + 1)\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i}\lambda,$$

and set

$$P_1 = -\sum_{i=1}^{2}\frac{\partial^2}{\partial x_1^2} - (x_1 + x_2)\frac{\partial^2}{\partial x_1\partial x_2} - C\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\Big)$$
$$= -\Big(\sum_{i=1}^{2} x_i\frac{\partial}{\partial x_i} + C\Big)\cdot\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\Big).$$

Then by direct computation, we have an equality between operators:

$$P + 2(x_1 - 1)x_2Q_{B_1,B_2} - (x_1 - x_2)Q_{B_1,B_2} = P_0 + P_1.$$

The power series P_0h is given by

$$\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\{(m_1+m_2)^2+(A+B_1+B_2)(m_1+m_2)-\lambda\} \\ \times \xi(m_1+m_2)x_1^{m_1}x_2^{m_2},$$

and the power series P_1h is given by

$$-\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}(m_1+m_2+B_1+B_2)(m_1+m_2+C) \\ \times \xi(m_1+m_2+1)x_1^{m_1}x_2^{m_2}.$$

Hence, $h_1 = Ph$ is equal to

$$\sum \frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\eta(m_1+m_2)x_1^{m_1}x_2^{m_2},$$

with

$$\eta(k) = \{k^2+(A+B_1+B_2)k-\lambda\}\xi(k)-(k+B_1+B_2)(k+C)\xi(k+1)$$

for each natural number k.

Now it is obvious that $Q_{B_1,B_2}h_1 = 0$ from the last power series expression of h_1, and for F_{h_1} we have

$$\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_{h_1}(z) = \sum_{k\geq 0}\frac{(B_1+B_2)_k}{k!}\eta(k)z^k$$

by definition. In turn, it is equal to the sum of

$$\sum \frac{(B_1+B_2)_k}{k!}\{k^2+(A+B_1+B_2)k-\lambda\}\xi(k)z^k \\ = \left\{\left(z\frac{d}{dz}\right)^2+(A+B_1+B_2)\left(z\frac{d}{dz}\right)-\lambda\right\}\left\{\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_h(z)\right\}$$

and

$$-\sum_{k\geq 0}\frac{(B_1+B_2)_k}{k!}(k+B_1+B_2)(k+C)\xi(k+1)z^k \\ = -\sum_{k=0}^{\infty}\frac{(B_1+B2)_{k+1}}{(k+1)!}(k+1+C-1)\xi(k+1)\frac{d}{dz}(z^{k+1})$$

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$$= -\frac{d}{dz}\Big\{\sum_{n=0}^{\infty}(n+C-1)\xi(n)z^n\Big\}$$

$$= -\frac{d}{dz}\Big(z\frac{d}{dz}+C-1\Big)\frac{\Gamma(B_1)\Gamma(B_2)}{\Gamma(B_1+B_2)}F_h(z).$$

Therefore, canceling the same factor $\Gamma(B_1)\Gamma(B_2)/\Gamma(B_1+B_2)$, we have

$$F_{h_1}(z) = \Big\{\Big(z\frac{d}{dz}\Big)^2 + (A+B_1+B_2)\Big(z\frac{d}{dz}\Big) - \lambda - \frac{d}{dz}\Big(z\frac{d}{dz}+C-1\Big)\Big\}F_h(z),$$

as desired. □

5.4. The solutions for the extremal entries

Now we can describe the solutions for the extremal entries $h_{0,0}$ (when d is even) and $h_{0,1}, h_{1,0}$ (when d is odd). We see in Section 5.2 that for even d,

$$Q_{(d+1)/2,(d+1)/2}h_{0,0} = 0 \qquad \text{and} \qquad Q_{(d+1)/2,(d+1)/2}h_{1,1} = 0,$$

and for odd d,

$$Q_{d/2,(d+2)/2}h_{0,1} = 0 \qquad \text{and} \qquad Q_{(d+2)/2,d/2}h_{1,0} = 0.$$

Thus, we have the following immediately by Lemma 5.2.

Lemma 5.4.

(i) *If d is even, $h_{0,0}$ and $h_{1,1}$ are constant multiples of the formal power series $h^P_{0,0}$ and $h^P_{1,1}$ of the form*

$$h^P_{i,i} = \sum_{m_1\geq 0, m_2\geq 0}\frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\xi_{i,i}(m_1+m_2)x_1^{m_1}x_2^{m_2} \quad (i=0,1),$$

with $B_1 = B_2 = (d+1)/2$, for some series $\{\xi_{i,i}(k) \mid k \in \mathbf{N}\}$ $(i=0,1)$.

(ii) *If d is odd, $h_{0,1}$ and $h_{1,0}$ have power series expressions:*

$$h^P_{i,1-i} = \sum_{m_1,m_2\geq 0}\frac{(B_1)_{m_1}(B_2)_{m_2}}{m_1!m_2!}\xi_{i,1-i}(m_1+m_2)x_1^{m_1}x_2^{m_2} \quad (i=0,1),$$

with $B_1 = (d/2)+i, B_2 = (d/2)+1-i$, for some series $\{\xi_{i,1-i}(k) \mid k \in \mathbf{N}\}$ $(i=0,1)$.

We can apply Lemma 5.3 here to get the following.

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PROPOSITION 5.3.

(i) *The series* $\{\xi_{0,0}(k)\}$ *is given by*

$$\xi_{0,0}(k)=\xi_{1,1}(k+1)=c_0\frac{\left(\frac{d+2}{2}\right)_k\left(\frac{d+2}{2}+\frac{d-L}{2}\right)_k}{\left(d+\frac{3}{2}\right)_k(d+1)_k} \quad \text{for } k\in\{0,\infty\}$$

with some constant c_0.

(ii) *For the series* $\{\xi_{0,1}(k)\}$ *we have*

$$\xi_{0,1}(k)=-\xi_{1,0}(k)=c_1\frac{\left(\frac{d+1}{2}\right)_k\left(\frac{d+1}{2}+\frac{d-L}{2}\right)_k}{\left(d+\frac{1}{2}\right)_k(d+1)_k} \quad \text{for } k\in\{0,\infty\}$$

with some constant c_1.

Proof. Since $h_{0,0}$ satisfies the Poisson equation $Ph_{0,0}=0$ with parameters

$$A=\frac{d-L}{2}+1, \qquad B_1=B_2=(d+1)/2,$$
$$C=d+\frac{3}{2}, \qquad \lambda=-\left(\frac{d}{2}+1\right)\left(\frac{d-L}{2}+\frac{d}{2}+1\right),$$

the associated function $F_{h_{0,0}}(z)$, which is a solution of the hypergeometric equation $LF=0$ regular at the origin, should be a constant multiple of the hypergeometric series

$${}_2F_1\left(\frac{d}{2}+1,\frac{d-L}{2}+\frac{d}{2}+1;d+\frac{3}{2};z\right).$$

Therefore, the series $\{\xi_{0,0}(k)\}$ is given as above.

Similarly for d odd, $h_{0,1}$ satisfies the Poisson equation $Ph_{0,1}=0$ with parameters

$$A=\frac{d-L}{2}, \qquad B_1=d/2, \qquad B_2=d/2+1,$$
$$C=d+1, \qquad \lambda=-\left(\frac{d-1}{2}+1\right)\left(\frac{d-L}{2}+\frac{d-1}{2}+1\right).$$

Hence, we have the series $\{\xi_{0,1}(k)\}$ above. The equality

$$\xi_{1,0}(k)=-\xi_{0,1}(k) \quad \text{for any } k$$

follows immediately from

$$\frac{\partial h_{0,1}}{\partial x_1}+\frac{\partial h_{1,0}}{\partial x_2}=0. \qquad \square$$

Now we have the following formulas for $h_{0,0}$ for even d and $h_{0,1},h_{1,0}$ for odd d.

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PROPOSITION 5.4.

(i) *If d is even,*

$$h_{0,0}(x_1,x_2) = c_0 \int_0^1 {}_2F_1\Big(\frac{d}{2}+1, \frac{d-L}{2}+\frac{d}{2}+1; d+\frac{3}{2}; x_1t+x_2(1-t)\Big) \times t^{(d+1)/2}(1-t)^{(d+1)/2}\,dt.$$

(ii) *If d is odd,*

$$h_{0,1}(x_1,x_2) = c_1 \int_0^1 {}_2F_1\Big(\frac{d-1}{2}+1, \frac{d-L}{2}+\frac{d-1}{2}+1; d+\frac{1}{2}; x_1t+x_2(1-t)\Big) \times t^{(d-2)/2}(1-t)^{d/2}\,dt.$$

REMARK 5.2. We do not know the value $\xi_{1,1}(0)$ for even d. But this will be known by Lemma 6.2 in the next section.

Now the chess game part is finished. To have solutions for all entries, next we face a kind of a jigsaw puzzle: to guess the next pieces, starting from the corners. We need some patience here.

§6. Power series solutions and integral solutions for general entries

6.1. The power series solutions for general h_I

In this section, we determine the (unique) power series expansion of the solutions of the holonomic system for $\{h_I \mid s(I) \geq 0\}$, which is regular at the origin.

THEOREM 6.1. *The holonomic system for $\{h_I \mid d \equiv i+j\,(mod\ 2)\}$ given by Proposition 4.1 has a unique system of solutions regular at the origin $(x_1,x_2)=(0,0)$, up to nonzero constant multiple. Moreover, these power series expansions are given as follows.*

(i) *If d is even and $s(L) \geq 0$ for $I=(i,j)$,*

$$h_I(x_1,x_2) = c_0'(-1)^{(i-j+d)/2} \cdot \sum_{m_1,m_2\geq 0} \frac{\left(\frac{d+1}{2}\right)_{m_1-[j/2]}\left(\frac{d+1}{2}\right)_{m_2-[i/2]}}{m_1!m_2!}$$

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$$\prod_{\ell=1}^{[i/2]}(m_1+s(I)+\ell)\cdot\prod_{\ell=1}^{[j/2]}(m_2+s(I)+\ell)$$
$$\times\,\xi_{00}\big(m_1+m_2-(i+j)/2\big)x_1^{m_1}x_2^{m_2}.$$

Here

$$\xi_{00}(k)=\frac{\left(\frac{d+2}{2}\right)_k\left(\frac{d+2}{2}+\frac{d-L}{2}\right)_k}{\left(d+\frac{3}{2}\right)_k(d+1)_k}\quad \textit{for } k\in\{-d/2,\infty\}.$$

(ii) *If d is odd and $s(I)\geq 0$,*

$$h_I(x_1,x_2)=c_1'(-1)^{(i-j+d)/2}\sum_{m_1,m_2\geq 0}\frac{\left(\frac{d}{2}\right)_{m_1+1-[(j+1)/2]}\left(\frac{d}{2}\right)_{m_2+1-[(i+1)/2]}}{m_1!m_2!}$$
$$\prod_{\ell=1}^{[i/2]}\big(m_1+s(I)+\ell\big)\prod_{\ell=1}^{[j/2]}\big(m_2+s(I)+\ell\big)$$
$$\times\,\xi_{01}\big(m_1+m_2-(i+j-1)/2\big)x_1^{m_1}x_2^{m_2}.$$

Here

$$\xi_{01}(k)=\frac{\left(\frac{d+1}{2}\right)_k\left(\frac{d+1}{2}+\frac{d-L}{2}\right)_k}{\left(d+\frac{1}{2}\right)_k(d+1)_k}\quad \textit{for } k\in\{-d/2,\infty\}.$$

Finally, c_0',c_1' are constants independent of i,j.

The proof is given in the following way. For the extremal entries $h_{(0,0)}$ or $h_{(1,0)},h_{(0,1)}$, we know the solution already, which is regular at the origin $(x_1,x_2)=(0,0)$ and unique up to nonzero constant multiple. The inductive equations (Lemmas 4.2 and 4.3) determine almost all the coefficients of the formal power series solutions for h_I. The problem is that we cannot get the information on the coefficients h_I with peripheral indices directly, that is, those $I=(i,j)$ with $ij=0$. On the other hand, the linear relations give the initial values $h_I(0,0)$. Thus, the problem was to know h_I when $ij=(0,0)$ but with $I\neq(0,0)$. Anyway, we have to guess at the correct system of power series solutions somehow. And once this is obtained, we can prove our theorem in the following three steps.

Step 1: Let $\{h_I^P\}$ denote the system of the power series defined by the right-hand sides of our theorem. Then, Lemma 6.2 shows that the initial values $h_I(0)$ coincide with the constant terms $h_I^P(0)$ of the power series solutions h_I^P.

Step 2: Proposition 6.1 claims that the system $\{h_I^P\}$ is a special solution of our holonomic system.

Step 3: We show that the difference $h_I^P(x_1,x_2) - h_I(x_1,x_2)$ is zero.

This is the outline of our proof.

6.2. Integral expression of the solutions

In this section, we give an integral expression of Euler type for the system of solutions for $\{h_I\}$, because it is deduced rather immediately from the power series expression given in the previous section. The first step in this procedure is to represent h_I by simpler power series.

LEMMA 6.1. *Assume that $s(I) \geq 0$. Let c_0' (resp., c_1') be the constant defined in Theorem 6.1 for even (resp., odd) d. We set for $I=(i_1,i_2)$*

$$H_I(x_1,x_2) = \sum_{m_1,m_2\geq 0} \frac{1}{m_1!m_2!} \begin{cases} \left(\frac{d+1}{2}\right)_{m_1-[i_2/2]}\left(\frac{d+1}{2}\right)_{m_2-[i_1/2]} \\ \quad \times \xi_{00}\Big(m_1+m_2-\frac{1}{2}(i+j)\Big)x_1^{m_1}x_2^{m_2} \quad (\textit{if } d \textit{ is even}), \\ \left(\frac{d}{2}\right)_{m_1+1-[(i_2+1)/2]}\left(\frac{d}{2}\right)_{m_2+1-[(i_1+1)/2]} \\ \quad \times \xi_{01}\Big(m_1+m_2-\frac{1}{2}(i+j-1)\Big)x_1^{m_1}x_2^{m_2} \quad (\textit{if } d \textit{ is odd }). \end{cases}$$

Then

$$h_I = c_*' \times (x_1x_2)^{-s(I)}\left(\frac{\partial}{\partial x_1}\right)^{[i_1/2]}\left(\frac{\partial}{\partial x_2}\right)^{[i_2/2]}\big\{(x_1x_2)^{s(I)}x_1^{[i_1/2]}x_2^{[i_2/2]}H_I\big\}.$$

Proof. It suffices to check the corresponding statement termwise:

$$\begin{aligned}(x_1x_2)^{-s(I)}&\left(\frac{\partial}{\partial x_1}\right)^{[i_1/2]}\left(\frac{\partial}{\partial x_2}\right)^{[i_2/2]}\big\{(x_1x_2)^{s(I)}x_1^{[i_1/2]}x_2^{[i_2/2]}x_1^{m_1}x_2^{m_2}\big\} \\ &= \prod_{\ell=1}^{[i_1/2]}(m_1+s(I)+\ell)\prod_{\ell=1}^{[i_2/2]}\big(m_2+s(I)+\ell\big)x_1^{m_1}x_2^{m_2}.\end{aligned}$$

□

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The next step is to find an integral expression for H_I. Note that now it satisfies the Euler-Darboux equations with

$$B_1 = (d+1)/2 - [i_2/2], \qquad B_2 = (d+1)/2 - [i_1/2] \quad \text{if } d \text{ is even,}$$
$$B_1 = (d+2)/2 + [-i_2/2], \qquad B_2 = (d+2)/2 + [-i_1/2] \quad \text{if } d \text{ is odd.}$$

THEOREM 6.2. *We have an integral expression for* $\{H_I\}$.

(a) *The case of even* d*:*

(i) *If both* i_1 *and* i_2 *are even,*

$$\begin{aligned} H_I(x_1,x_2) = H_I(0)&\frac{\Gamma\left(d+1-\frac{i_1+i_2}{2}\right)}{\Gamma\left(\frac{d+1-i_1}{2}\right)\Gamma\left(\frac{d+1-i_2}{2}\right)} \\ &\times \int_0^1 {}_2F_1\Big(s(I)+1, \frac{1}{2}(d-L)+s(I)+1; \\ &\frac{d+3}{2}+s(I); tx_1+(1-t)x_2\Big) \\ &\times t^{\frac{d+1-i_2}{2}-1}(1-t)^{\frac{d+1-i_1}{2}-1}\,dt. \end{aligned}$$

Here the constant $H_I(0)$ *is given by*

$$\begin{aligned} &H_I(0)\frac{\Gamma\left(d+1-\frac{i_1+i_2}{2}\right)}{\Gamma\left(\frac{d+1-i_1}{2}\right)\Gamma\left(\frac{d+1-i_2}{2}\right)} \\ &\quad = c_0'(-1)^{(i-j+d)/2}\frac{\Gamma\left(d+1-\frac{i_1+i_2}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}\xi_{00}\Big(-\frac{i_1+i_2}{2}\Big) \\ &\quad = c_0'(-1)^{(i-j+d)/2} \\ &\qquad \times \frac{\Gamma(d+1)}{\Gamma\left(\frac{d+1}{2}\right)^2}\frac{\left(\frac{d+3}{2}+s(I)\right)_{(i_1+i_2)/2}}{(s(I)+1)_{(i_1+i_2)/2}\left(\frac{1}{2}(d-L)+s(I)+1\right)_{(i_1+i_2)/2}}. \end{aligned}$$

(ii) *If both* i_1 *and* i_2 *are odd,*

$$\begin{aligned} H_I(x_1,x_2) = H_I(0)&\frac{\Gamma\left(d+1-\frac{i_1+i_2}{2}+1\right)}{\Gamma\left(\frac{d+2-i_1}{2}\right)\Gamma\left(\frac{d+2-i_2}{2}\right)G(0)} \\ &\times \int_0^1 G\big(tx_1+(1-t)x_2\big)t^{\frac{d+2-i_2}{2}-1}(1-t)^{\frac{d+2-i_1}{2}-1}\,dt, \end{aligned}$$

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where

$$G(z) = \Big(z\frac{d}{dz} + d + 1 - \frac{i_1+i_2}{2}\Big) \times {}_2F_1\Big(s(I)+1, \frac{1}{2}(d-L)+s(I)+1; \frac{d+3}{2}+s(I); z\Big).$$

Moreover, the constant $H_I(0)$ *is given by*

$$\begin{aligned} H_I(0)&\frac{\Gamma\big(d+2-\frac{i_1+i_2}{2}\big)}{\Gamma\big(\frac{d+2-i_1}{2}\big)\Gamma\big(\frac{d+2-i_2}{2}\big)G(0)} \\ &= c_0'(-1)^{(i-j+d)/2}\frac{\Gamma\big(d+1-\frac{i_1+i_2}{2}\big)}{\Gamma\big(\frac{d+1}{2}\big)\Gamma\big(\frac{d+1}{2}\big)}\xi_{00}\Big(-\frac{i_1+i_2}{2}\Big) \\ &= c_0'(-1)^{(i-j+d)/2} \\ &\quad\times \frac{\Gamma(d+1)}{\Gamma(\frac{d+1}{2})^2}\frac{\big(\frac{d+3}{2}+s(I)\big)_{(i_1+i_2)/2}}{(s(I)+1)_{(i_1+i_2)/2}\big(\frac{1}{2}(d-L)+s(I)+1\big)_{(i_1+i_2)/2}}. \end{aligned}$$

(b) *The case of odd* d*:*

$$\begin{aligned} H_I(x_1,x_2) = H_I(0)&\frac{\Gamma\big(d+1-\frac{i_1+i_2}{2}+\frac{1}{2}\big)}{\Gamma\big(\frac{d}{2}-\big[\frac{i_1}{2}\big]\big)\Gamma\big(\frac{d}{2}-\big[\frac{i_2}{2}\big]\big)} \\ &\times \int_0^1 {}_2F_1\Big(s(I)+1, \frac{1}{2}(d-L)+s(I)+1; \\ &\frac{d+2}{2}+s(I); tx_1+(1-t)x_2\Big) \\ &\times \begin{cases} t^{\frac{d+1-i_2}{2}-1}(1-t)^{\frac{d+2-i_1}{2}-1}\,dt & (\text{if } i_1 \text{ even, } i_2 \text{ odd}), \\ t^{\frac{d+2-i_2}{2}-1}(1-t)^{\frac{d+1-i_1}{2}-1}dt & (\text{if } i_1 \text{ odd, } i_2 \text{ even}). \end{cases} \end{aligned}$$

The constant $H_I(0)$ *is given by*

$$\begin{aligned} H_I(0)&\frac{\Gamma\big(d+\frac{3}{2}-\frac{i_1+i_2}{2}\big)}{\Gamma\big(\frac{d}{2}-\big[\frac{i_1}{2}\big]\big)\Gamma\big(\frac{d}{2}-\big[\frac{i_2}{2}\big]\big)} \\ &= c_1'(-1)^{(i-j+d)/2}\frac{\Gamma\big(d+\frac{3}{2}-\frac{i_1+i_2}{2}\big)}{\Gamma\big(\frac{d}{2}\big)\Gamma\big(\frac{d}{2}\big)}\xi_{01}\Big(-\frac{i_1+i_2}{2}+\frac{1}{2}\Big). \end{aligned}$$

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6.3. Preparation for the proof of Theorem 6.1

We denote by $h^P_{i,j}$ the power series of the right-hand side of the equalities in Theorem 6.1.

We first confirm that the constant terms $h^P_{i,j}(0)$ of $h^P_{i,j}$ are consistent with the known values of the constants $h_{i,j}(0)$.

LEMMA 6.2. *We have $h_I(0) = h^P_I(0)$ for all I.*

Proof. It suffices to show the following two statements.

(i) If $h_{i,j}(0) = h^P_{i,j}(0)$, then $h_{i\mp 1,j\pm 1}(0) = h^P_{i\mp 1,j\pm 1}(0)$ for $i+j \le d$.

(ii) If $h_{i,j}(0) = h^P_{i,j}(0)$, then

$$h_{i+1,j+1}(0) = h^P_{i+1,j+1}(0) \quad \text{for } i+j \le d-2.$$

To show (i), it suffices to show an equivalent (single) statement:

$$h_{i-1,j+1}(0)/h_{i,j}(0) = -(d+1-i)/(d-j) = h^P_{i-1,j+1}(0)/h^P_{i,j}(0).$$

The first equality is immediate from Proposition 4.2. We want to show the second one. Set

$$f_1(d;i,j) = \begin{cases} \left(\frac{d+1}{2}\right)_{-[j/2]}\left(\frac{d+1}{2}\right)_{-[i/2]} & \text{if } d \text{ is even,}\\ \left(\frac{d}{2}\right)_{-[(j-1)/2]}\left(\frac{d}{2}\right)_{-[(i-1)/2]} & \text{if } d \text{ is odd.}\end{cases}$$

Moreover, set

$$f_2(d;i,j) = \prod_{\ell=1}^{[i/2]}\left(\frac{1}{2}(d-i-j)+\ell\right)\prod_{\ell=1}^{[j/2]}\left(\frac{1}{2}(d-i-j)+\ell\right).$$

Then by definition,

$$h^P_{i,j}(0) = \begin{cases} c'_0(-1)^{(i-j+d)/2}\xi_{00}\left(-\frac{i+j}{2}\right)f_1(d;i,j)f_2(d;i,j) & \text{if } d \text{ is even,}\\ c'_1(-1)^{(i-j+d)/2}\xi_{01}\left(-\frac{i+j}{2}+\frac{1}{2}\right)f_1(d;i,j)f_2(d;i,j) & \text{if } d \text{ is odd.}\end{cases}$$

Hence,

$$h^P_{i-1,j+1}(0)/h^P_{i,j}(0) = (-1)\frac{f_1(d;i-1,j+1)}{f_1(d;i,j)}\frac{f_2(d;i-1,j+1)}{f_2(d;i,j)}.$$

Since

$$f_1(d;i-1,j+1)/f_1(d;i,j) = \begin{cases} \frac{d-i}{2} & (d,i,j \text{ all even}),\\ \left(\frac{d-j}{2}\right)^{-1} & (d \text{ even}, i,j \text{ odd}),\\ \left(\frac{d-j}{2}\right)^{-1}\left(\frac{d-i+1}{2}\right) & (d,i \text{ odd}, j \text{ even}),\\ 1 & (d,j \text{ odd}, i \text{ even})\end{cases}$$

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and

$$f_2(d;i-1,j+1)/f_2(d;i,j) = \begin{cases} \left(\frac{d-j}{2}\right)^{-1} & (d,i,j \text{ all even}), \\ \left(\frac{d+1-i}{2}\right) & (d \text{ even}, i,j \text{ odd}), \\ 1 & (d,i \text{ odd}, j \text{ even}), \\ \left(\frac{d-j}{2}\right)^{-1}\left(\frac{d-i+1}{2}\right) & (d,j \text{ odd}, i \text{ even}), \end{cases}$$

we have the desired equality:

$$h^P_{i-1,j+1}(0)/h^P_{i,j}(0) = -(d+1-i)/(d-j).$$

(ii) We know already that

$$\frac{h_{i+1,j+1}(0)}{h_{i,j}(0)} = \frac{\left(d+\frac{1}{2}-\frac{i+j}{2}\right)\left(d-\frac{i+j}{2}\right)\left(\frac{d}{2}-\frac{i+j}{2}\right)}{\left(d-\frac{1}{2}-\frac{i+j}{2}\right)\left(\frac{d}{2}-\frac{i}{2}\right)\left(\frac{d}{2}-\frac{j}{2}\right)}.$$

It suffices to compute $h^P_{i+1,j+1}(0)/h^P_{i,j}(0)$. When d is even, it is equal to

$$\xi_{0,0}\Big(-\frac{i+j}{2}-1\Big)\Big/\xi_{0,0}\Big(-\frac{i+j}{2}\Big)$$
$$\times \begin{cases} 1\times 1\times \frac{\frac{1}{2}(d-i-j)}{\{\frac{1}{2}(d-i-j)+\frac{i}{2}\}}\frac{\frac{1}{2}(d-i-j)}{\left\{\frac{1}{2}(d-i-j)+\frac{j}{2}\right\}} \\ \quad (\text{if } i,j \text{ even}), \\ \left(\frac{d+1}{2}-\frac{j+1}{2}\right)^{-1}\times\left(\frac{d+1}{2}-\frac{i+1}{2}\right)^{-1}\times\frac{1}{2}(d-i-j)\times\frac{1}{2}(d-i-j) \\ \quad (\text{if } i,j \text{ odd}), \end{cases}$$

which is in turn equal to

$$\frac{\left(c_0-\frac{i+j}{2}-1\right)\left(d+1-\frac{i+j}{2}-1\right)}{\left(a_0-\frac{i+j}{2}-1\right)\left(b_0-\frac{i+j}{2}-1\right)}\times\frac{(d-i-j)^2}{(d-i)(d-j)} = \frac{h_{i+1,j+1}(0)}{h_{i,j}(0)},$$

as desired.

When d is odd,

$$h^P_{i+1,j+1}(0)/h^P_{i,j}(0) = \xi_{01}\Big(-\frac{i+j}{2}-\frac{1}{2}\Big)\Big/\xi_{01}\Big(-\frac{i+j}{2}+\frac{1}{2}\Big)$$
$$\times \begin{cases} \left(\frac{d}{2}-\frac{j}{2}\right)^{-1}\times 1\times\frac{1}{2}(d-i-j)\times\frac{\frac{1}{2}(d-i-j)}{\{\frac{1}{2}(d-i-j)+\frac{j}{2}\}} \\ \quad (\text{if } i \text{ odd, } j \text{ even}), \\ 1\times\left(\frac{d}{2}-\frac{i}{2}\right)\times\frac{\frac{1}{2}(d-i-j)}{\{\frac{1}{2}(d-i-j)+\frac{i}{2}\}}\times\frac{1}{2}(d-i-j) \\ \quad (\text{if } i \text{ even, } j \text{ odd}). \end{cases}$$

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It in turn is equal to

$$\frac{\left(d-\frac{i+j}{2}\right)\left(d+\frac{1}{2}-\frac{i+j}{2}\right)}{\left(\tilde{a}_0-\frac{i+j}{2}\right)\left(\tilde{b}_0-\frac{i+j}{2}\right)}\times\frac{(d-i-j)^2}{(d-i)(d-j)}=h_{i+1,j+1}(0)/h_{i,j}(0).$$

Thus, the proof of the lemma is completed. $\square$

Here is another ingredient for the proof of Theorem 6.1.

PROPOSITION 6.1. *The system of power series $\{h^P_{i,j}\mid i+j\le d\}$ satisfies the following equations:*

$$\text{(i)}\qquad \frac{\partial}{\partial x_1}h^P_{i,j+2}=-\Big(x_2\frac{\partial}{\partial x_2}+\frac{d-i-j}{2}\Big)h^P_{i,j},$$

$$\text{(ii)}\qquad \frac{\partial}{\partial x_2}h^P_{i;2,j}=-\Big(x_1\frac{\partial}{\partial x_1}+\frac{d-i-j}{2}\Big)h^P_{i,j},$$

$$\text{(iii)}\qquad \begin{aligned}&(x_2-1)\Big\{\frac{\partial}{\partial x_2}+(d-j-1)-\frac{L+i-j-2}{2}\frac{1}{x_2-1}\Big\}h^P_{i,j+2}\\&=-i\cdot h^P_{i-1,j+1}+(x_1-1)\\&\Big\{x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+\frac{3}{2}(d-i-j)+(j+1)\\&\quad-\frac{L-i+j}{2}\frac{x_1}{x_1-1}\Big\}h^P_{i,j},\end{aligned}$$

$$\text{(iv)}\qquad \begin{aligned}&(x_1-1)\Big\{\frac{\partial}{\partial x_1}+(d-i-1)-\frac{L-i+j-2}{2}\frac{1}{x_1-1}\Big\}h^P_{i+2,j}\\&=-j\cdot h^P_{i+1,j-1}+(x_2-1)\\&\Big\{2x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+\frac{3}{2}(d-i-j)+(i+1)\\&\quad-\frac{L+i-j}{2}\frac{x_2}{x_2-1}\Big\}h^P_{i,j}.\end{aligned}$$

Proof. We have the following.

(i) The coefficient of $x_1^{m_1}x_2^{m_2}$ in the right-hand side is given as

$$\begin{aligned}&c_0'(-1)^{(i-j+d)/2}\cdot(-1)\cdot\frac{(B_1)_{m_1-[j/2]}(B_2)_{m_2-[i/2]}}{m_1!m_2!}\\&\times\prod_{\ell=1}^{[i/2]}\Big(m_1+\frac{1}{2}(d-i-j)+\ell\Big)\end{aligned}$$

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$$\times \prod_{\ell=1}^{[j/2]} \Big\{ \Big(m_2 + \frac{1}{2}(d-i-j) + \ell \Big) \cdot \Big(m_2 + \frac{1}{2}(d-i-j) \Big) \Big\}.$$

On the other hand, the coefficients at $x_1^{m_1+1} x_2^{m_2}$ of $h^P_{i,j+2}$ times (m_1+1) equal

$$\begin{aligned} &c_0'(-1)^{(i-j+d-2)/2} \cdot \frac{(B_1)_{m_1+1-[(j+2)/2]}(B_2)_{m_2-[i/2]}}{m_1! m_2!} \\ &\times \prod_{\ell=1}^{[i/2]} \Big((m_1+1) + \frac{1}{2}(d-i-j-2) + \ell \Big) \\ &\times \prod_{\ell=1}^{[(j+2)/2]} \Big\{ \Big(m_2 + \frac{1}{2}(d-i-j-2) + \ell \Big) \cdot \Big(m_2 + \frac{1}{2}(d-i-j) \Big) \Big\}. \end{aligned}$$

Since these two numbers are equal, (i) is true. The proof of (ii) is the same, changing the roles of x_1 and x_2. □

(iii) We first rewrite the equation: set

$$f_1 = \Big\{ x_2 \frac{\partial}{\partial x_2} + \Big(d - \frac{L}{2} + \frac{i+j}{2} \Big) \Big\} h^P_{i,j+2}$$

$$- x_1 \Big\{ x_1 \frac{\partial}{\partial x_1} 2 x_2 \frac{\partial}{\partial x_2} + \frac{3}{2} d - \frac{L}{2} - i - j + 1 \Big\} h^P_{i,j}$$

and

$$\begin{aligned} f_2 = {} & \frac{\partial}{\partial x_2} h^P_{i,j+2} - h^P_{i-1,j+1} \\ & - \Big\{ x_1 \frac{\partial}{\partial x_1} 2 x_2 \frac{\partial}{\partial x_2} + \frac{3}{2} d - \frac{3}{2} i - \frac{1}{2} j + 1 \Big\} h^P_{i,j}. \end{aligned}$$

Then the equality (iii) is equivalent to $f_1 = f_2$. First, we compute the coefficients of the power series f_1.

Lemma 6.3. *The coefficient of $x_1^{m_1} x_2^{m_2}$ in the power series f_1 is given by the following.*

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(i) *If d is even*

$$f^\lambda_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}-1\Big)(m_1+m_2+a_0-1)$$
$$\cdot\Big(m_1+m_2+a_0-1-\frac{i+j}{2}\Big)\Big(m_1+m_2+b_0-1-\frac{i+j}{2}\Big),$$

with

$$f^\lambda_{m_1,m_2}=c_0'(-1)^{(i-j+d)/2-1}\frac{\left(\frac{d+1}{2}\right)_{m_1-[j/2]-1}\left(\frac{d+1}{2}\right)_{m_2-[i/2]}}{m_1!m_2!}$$
$$\times\prod_{\ell=1}^{[i/2]}\Big(m_1+\frac{1}{2}(d-i-j)+\ell-1\Big)\prod_{\ell=1}^{[j/2]}\Big(m_2+\frac{1}{2}(d-i-j)+\ell\Big)$$

and

$$a_0=\frac{d+2}{2},\qquad b_0=\frac{d+2}{2}+\frac{d-L}{2}.$$

(ii) *If d is odd,*

$$f^\lambda_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}-\frac{1}{2}\Big)\Big(m_1+m_2+\tilde a_0-\frac{1}{2}(i+j)\Big)$$
$$\times\Big(m_1+m_2+\tilde b_0-\frac{1}{2}(i+j)\Big),$$

with

$$f^\lambda_{m_1,m_2}=c_1'(-1)^{(i-j+d)/2-1}\Big(\frac{d}{2}\Big)_{m_1-[(j+1)/2]}\Big(\frac{d}{2}\Big)_{m_2+1-[(i+1)/2]}$$
$$\times\prod_{\ell=1}^{[i/2]}\Big(m_1+\frac{1}{2}(d-i-j)+\ell-1\Big)\prod_{\ell=1}^{[j/2]}\Big(m_2+\frac{1}{2}(d-i-j)+\ell\Big)$$

and

$$\tilde a_0=\frac{d}{2},\qquad \tilde b_0=\frac{d}{2}+\frac{d-L}{2}.$$

Proof. We have the following.

(i) If d is even, the first term of f_1 equals

$$\sum_{m_1,m_2\ge 0}f^\lambda_{m_1,m_2}\Big(m_2+\frac{1}{2}(d-i-j)\Big)\Big(m_2+b_0-1-\frac{i+j}{2}\Big)$$
$$\times\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}-1\Big)x_1^{m_1}x_2^{m_2},$$

and the second term equals

$$\sum_{m_1,m_2\geq 0} f^{\lambda}_{m_1,m_2} m_1\Big(m_1-1+2m_2+\frac{d}{2}+b_0-i-j\Big)$$
$$\times\,\xi\Big(m_1+m_2-\frac{i+j}{2}-1\Big)x_1^{m_1}x_2^{m_2}.$$

Hence, the sum of the coefficients at $x_1^{m_1}x_2^{m_2}$ is equal to

$$\begin{aligned}
&f^{\lambda}_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}-1\Big)\\
&\qquad\times\Big(m_2+\frac{1}{2}(d-i-j)\Big)\Big(m_2+b_0-1-\frac{i+j}{2}\Big)\\
&\qquad+m_1\Big(m_1+2m_2+\frac{d}{2}+b_0-1-i-j\Big)\\
&\quad=(m_1+\tilde m_2)^2+\Big(\frac{d}{2}+b_0-1\Big)\tilde m_2+\Big(\frac{d}{2}+b_0-1\Big)m_1+\frac{d}{2}(b_0-1)\\
&\quad=\Big(m_1+\tilde m_2+\frac{d}{2}\Big)(m_1+\tilde m_2+b_0-1),
\end{aligned}$$

if we write $\tilde m_2=m_2-(i+j)/2$. Thus, the case of even d is proved.

(ii) For odd d, the coefficient at $x_1^{m_1}x_2^{m_2}$ of the first term in f_1 is given by

$$f^{\lambda}_{m_1,m_2}\Big(m_2+\frac{1}{2}(d-i-j)\Big)\Big(m_2+\tilde b_0-\frac{i+j}{2}\Big)\xi\Big(m_1+m_2-\frac{i+j}{2}-\frac{1}{2}\Big),$$

and that of the second term by

$$f^{\lambda}_{m_1,m_2}\Big(m_1+2m_2+\tilde b_0+\frac{d}{2}-i-j\Big)\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}-\frac{1}{2}\Big).$$

The sum of these two coefficients equals

$$\begin{aligned}
&f^{\lambda}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}-\frac{1}{2}\Big)\\
&\qquad\times\Big(m_2+\frac{1}{2}(d-i-j)\Big)\Big(m_2+\tilde b_0-\frac{i+j}{2}\Big)\\
&\qquad+m_1\Big(m_1+2m_2+\frac{d}{2}+\tilde b_0-i-j\Big)\\
&\quad=\Big(m_1+m_2+\frac{d}{2}-\frac{i+j}{2}\Big)\cdot\Big(m_1+m_2+\tilde b_0-\frac{i+j}{2}\Big),
\end{aligned}$$

and our lemma is shown. $\square$

Now we compute the other side, f_2.

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LEMMA 6.4. *The coefficient of $x_1^{m_1}x_2^{m_2}$ in the power series f_2 is given by the following.*

(i) *For even d,*

$$f^{\rho}_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}\Big)\Big(m_1+\frac{1}{2}(d-i-j)\Big)$$
$$\times\Big(m_1+m_2+d-\frac{i+j}{2}\Big)\Big(m_1+m_2+d-\frac{i+j}{2}+\frac{1}{2}\Big),$$

with

$$f^{\rho}_{m_1,m_2}=c_0'(-1)^{(i-j+d)/2-1}\Big(\frac{d+1}{2}\Big)_{m_1-[j/2]-1}\Big(\frac{d+1}{2}\Big)_{m_2-[i/2]}$$
$$\times\prod_{\ell=1}^{[i/2]-1}\Big(m_1+\frac{1}{2}(d-i-j)+\ell\Big)\prod_{\ell=1}^{[j/2]}\Big(m_2+\frac{1}{2}(d-i-j)+\ell\Big).$$

(ii) *For odd d,*

$$f^{\rho}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}\Big)\Big(m_1+\frac{1}{2}(d-i-j)\Big)$$
$$\times\Big(m_1+m_2+d-\frac{i+j}{2}-\frac{1}{2}\Big)\Big(m_1+m_2+d-\frac{i+j}{2}+\frac{1}{2}\Big),$$

with

$$f^{\rho}_{m_1,m_2}=c_1'(-1)^{(i-j+d)/2-1}\Big(\frac{d}{2}\Big)_{m_1-[(j+1)/2]}\Big(\frac{d}{2}\Big)_{m_2+1-[(i+1)/2]}$$
$$\times\prod_{\ell=1}^{[i/2]-1}\Big(m_1+\frac{1}{2}(d-i-j)\Big)+\ell)\prod_{\ell=1}^{[j/2]}\Big(m_2+\frac{1}{2}(d-i-j)+\ell\Big).$$

Proof. First, for $(\frac{\partial}{\partial x_2})h_{i,j+2}$, we have the following.

SUBLEMMA 2A. *The coefficient of $x_1^{m_1}x_2^{m_2}$ in the power series $\frac{\partial}{\partial x_2}h^P_{i,j+2}$ equals*

$$f^{\rho}_{m_1,m_2}\times\begin{cases}\xi_{0,0}\big(m_1+m_2-\frac{i+j}{2}\big) & (\textit{if } d \textit{ even}),\\ \xi_{0,1}\big(m_1+m_2-\frac{i+j}{+}\frac{1}{2}\big) & (\textit{if } d \textit{ odd})\end{cases}$$

times

$$\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i-j)+1\Big)\Big(m_1+\frac{1}{2}(d-i-j)\Big).$$

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Proof of Sublemma 2A. If d is even, $\frac{\partial}{\partial x_2} h_{i,j+2}$ equals

$$\sum f^{\rho}_{m_1,m_2}\Big(m_2+\frac{d+2}{2}-\Big[\frac{i}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j}{2}\Big]+1\Big)$$
$$\times\Big(m_1+\frac{1}{2}(d-i-j)\Big)\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}\Big)x_1^{m_1}x_2^{m_2}.$$

We can confirm that

$$\Big(m_2+\frac{d+1}{2}-\Big[\frac{i}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j}{2}\Big]+1\Big)$$
$$=\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+1\Big)$$

either if i,j are both odd or if i,j are both even.

If d is odd, $(\frac{\partial}{\partial x_2})h_{i,j+2}$ equals

$$\sum f^{\rho}_{m_1,m_2}\Big(m_2+\frac{d}{2}+1-\Big[\frac{i+1}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j}{2}\Big]+1\Big)$$
$$\times\Big(m_1+\frac{1}{2}(d-i-j)\Big)\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\Big)x_1^{m_1}x_2^{m_2}.$$

Note here that

$$\Big(m_2+\frac{d}{2}+1-\Big[\frac{i+1}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j}{2}\Big]+1\Big)$$
$$=\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+1\Big),$$

either if i is odd and j is even or if i is even and j is odd.

Thus, Sublemma 2A is proved. $\square$

For $-ih^{P}_{i-1,j+1}$, we have the following.

SUBLEMMA 2B. *The coefficient of $x_1^{m_1}x_2^{m_2}$ in the power series $h^{P}_{i-1,j+1}$ is equal to*

$$f^{\rho}_{m_1,m_2}\times\begin{cases}\xi_{0,0}\big(m_1+m_2-\frac{i+j}{2}\big) & (\textit{if } d \textit{ even}),\\ \xi_{0,1}\big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\big) & (\textit{if } d \textit{ odd})\end{cases}$$

times

$$-i\cdot\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big).$$

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Proof of Sublemma 2B. If d, i, j are all even, the coefficient in question is

$$f^{\rho}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}\Big)\\ \times(-i)\Big(\frac{d+1}{2}+m_1-\frac{j}{2}-1\Big)\Big(\frac{d+1}{2}+m_2-\frac{i}{2}\Big)$$

by definition of $h^P_{i-1,j+1}$. If d is even and i, j are both odd, it reads

$$f^{\rho}_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}\Big)\\ \times(-i)\Big(m_1+\frac{1}{2}(d-i-j)+\Big[\frac{i}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j+1}{2}\Big]\Big).$$

Since

$$\Big(m_1+\frac{1}{2}(d-i-1)+\Big[\frac{i}{2}\Big]\Big)\Big(m_2+\frac{1}{2}(d-i-j)+\Big[\frac{j+1}{2}\Big]\Big)\\ =\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\quad\text{for } i,j \text{ odd},$$

Sublemma 2B is true for even d.

If d is odd, the coefficient in question is equal to

$$f^{\rho}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\Big)$$

times

$$(-i)\cdot\begin{cases}\big(\frac{d}{2}+m_2-\big[\frac{i}{2}\big]\big)\big(m_1+\frac{1}{2}(d-i-j)+\big[\frac{i-1}{2}\big]\big) & (i \text{ odd}, j \text{ even}),\\ \big(\frac{d}{2}+m_1-\big[\frac{j+2}{2}\big]\big)\big(m_2+\frac{1}{2}(d-i-j)+\big[\frac{j+1}{2}\big]\big) & (i \text{ even}, j \text{ odd}).\end{cases}$$

In cases where either i is odd and j is even or i is even and j is odd, the last number is the same:

$$(-i)\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big).$$

Hence, Sublemma 2B is shown for the case of odd d, too. $\square$

For

$$-\Big(x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big)h^P_{i,j},$$

we have the following.

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SUBLEMMA 2C. *The coefficient of* $x_1^{m_1}x_2^{m_2}$ *in the power series*

$$-\Big(x_1\frac{\partial}{\partial x_1}+2x_2\frac{\partial}{\partial x_2}+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big)h^P_{i,j}$$

is equal to

$$f^\rho_{m_1,m_2}\times\begin{cases}\xi_{0,0}\big(m_1+m_2-\frac{i+j}{2}\big) & (\text{if } d \text{ even}),\\ \xi_{0,1}\big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\big) & (\text{if } d \text{ odd})\end{cases}$$

times

$$\Big(m_1+\frac{1}{2}(d-j)\Big)\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_1+2m_2+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big).$$

Proof of Sublemma 2C. If d is even, the coefficient in question is

$$f^\rho_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}\Big)\Big(m_1+\frac{d+1}{2}-\Big[\frac{j}{2}\Big]-1\Big)$$
$$\times\Big(m_1+(d-i-j);\Big[\frac{i}{2}\Big]\Big)\Big(m_1+2m_2+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big).$$

Thus, the equality

$$\Big(m_1+\frac{d+1}{2}-\Big[\frac{j}{2}\Big]-1\Big)\Big(m_1+\frac{1}{2}(d-i-j)+\Big[\frac{i}{2}\Big]\Big)$$
$$=\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_1+\frac{1}{2}(d-i)\Big),$$

which is valid either when both i and j are even or when both i and j are odd, implies Sublemma 2C.

If d is odd, the coefficient in question is

$$f^\rho_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\Big)\Big(m_1+\frac{d}{2}-\Big[\frac{j}{2}\Big]\Big)$$
$$\times\Big(m_1+2m_2+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big)\Big(m_1+\frac{1}{2}(d-i-j)+\Big[\frac{i}{2}\Big]\Big).$$

The equality

$$\Big(m_1+\frac{d}{2}-\Big[\frac{j}{2}\Big]\Big)\Big(m_1+\frac{1}{2}(d-i-j)+\Big[\frac{i}{2}\Big]\Big)$$
$$=\Big(m_1+\frac{1}{2}(d-j)\Big)\Big(m_2+\frac{1}{2}(d-j)-\frac{1}{2}\Big)$$

is valid either when i is odd and j is even or when i is even and j is odd. Hence, Sublemma 2C is proved. □

6.3.1. Conclusion of the proofs of Lemma 6.4 and Proposition 6.1. In view of Sublemmas 2A, 2B, and 2C, the proof of Lemma 6.4 is reduced to show the following equality:

$$\begin{aligned}
&\Big(m_2+\frac{1}{2}(d-i)+1\Big)\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\Big(m_1+\frac{1}{2}(d-i-j)\Big)\\
&\qquad -i\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\Big(m_2+\frac{1}{2}(d-i)+\frac{1}{2}\Big)\\
&\qquad +\Big(m_1+\frac{1}{2}(d-j)\Big)\Big(m_1+2m_2+\frac{3}{2}d-\frac{3}{2}i-\frac{1}{2}j+1\Big)\\
&\qquad \times\Big(m_1+\frac{1}{2}(d-j)-\frac{1}{2}\Big)\\
&\quad=\Big(m_1+\frac{1}{2}(d-i-j)\Big)\Big(m_1+m_2+d-\frac{i+j}{2}\Big)\\
&\qquad \times\Big(m_1+m_2+d-\frac{i+j}{2}+\frac{1}{2}\Big),
\end{aligned}$$

which is confirmed by direct computation.

Now let us return to the proof of Proposition 6.1(iii). We have to show that $f_1=f_2$. By Lemmas 6.3 and 6.4, it is reduced to the equality

$$\begin{aligned}
&f^{\lambda}_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}-1\Big)\\
&\qquad\times\Big(m_1+m_2+a_0-1-\frac{i+j}{2}\Big)\Big(m_1+m_2+b_0-1-\frac{i+j}{2}\Big)\\
&\quad=f^{\rho}_{m_1,m_2}\xi_{0,0}\Big(m_1+m_2-\frac{i+j}{2}\Big)\\
&\qquad\times\Big(m_1+\frac{1}{2}(d-i-j)\Big)\Big(m_1+m_2+d-\frac{i+j}{2}\Big)\\
&\qquad\times\Big(m_1+m_2+d-\frac{i+j}{2}+\frac{1}{2}\Big)
\end{aligned}$$

if d is even, and to

$$\begin{aligned}
&f^{\lambda}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}-\frac{1}{2}\Big)\\
&\qquad\times\Big(m_1+m_2+\tilde{a}_0-\frac{i+j}{2}\Big)\Big(m_1+m_2+\tilde{b}_0-\frac{i+j}{2}\Big)\\
&\quad=f^{\rho}_{m_1,m_2}\xi_{0,1}\Big(m_1+m_2-\frac{i+j}{2}+\frac{1}{2}\Big)
\end{aligned}$$

$$\times \left(m_1+\frac{1}{2}(d-i-j)\right)\left(m_1+m_2+d-\frac{i+j}{2}\right)$$
$$\times \left(m_1+m_2+d-\frac{i+j}{2}+\frac{1}{2}\right)$$

if d is odd.

Either d is even or it is odd; thus, we have

$$f^{\lambda}_{m_1,m_2}=f^{\rho}_{m_1,m_2}\left(m_1+\frac{1}{2}(d-i-j)\right)$$

by definition. So at last we have to show that

$$\xi_{0,0}(k-1)(k+a_0-1)(k+b_0-1)=\xi_{0,0}(k+d)\left(k+d+\frac{1}{2}\right),$$

for $k=m_1+m_2-(i+j)/2$, if d is even. If d is odd, we have to check that

$$\xi_{0,1}(k-1)\left(k-1+\tilde{a}_0+\frac{1}{2}\right)\left(k-1+\tilde{b}_0+\frac{1}{2}\right)=\xi_{0,1}(k)\left(k+d-\frac{1}{2}\right)(k+d)$$

with $k=m_1+m_2-(i+j)/2+1/2\in\mathbf{Z}$. Since these are exactly the recurrence relations defining $\xi_{0,0}(k)$ and $\xi_{0,1}(k)$, respectively, we complete the proof of $f_1=f_2$. □

6.4. Proof of Theorem 6.1

When d is even, Theorem 6.1 is true for $h_{0,0}$ by Proposition 5.3, and when d is odd, it is also true for $h_{0,1}$ and $h_{1,0}$ by Proposition 5.3.

We proceed by induction on the indices $I=(i,j)$. The first step of the induction is to show Theorem 6.1 when $i\le 1$ or $j\le 1$. By symmetry, it suffices to discuss only the case $i\le 1$. Moreover, we may assume that $j\ge 2$. Since

$$\frac{\partial}{\partial x_1}h_{i,j}=-\left\{x_2\frac{\partial}{\partial x_2}+\frac{d-i-j+2}{2}\right\}h_{i,j-2}$$

by an inductive equation of type I (Lemma 4.2), and since

$$\frac{\partial}{\partial x_1}h^P_{i,j}=-\left\{x_2\frac{\partial}{\partial x_2}+\frac{d-i-j+2}{2}\right\}h^P_{i,j-2}$$

by Proposition 6.1, the hypothesis of the induction

$$h_{i,j-2}=h^P_{i,j-2}$$

implies that

$$\frac{\partial}{\partial x_1}(h_{i,j}-h^P_{i,j})=0.$$

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Hence,

$$h_{i,j} = h^P_{i,j} + F(x_2),$$

where $F(x_2)$ is a power series only in the variable x_2, regular at $x_2 = 0$. By Proposition 6.1(iii) and by the corresponding equation for $h_{i,j}$, we have

$$(x_2 - 1)\Big\{\frac{\partial}{\partial x_2} + \Big(d - j - 1 - \frac{L + i - j}{2}\Big)\frac{1}{x_2 - 1}\Big\}(h_{i,j} - h^P_{i,j}) = 0,$$

since $h_{i,j-2} = h^P_{i,j-2}$ and $h_{i-1,j-1} = h^P_{i-1,j-1}$ by the hypothesis of induction.

Therefore, we have

$$(x_2 - 1)\Big\{\frac{\partial}{\partial x_2} + \Big(d - j - 1 - \frac{L + i - j}{2}\Big)\frac{1}{x_2 - 1}\Big\}F(x_2) = 0;$$

that is,

$$F(x_2) = C'(x_2 - 1)^{d+1-L/2-(i+j)/2}$$

with a constant of integration C'. Then

$$h_{i,j}(0) = h^P_{i,j}(0) + F_2(0) = h^P_{i,j}(0) + C'.$$

But we already know that $h_{i,j}(0) = h^P_{i,j}(0)$ by Lemma 6.2. Thus, $C' = 0$, that is, $F_2(x_2) = 0$, which means that $h_{i,j} = h^P_{i,j}$.

The next and last step of induction is to discuss the case when $i, j \geq 2$. In this case we use the equations

$$\frac{\partial}{\partial x_1}h_{i,j} = -\Big\{x_2\frac{\partial}{\partial x_2} + \frac{d - i - j + 2}{2}\Big\}h_{i,j-2}$$

and

$$\frac{\partial}{\partial x_2}h_{i,j} = -\Big\{x_1\frac{\partial}{\partial x_1} + \frac{d - i - j + 2}{2}\Big\}h_{i-2,j}.$$

Since the corresponding equations are valid for $h^P_{i,j}$, the hypothesis of induction,

$$h_{i,j-2} = h^P_{i,j-2}, \qquad h_{i-2,j} = h^P_{i-2,j},$$

implies that

$$\frac{\partial}{\partial x_1}(h_{i,j} - h^P_{i,j}) = \frac{\partial}{\partial x_2}(h_{i,j} - h^P_{i,j}) = 0;$$

that is,

$$h_{i,j} = h^P_{i,j} + C \quad \text{with a constant of integration } C.$$

Since Lemma 6.2 implies that $C = h_{i,j}(0) - h^P_{i,j}(0) = 0$, we complete the proof of the equality $h_{i,j} = h^P_{i,j}$ for $i + j \leq d$. ∎

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§7. Postscript

7.1. Comparison with the case SU(2, 2)

Here we discuss a problem of the behavior of the matrix coefficients at infinity.

When G/K is of Hermitian type, the matrix coefficients of holomorphic and antiholomorphic discrete series with the minimal K-types are the Bergmann kernel. In this case, its A-radial part is a Laurent polynomial function in the coordinates (i.e., in the matrix entries of $A \subset G \hookrightarrow \mathrm{GL}(N, \mathbf{C})$). No transcendental function appears in this case.

Some years ago, I together with Takahiro Hayata and Harutaka Koseki had an explicit formula of the radial part of the middle discrete series of $\mathrm{SU}(2,2)$ (see [3]). As a corollary of this result, we can say that the matrix coefficients have logarithmic singularities at infinity (see [4]).

Our main results also seem to give the asymptotic expansion at infinity. But the characteristic indices are always halves of integers; hence, we have no logarithmic singularities.

7.2. A conjecture, or a hope?

CONJECTURE. Given a discrete series representation π of a semisimple Lie group G with Cartan decomposition $G = KAK$, the A-radial part of the matrix coefficients of K-finite types of π is an elementary transcendental function in the matrix variables in the coordinates of A. Here elementary transcendental functions are rational expressions of logarithms, hyperbolic sines, and algebraic functions.

This means that our job is not finished even for the case of the group $\mathrm{Sp}(2, \mathbf{R})$. We have to find a better expression than the main theorems, Theorems 6.1 and 6.2.

Acknowledgments. The author thanks Takashi Taniguchi for a careful reading of the first draft of this paper to correct many typos and for helpful comments. The author thanks the organizers of the special year on automorphic forms at the Mathematical Sciences Research Institute in the fall of 1994 for giving him the chance to concentrate on the computation of the holonomic system for c_I in Section 2 of this paper for an entire month. The author also thanks Professor Takeshi Hirai for his ongoing communications over more than three decades.

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https://doi.org/10.1017/S0027763000010631 Published online by Cambridge University Press