SOME TOPOLOGICAL PROPERTIES OF RESIDUALLY ČERNIKOV GROUPS

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1. Introduction. In this paper we shall indicate how to generalise the concept of a cofinite group (see [**7**]). We recall that any residually finite group can be made into a topological group by taking as a basis of neighbourhoods of the identity precisely the normal subgroups of finite index. The class of compact cofinite groups is then easily seen to be the class of profinite groups, where a group is profinite if and only if it is an inverse limit of finite groups. It turns out that every cofinite group can be embedded as a dense subgroup of a profinite group. This has important consequences for the class of countable locally finite-soluble groups with finite Sylow p-subgroups for all primes p, as shown in [**7**] and [**14**].

Our generalisation is as follows. By a *separating filter base* $\mathcal{N}$ of a group G we shall mean a set of normal subgroups satisfying:

(i) if $N \in \mathcal{N}$, G/N is a Černikov group;

(ii) if $L, M \in \mathcal{N}$ there exists $N \in \mathcal{N}$ such that $N \leq L \cap M$;

(iii) $\cap\{N : N \in \mathcal{N}\} = 1$.

Thus G possesses a separating filter base if and only if G is a residually Černikov group. We shall call G a *co-Černikov group relative to* $\mathcal{N}$ and regard G as a topological space with

$$\{Hx : x \in G \text{ and there exists } N \in \mathcal{N} \text{ such that } N \leq H \leq G\}$$

as a closed sub-base. Thus the closed subsets of G are intersections of finite unions of certain cosets of G. We shall let $(G, \mathcal{N})$ denote that G is a co-Černikov group relative to $\mathcal{N}$ and the topology determined by $\mathcal{N}$ will be called a *co-Černikov topology*. Of course, G will possess many such topologies, depending on $\mathcal{N}$. By a *pro-Černikov group* we shall simply mean an inverse limit of Černikov groups.

Thus, it is straightforward to show that a cofinite group with any of its cofinite topologies is a co-Černikov group with that topology. However we shall show that, in general, co-Černikov groups need not be topological groups.

In Section 2 we give many of the elementary properties of co-Černikov groups. We show that if G is a Černikov group then G (with its unique co-Černikov topology) is a compact T_1-space. (By a T_1-*space* we mean a space in which points are closed.) Many properties of cofinite groups have their analogue in the study of co-Černikov groups, and we exploit the very similar nature of these two classes of groups as often as possible. In particular, in 2.8 we prove that the compact co-Černikov groups are precisely the pro-Černikov groups. It is then easily seen that any co-Černikov group can be embedded as a dense subgroup of a compact co-Černikov group.

In Section 3 we study more closely the Sylow theory of compact co-Černikov groups.

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By using the methods of [7] it is relatively easy to show that the well known results of Sylow and Hall extend to the class of compact co-Černikov groups. However the idea of a Sylow p-subgroup is modified so that we no longer characterise them as being the maximal p-subgroups. Instead we define most of the new concepts via the Černikov factor groups.

Our proofs are fairly standard generalisations of those that occur in the analogous case for cofinite groups. However we have been forced, much of the time, to use the following generalisation of the theorem of Kuroš [11] concerning the inverse limit of a system of non-empty finite sets.

THEOREM 1.1. *Let* $\{S_i, \alpha_{ij} : i, j \in I, i \leq j\}$ *be an inverse system of non-empty compact topological* T_1*-spaces and closed, continuous maps. Then:*

(a) $S = \varprojlim S_i \neq \emptyset$;

(b) $S = \varprojlim \beta_i(S)$ *and the restriction of* $\alpha_{ij} : \beta_j(S) \to \beta_i(S)$ *is surjective;*

(c) *the image of the canonical projection* $\beta_i : S \to S_i$ *is* $\beta_i(S) = \bigcap_{i \leq j} \alpha_{ij}(S_j)$;

(d) *if* $T \subseteq S$ *then* $\bar{T} = \varprojlim \overline{\beta_i(T)}$ *and if* $T \subseteq_c S$ *then* $T = \varprojlim \beta_i(T) = \varprojlim \overline{\beta_i(T)}$;

(e) S *is compact.*

The proof of this result can be found in [16, Theorem 2.1].

The study of the class of co-Černikov groups is also of interest because of some of the group classes it contains. It is straightforward to show (from 3.13 and 3.17 of [9]) that the class $\mathfrak{X}$ of countable locally finite-soluble groups satisfying min-p for all primes p is a sub-class of the class of co-Černikov groups. No applications of the results we present here are given, but we mention some results that appear in [3]. There it was shown that the Sylow generating bases of an $\mathfrak{X}$-group are locally conjugate, thus generalising a result of Baer [1]. (Our terminology is as in [4].) However our method of proof was similar to that of [7], where the full force of the topological arguments is seen. In [4], a more straightforward proof of this result is given.

Furthermore, we also showed in [3] that the Carter subgroups of an $\mathfrak{X}$-group are isomorphic and one way of doing this is by use of the topological methods established here. Actually our result could be deduced from a theorem of Massey [12], but we think the method used in [3] is of some interest.

Our notation and terminology is mostly standard. If G is a Černikov group, G^0 will denote the unique minimal subgroup of finite index in G and will be called the *radicable part* of G. If $K \leq (G, \mathcal{N})$, a co-Černikov group, then we shall write $K \leq_c G$, $K \leq_o G$ and $K \leq_d G$ to denote that K is a closed subgroup of G, K is an open subgroup of G and K is a dense subgroup of G respectively. If U is a subset of a set V then $\mathscr{C}U$ will denote the complement of U in V.

This paper is part of my Ph.D. thesis, and the results were obtained whilst I was a student at the University of Warwick. I should like to thank my supervisor, Dr. S. E. Stonehewer, for all his help and suggestions. I would also like to thank Professor B.

Hartley for his many suggestions which made this paper possible. Finally, I should like to thank the Science Research Council for its financial support.

2. Elementary properties of co-Černikov groups. In this section we shall give the basic definitions and elementary properties of co-Černikov groups. Many of our results are analogous to those obtained for arbitrary topological groups. However we shall give a straightforward example to show that co-Černikov groups are not, in general, topological groups.

If G is a Černikov group then we can regard G as a topological space with $\{Hx : x \in G, H \leq G\}$ as a closed sub-base. Thus G is a co-Černikov group relative to $\{1\}$ and every co-Černikov topology on G gives rise to this topology. The topology defined on G will be called the *coset topology* of G. It is analogous to the $\mathscr{W}$-topology, a subtopology of the Zariski topology of an affine algebraic group, defined in [**16**, p. 188]. We shall show that G, together with the coset topology, is a compact T_1-space. A topological space T is said to be *Noetherian* if every ascending chain of open subsets of T terminates in finitely many steps or, equivalently, if every descending chain of closed subsets of T terminates in finitely many steps.

Lemma 2.1. *Let G be a Černikov group with coset topology. Then:*
(i) *G is a Noetherian, compact, T_1-space;*
(ii) *every closed subset of G is a finite union of cosets of G.*

Proof. (i) Let $\mathscr{S} = \{Hx : x \in G, H \leq G\}$, a closed sub-base for the coset topology. Clearly $\mathscr{S}$ is closed under finite intersections and since G satisfies the minimum condition, $\mathscr{S}$ has the minimum condition also. By a result of R. Bryant [**2**, Lemma 3.2], the coset topology of G is a Noetherian topology. Hence G is compact and, from the definition, G is a T_1-space.

(ii) This follows from (i) and the laws of set theory.

This completes the proof.

I am indebted to both Professor Hartley and the referee for bringing R. Bryant's result to my attention.

We can now obtain some information concerning the continuity properties of homomorphisms of Černikov groups.

Lemma 2.2. *Let G, H be Černikov groups with coset topologies.*
(i) *If $K \leq G$ then $xK \subseteq_c G$, for all $x \in G$.*
(ii) *If $\phi : G \to G$ is defined by $\phi(x) = x^{-1}$, for all $x \in G$, then ϕ is closed and continuous.*
(iii) *If $\theta : G \to H$ is a homomorphism then θ is closed and continuous.*
(iv) *If $y \in G$ and α_y, $\beta_y : G \to G$ are defined by $\alpha_y(x) = xy$, $\beta_y(x) = yx$, for all $x \in G$, then α_y and β_y are both closed and continuous.*

Since (iii) can be handled by using 2.1 (ii), the proof of the above result is straightforward and is omitted.

One might hope that a Černikov group G with coset topology was Hausdorff also, but even the most straightforward example shows that this is generally not the case. For let $G \cong C_{p^\infty}$, the Prüfer p-group, and suppose $x, y \in G$ with $x \neq y$. Suppose there exist open sets $U, V \neq \emptyset$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then $\mathscr{C}U \cup \mathscr{C}V = G$. Since $\mathscr{C}U, \mathscr{C}V$ are closed they are finite unions of cosets by 2.1 (ii). It follows from [**13**, (4.4)] that G has a subgroup of finite index. Hence $G = \mathscr{C}U$ or $G = \mathscr{C}V$, a contradiction. This example also shows that Černikov groups with coset topology need not be topological groups in this topology. For, every co-Černikov group with a co-Černikov topology is evidently T_1; but the above example cannot be a topological group since this would contradict the equivalence of (i) and (v) in Proposition 3($\mathscr{T}\mathscr{G}$) of [**8**]. However in applications it is the compactness that will turn out to be important.

PROPOSITION 2.3. *Let $(G, \mathcal{N})$ be a co-Černikov group and let $\mathscr{F}$ be the closed sub-base determined by $\mathcal{N}$. If $H \leq G$ then*

$$\bar{H} = \cap\{HN : N \in \mathcal{N}\} = \cap\{HK : K \in \mathscr{F} \quad \text{and} \quad K \leq G\}.$$

In particular $\bar{H} \leq_c G$ and if $H \lhd G$ then $\bar{H} \lhd_c G$.

Proof. If $N \in \mathcal{N}$, $N \leq NH \leq G$; so $HN \leq_c G$ by definition of $(G, \mathcal{N})$. Hence $H \leq \cap\{HN : N \in \mathcal{N}\} \leq_c G$; so

$$\bar{H} \subseteq \cap\{HN : N \in \mathcal{N}\}.$$

Conversely, $\bar{H}$ is closed and hence

$$\bar{H} = \bigcap_{i \in I} \bigcup_{j=1}^{n_i} K_{ij} x_{ij},$$

for some index set I, elements $x_{ij} \in G$, subgroups K_{ij} of G with $N_{ij} \leq K_{ij}$ (for some $N_{ij} \in \mathcal{N}$) and $n_i \in \mathbb{N}_0$. Put $A_i = \bigcup_{j=1}^{n_i} K_{ij} x_{ij}$. Since $\mathcal{N}$ is a separating filter base, it follows that for each $i \in I$ there exists $N_i \in \mathcal{N}$ such that $N_i \leq K_{ij}$ for $j = 1, \ldots, n_i$.

Now

$$H = \bigcup_{j=1}^{n_i} (H \cap K_{ij} x_{ij}) = \bigcup_{j=1}^{m_i} (H \cap K_{ij}) y_{ij}$$

say. Here the K_{ij} have been renumbered, if necessary, $m_i \leq n_i$ is the number of non-empty intersections $H \cap K_{ij} x_{ij}$ and $y_{ij} \in H \cap K_{ij} x_{ij}$. Hence

$$HN_i = \bigcup_{j=1}^{m_i} (H \cap K_{ij}) N_i y_{ij} \subseteq \bigcup_{j=1}^{m_i} K_{ij} N_i y_{ij} \subseteq A_i,$$

for all i. Therefore $\cap\{HN : N \in \mathcal{N}\} \subseteq \bigcap_{i \in I} A_i = \bar{H}$ and the result follows. It is now clear from the definitions that $\bar{H} = \cap\{HK : K \in \mathscr{F}, K \leq G\}$. This completes the proof.

The next corollary is an easy consequence.

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COROLLARY 2.4. *If $(G, \mathcal{N})$ is a co-Černikov group then $H \leq_d G$ if and only if $G = NH$ for all $N \in \mathcal{N}$.*

The following extension of 2.2 is easily established.

LEMMA 2.5. *Let $(G, \mathcal{N})$ and $(H, \mathcal{M})$ be co-Černikov groups.*

(i) *If $K \leq G$ and there exists $N \in \mathcal{N}$ such that $N \leq K$ then, for all $x \in G$, xK is a closed set.*

(ii) *If $\phi : G \to G$ denotes inversion then ϕ is closed and continuous.*

(iii) *If $\alpha_y, \beta_y : G \to G$ are defined by $\alpha_y(x) = xy$ and $\beta_y(x) = yx$, for $x, y \in G$, then α_y and β_y are both closed and continuous.*

Our immediate aim is to show that pro-Černikov groups and compact co-Černikov groups are the same thing. We shall fix the following notation for the rest of this section. Let $\{G_i, \theta_{ij} : i, j \in I\}$ be an inverse system of Černikov groups, with coset topologies, and group homomorphisms indexed by a set I. So if $i \geq j \geq k$ there exist homomorphisms $\theta_{ji} : G_i \to G_j$ such that $\theta_{ki} = \theta_{kj} \circ \theta_{ji}$ and θ_{ii} is the identity map on G_i. Let $G = \varprojlim G_i$, a pro-Černikov group, and let $H = \operatorname{Cr}_{i \in I} G_i$, the cartesian product of the G_i. Give G and H their usual topologies. Let $\alpha : G \to H$ denote the inclusion map, $\beta_i : H \to G_i$ the ith projection map and $\gamma_i = \beta_i \circ \alpha$. Put $M_i = \ker \beta_i$, $N_i = \ker \gamma_i$ and $\mathcal{N} = \{N_i : i \in I\}$.

The following properties are easily verified.

LEMMA 2.6. (i) *$M_i \cong \operatorname{Cr}_{j \neq i} G_j$ for all $i \in I$.*

(ii) *$M_i \cap G = N_i$ for all $i \in I$.*

(iii) $\bigcap_{i \in I} M_i = 1$.

(iv) $\bigcap_{i \in I} N_i = 1$.

(v) *$N_i \leq N_j$ if $i \geq j$.*

LEMMA 2.7. (i) *The maps α, β_i, γ_i, θ_{ij} are continuous for $i, j \in I$.*

(ii) *The left and right translation maps in G and H are continuous.*

(iii) *For each $i \geq j$, θ_{ji} is a closed map.*

(iv) *If $M_i \leq L \leq H$ then $L \leq_c H$ for each $i \in I$.*

(v) *If $N_i \leq L \leq G$ then $L \leq_c G$ for each $i \in I$.*

Proof. Parts (i), (ii) and (iii) are trivial to prove using elementary topology (see **[15]**, for example) and 2.2.

(iv) With the usual identifications, $H = M_i \times G_i$; so $L = M_i \times (G_i \cap L)$ by Dedekind's law. Since G_i has the coset topology and H has its usual topology, L is a cartesian product of closed sets and so is closed.

(v) If $N_i \leq L \leq G$ then $M_i \leq M_iL \leq H$; so $M_iL \leq_c H$ by (iv). Since G has the subspace topology, $M_iL \cap G \leq_c G$. By the Dedekind law and 2.6(ii), $M_iL \cap G = L$ and the result follows.

To prove the equivalence that we seek, we shall require the theorem concerning inverse limits of topological spaces mentioned in the introduction.

THEOREM 2.8. *A group K is a pro-Černikov group if and only if, for some separating filter base $\mathcal{M}$, $(K, \mathcal{M})$ is a compact co-Černikov group.*

Proof. $(\Rightarrow)$ With the notation introduced after 2.5, we may write $K = G = \varprojlim G_i$. By 2.1(i) and 2.2(iii), the hypotheses of 1.1 are satisfied and by 1.1(b) and (c) we may assume that the γ_i are surjective.

Let τ be the co-Černikov topology induced on G by $\mathcal{N}$ and let σ denote the natural subspace topology on G. Then, by 1.1(e), (G, σ) is a compact space. To prove the result it now suffices to show $\sigma = \tau$. By 2.7(v), if $N_i \le L \le G$ then $L \le_c (G, \sigma)$. Hence sub-basic closed sets in τ are closed in σ; so $\tau \subseteq \sigma$. On the other hand if $L \le G_i$ and $g \in G_i$ then $(Lg \times M_i) \cap G$ is a sub-basic closed set in σ and

$$N_i = M_i \cap G \le (L \times M_i) \cap G.$$

Thus $(L \times M_i) \cap G \le_c (G, \tau)$. Since γ_i is surjective, 2.5(iii) implies that $(Lg \times M_i) \cap G \subseteq_c (G, \tau)$ and hence $\sigma = \tau$ as required.

$(\Leftarrow)$ Our proof is similar to that given in [**8**] for profinite groups.

Let $(K, \mathcal{M})$ be a compact co-Černikov group, for some separating filter base $\mathcal{M}$. Put $\mathcal{M} = \{H_i : i \in J\}$, for some index set J, and order J via:

$$j \le i \quad \Leftrightarrow \quad H_i \le H_j.$$

Thus for $j \le i$ there is a map $\psi_{ji} : K/H_i \to K/H_j$ and $\{K/H_i, \psi_{ji} : i, j \in J\}$ is an inverse system of Černikov groups and group homomorphisms. Put $L = \varprojlim K/H_i$ and define $\phi : K \to L$ by:

$$\phi(g) = (gH_i) \in L, \quad \text{for each } g \in K.$$

The map ϕ is clearly a monomorphism since $\mathcal{M}$ is a filter base. It now suffices to show that ϕ is a surjection; so let $(g_iH_i) \in L$. If $\{H_{i_j} : 1 \le j \le r\}$ is any finite set of elements of $\mathcal{M}$, there exists $H_k \in \mathcal{M}$ such that

$$H_k \le H_{i_1} \cap \dots \cap H_{i_r};$$

whence $i_1, \dots, i_r \le k$. Also $\psi_{ji}(g_iH_i) = g_iH_j = g_jH_j$ if $j \le i$; so $g_kH_k \subseteq g_{i_j}H_{i_j}$ for $j = 1, \dots, r$. Hence the set $\{g_iH_i : i \in J\}$ has the finite intersection property and since $(K, \mathcal{M})$ is compact it follows that

$$\cap\{g_iH_i : i \in J\} \ne \varnothing.$$

If g is an element of this intersection then it is clear that $\phi(g) = (g_iH_i)$. Also ϕ is a homeomorphism since $\phi(H_j) = L \cap \mathop{\mathrm{Cr}}_{i \ne j} G/H_i$, which is in the filter base for L. This proves the result.

We now show that a co-Černikov group can always be embedded as a dense subgroup of a compact co-Černikov group.

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COROLLARY 2.9. *Let $(K, \mathcal{M})$ be a co-Černikov group. Then K can be embedded as a dense subgroup of a pro-Černikov group.*

Proof. Let $\mathcal{M} = \{H_i : i \in J\}$ and let J be an index set ordered by

$$j \leq i \quad \Leftrightarrow \quad H_i \leq H_j \quad \text{for} \quad i, j \in J.$$

Put $L = \varprojlim K/H_i$. By 2.8, L together with a suitable separating filter base (which can easily be written down) is a compact co-Černikov group when, for each i, K/H_i is given its coset topology and L is given its natural topology. The map $\phi : K \to L$ given by $\phi(g) = (gH_i)$ (for $g \in K$) is certainly an embedding of K in L, since $\mathcal{M}$ is a separating filter base; so it suffices to prove $\phi(K)$ is dense in L.

Let U be a basic open subset of L so that $U = L \cap \operatorname*{Cr}_{i \in J} X_i$ with $X_i \subseteq_o K/H_i$ and, for all but finitely many i, $X_i = K/H_i$. To show that $\phi(K)$ is dense in L, we need to show that $U \cap \phi(K) = \operatorname*{Cr}_{i \in J} X_i \cap \phi(K) \neq \emptyset$. If $(g_i H_i) \in \operatorname*{Cr}_{i \in J} X_i \cap L$ and $i_1, \ldots, i_r$ are the indices for which $X_i \neq K/H_i$, there exists $k \in J$ such that

$$H_k \leq H_{i_1} \cap \ldots \cap H_{i_r}.$$

Then, for this k, $(g_k H_i) \in \phi(K) \cap \operatorname*{Cr}_{i \in J} X_i$. Finally note that $\phi(H_j) = \phi(K) \cap \operatorname*{Cr}_{i \neq j} G/H_i$; so ϕ is closed and continuous. The result follows.

Using the notation of 2.9, let $\phi_i : L \to K/H_i$ be the natural projection map. If ϕ is the embedding defined in 2.9, we can prove the following lemma in a similar manner to the proof of that corollary.

LEMMA 2.10. *For each $i \in J$, $\overline{\phi(H_i)} = \ker \phi_i$.*

A compact co-Černikov group $(L, \mathcal{P})$ containing a subgroup $(K, \mathcal{M})$ as a dense subspace will be called a *completion* of $(K, \mathcal{M})$. We shall prove in 2.20 a result analogous to Theorem 2.1 of [7]. Before doing this we give some further elementary properties of co-Černikov groups. We first give two obvious methods of constructing co-Černikov groups from a given co-Černikov group $(K, \mathcal{M})$. If $L \leq K$ let $L \cap \mathcal{M} = \{L \cap M : M \in \mathcal{M}\}$ and if $L \lhd K$ let $\mathcal{M}L/L = \{ML/L : M \in \mathcal{M}\}$.

PROPOSITION 2.11. *Suppose $(K, \mathcal{M})$ is a co-Černikov group and $L \leq K$. Then $(L, \mathcal{M} \cap L)$ is a co-Černikov group and the co-Černikov topology induced by $\mathcal{M} \cap L$ is the subspace topology.*

The proof is trivial and is omitted.

PROPOSITION 2.12. *If $(K, \mathcal{M})$ is a co-Černikov group and $L \lhd_c K$ then $(K/L, \mathcal{M}L/L)$ is a co-Černikov group.*

Proof. This follows since $\mathcal{M}L/L$ is a separating filter base for K/L by 2.3. The co-Černikov topology defined on K/L then has as a closed sub-base the set

$$\{(F/L)\cdot Lx : \text{there exists } M\in\mathcal{M} \text{ such that } ML\le F\}.$$

This completes the proof.

It is easy see that the co-Černikov topology defined on K/L in 2.12 is the quotient topology. Because of 2.11 and 2.12 one might ask whether the product topology on a cartesian product of co-Černikov groups yields a co-Černikov topology. An affirmative answer would give, together with 2.11, a direct proof of the necessity of 2.8. However the following easy example shows this is not true.

Let $K\cong C_{p^\infty}$, the unique infinite locally cyclic p-group. Let τ denote the product topology on $K\times K$, induced by the coset topology on K and let σ be the coset topology on $K\times K$. Then $\tau\neq\sigma$. For let $A=\{(a,a): a\in K\}$. A is certainly σ-closed but is not τ-closed. If that were the case there would exist subgroups B_i, C_i of K and elements x_i, $y_i\in K$ such that

$$A=\bigcap\left\{\left(\bigcup_{\text{finite}}(B_ix_i\times K)\right)\cup\left(\bigcup_{\text{finite}}(K\times C_iy_i)\right)\right\}.$$

It follows from [**13**, 4.4] that there exists i such that either $|A:A\cap(B_i\times K)|<\infty$ or $|A:A\cap(K\times C_i)|<\infty$. Since A is radicable it follows that either $A\le B_i\times K$ or $A\le K\times C_i$, for this i, and hence $A=K\times K$, a contradiction.

The following three results, although straightforward, are very important for the applications in Section 3 and [**3**].

Lemma 2.13. *Let $(K,\mathcal{M})$ be a co-Černikov group and suppose $L\lhd_c K$ with K/L a Černikov group. Then there exists $M\in\mathcal{M}$ such that $M\le L$.*

Proof. If $\mathcal{M}=\{H_i : i\in J\}$ then by 2.3

$$L=\bigcap_{i\in J} LH_i.$$

But K/L has the minimal condition on subgroups and so there are subgroups $H_1,\ldots,H_n\in\mathcal{M}$ such that $L=\bigcap_{i=1}^{n} LH_i$. Since $\mathcal{M}$ is a separating filter base there is an $M\in\mathcal{M}$ such that

$$M\le\bigcap_{i=1}^{n} H_i\le L,$$

as required.

Corollary 2.14. *If $(K,\mathcal{M})$ is a co-Černikov group, $L\lhd_c K$ with K/L Černikov and $L\le M\le K$ then $M\le_c K$.*

Proof. The proof is clear from 2.13 and the definition of the co-Černikov topology induced on K by $\mathcal{M}$.

LEMMA 2.15. *Let $(K, \mathcal{M})$ be a co-Černikov group and $L \leq K$. If $(L, L \cap \mathcal{M})$ is compact then $L \leq_c K$.*

Proof. By 2.3, $\bar{L} = \bigcap\{LM : M \in \mathcal{M}\}$. If $x \in \bar{L}$ then $x \in LM$ for each $M \in \mathcal{M}$. Hence $L \cap xM \neq \emptyset$ and $L \cap xM \subseteq_c L$ since L has the subspace topology. Moreover, since $\mathcal{M}$ is a separating filter base, $\{L \cap xM : M \in \mathcal{M}\}$ has the finite intersection property. Hence by the compactness of L,

$$\emptyset \neq \bigcap\{L \cap xM : M \in \mathcal{M}\} = L \cap \{x\}.$$

So we must have $x \in L$ and L is closed, as required.

Thus in a compact co-Černikov group, closed subgroups and compact subgroups are the same thing.

We shall now prove our generalisation of 2.1 of [7]. We give some preliminary results first, all of which are well known in the cofinite case.

LEMMA 2.16. *Let $(K, \mathcal{M})$ be a co-Černikov group and $L \leq_c K$. Then $N_K(L) \leq_c K$.*

Proof. If $M \in \mathcal{M}$, define $N_M/M = N_{K/M}(LM/M)$. We shall show

$$N_K(L) = \bigcap\{N_M : M \in \mathcal{M}\} \leq_c K.$$

If $x \in \bigcap\{N_M : M \in \mathcal{M}\}$ then $L^x M = LM$ for all $M \in \mathcal{M}$. Since $L, L^x \leq_c K$, 2.3 implies $L^x = L$. Thus $x \in N_K(L)$. The reverse inclusion is obvious.

LEMMA 2.17. *Suppose $(K, \mathcal{M})$ is a co-Černikov group.*

(i) *If $N \lhd_c K$ then the natural map $\alpha : (K, \mathcal{M}) \to (K/N, \mathcal{M}N/N)$ is continuous.*

(ii) *If $(L, \mathcal{L})$ is a co-Černikov group and $\alpha : (K, \mathcal{M}) \to (L, \mathcal{L})$ is a continuous epimorphism then given $M \lhd_c L$ with L/M Černikov there exists $N \lhd_c K$ with K/N Černikov and $\alpha(N) = M$.*

Proof. (i) This is clear from the definitions.

(ii) Let $\alpha^{-1}(M) = N$. Since α is continuous, $N \lhd_c K$. Since α is an epimorphism, $\alpha(N) = M$ and clearly K/N is a Černikov group.

LEMMA 2.18. *Suppose $(K, \mathcal{M})$ is a co-Černikov group and A, B are subsets of K. Then $\bar{A}\bar{B} \subseteq \overline{AB}$.*

Proof. For each $b \in B$, $Ab \subseteq AB$ and $\bar{A}b \subseteq \overline{AB}$. Thus $\bar{A}B \subseteq \overline{AB}$. Hence $\overline{\bar{A}B} \subseteq \overline{AB}$. Applying the first part of the argument to B and $\bar{A}$ now gives the result.

LEMMA 2.19. *Let $(K, \mathcal{M})$ be a co-Černikov group and $(\bar{K}, \mathcal{L})$ a completion of K (thus $\mathcal{L} \cap K$ induces the same topology on K as $\mathcal{M}$ does).*

(i) *If $\mathcal{P} = \{M \lhd_c K : K/M$ is Černikov$\}$ then*

$$\bar{\mathcal{P}} = \{\bar{M} : M \in \mathcal{P}\} = \{M \lhd_c \bar{K} : \bar{K}/M \text{ is Černikov}\}.$$

(ii) *If $M \in \mathcal{P}$ then $M = \bar{M} \cap K$.*

Proof. (i) Let $\mathcal{L} = \{L_i : i \in J\}$ be the separating filter base. If $M \lhd_c \bar{K}$ and $\bar{K}/M$ is

Černikov there exists $L_k \le M$ by 2.13. Let $\mathfrak{Q} = \{L_i \in \mathcal{L} : L_i \le M\}$. Then $\mathfrak{Q}$ is a separating filter base for M. For if $L_j \in \mathcal{L}$, $L_j \cap L_k$ contains some $L_i \in \mathcal{L}$. Thus $L_i \in \mathfrak{Q}$. If $x \in \cap\{L_m : L_m \in \mathfrak{Q}\}$ then $x \in L_i$ and hence $x \in L_j$. Thus $x \in \cap\{L_m : L_m \in \mathcal{L}\} = 1$ and hence $\mathfrak{Q}$ is a filter base.

Now $N = M \cap K \lhd_c K$ and K/N is Černikov. Also $\bar{N} \le M$. We show $\bar{N} = M$. For each $L_i \in \mathfrak{Q}$, $\bar{K} = KL_i$ by 2.4; so by the Dedekind law

$$M = L_i(M \cap K) = L_i N. \tag{1}$$

Since $\mathfrak{Q}$ is a separating filter base for M, $N \le_d M$ and hence $M = \bar{N}$.

Suppose now $N \lhd_c K$ and K/N is Černikov. Since $\bar{N} \le_c \bar{K}$, $N_{\bar{K}}(\bar{N}) \le_c \bar{K}$. Also if $g \in K$ then $N = N^g \le \bar{N}^g$. Hence $\bar{N} = \bar{N}^g$ and $K \le N_{\bar{K}}(\bar{N})$. It follows, since $K \le_d \bar{K}$, that $\bar{N} \lhd_c \bar{K}$. Moreover, K has the subspace topology; so there exists $i \in J$ such that $L_i \cap K \le N$ by 2.13. It follows from the argument used in showing (1) that

$$L_i = \overline{(L_i \cap K)} \le \bar{N}.$$

Thus $\bar{K}/\bar{N}$ is a Černikov group since $\bar{K}/L_i$ is, and (i) follows.

(ii) Clearly $N \le \bar{N} \cap K$. However K has the subspace topology; so there exists $C \le_c \bar{K}$ such that $C \cap K = N$. Hence $\bar{N} \le C$ and $\bar{N} \cap K \le C \cap K = N$. The result follows.

We can now prove the result we have been seeking. Our proof, as with much that has gone before, is similar to that of Hartley [**7**].

THEOREM 2.20. *Let $(K, \mathcal{M})$ be a co-Černikov group contained as a dense subgroup of the compact co-Černikov group $(\bar{K}, \mathcal{P})$. Let $(L, \mathcal{L})$ be any compact co-Černikov group and $\alpha : (K, \mathcal{M}) \to (L, \mathcal{L})$ a continuous homomorphism. Then:*

(i) *α can be uniquely extended to a continuous homomorphism $\bar{\alpha} : (\bar{K}, \mathcal{P}) \to (L, \mathcal{L})$;*

(ii) *$\bar{\alpha}(\bar{K}) = \overline{\alpha(K)}$;*

(iii) *$\bar{\alpha}$ is injective if and only if α is an algebraic and topological embedding and in that case $\bar{\alpha}$ is an algebraic and topological isomorphism between $\bar{K}$ and $\overline{\alpha(K)}$.*

Proof. To begin we prove that if $\bar{\alpha}$ is any continuous extension of α to $\bar{K}$ then $\bar{\alpha}(\bar{K}) = \overline{\alpha(K)}$. $\bar{K}$ is compact; so $\bar{\alpha}(\bar{K})$ is compact and hence is closed by 2.15. Thus $\overline{\alpha(K)} \le \bar{\alpha}(\bar{K})$. Conversely, note that $K \le \bar{\alpha}^{-1}(\overline{\alpha(K)})$ and since $\overline{\alpha(K)}$ is closed and $\bar{\alpha}$ is continuous, it follows that $\bar{K} \le \bar{\alpha}^{-1}(\overline{\alpha(K)})$. Thus $\bar{\alpha}(\bar{K}) \le \overline{\alpha(K)}$ and $\bar{\alpha}(\bar{K}) = \overline{\alpha(K)}$. Thus (ii) is established and we may clearly also assume $\overline{\alpha(K)} = L$, so $\alpha(K)$ is dense in L.

We now show the existence of at most one continuous extension $\bar{\alpha}$ of α from K to $\bar{K}$. Let $\mathfrak{Q}$ be the set of closed normal subgroups M of K such that K/M is Černikov. If $g \in \bar{K}$, $\bar{\alpha}(g) \in \bar{\alpha}(g\bar{M})$ for all $M \in \mathfrak{Q}$. Thus

$$\{\bar{\alpha}(g)\} \subseteq \bigcap_{M \in \mathfrak{Q}} \bar{\alpha}(g\bar{M}). \tag{2}$$

We show the right hand side of (2) has just a single point. For, if $x \in \bigcap_{M \in \mathfrak{Q}} \bar{\alpha}(g\bar{M})$ then

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$g \in \bigcap_{M \in \mathfrak{Q}} \bar{\alpha}^{-1}(x)\bar{M}$. Since $\bar{\alpha}$ is continuous and L is a T_1-space, 2.3 and 2.19 imply $g \in \bar{\alpha}^{-1}(x)$. Hence $x = \bar{\alpha}(g)$; so the right hand side of (2) has a single point.

Since $\bar{K} = K\bar{M}$, $g\bar{M} \cap K \neq \varnothing$ and if $x \in g\bar{M} \cap K$,

$$x\bar{M} \cap K = x(\bar{M} \cap K) = xM$$

by 2.19(ii). Thus $x\bar{M} = \overline{(g\bar{M} \cap K)}$. Also $g(\bar{M} \cap K) \subseteq \bar{\alpha}^{-1}\overline{(\alpha(g\bar{M} \cap K))}$ and as shown below $\bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}$ has a single point. Since $\bar{\alpha}(g\bar{M}) = \bar{\alpha}\overline{(g\bar{M} \cap K)} \subseteq \overline{\alpha(g\bar{M} \cap K)}$, it follows that

$$\{\bar{\alpha}(g)\} = \bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}. \tag{3}$$

We have now determined $\bar{\alpha}$ uniquely in terms of α. To show $\bar{\alpha}$ exists, we shall show that the right hand side of (3) is a single point for all $g \in \bar{K}$.

If $M \in \mathfrak{Q}$ then 2.19(i) and $K \leq_d \bar{K}$ imply $g\bar{M} \cap K \neq \varnothing$ for each $g \in \bar{K}$. The set $\{g\bar{M} \cap K : M \in \mathfrak{Q}\}$ therefore has the finite intersection property and hence so does the set $\{\overline{\alpha(g\bar{M} \cap K)} : M \in \mathfrak{Q}\}$. It follows, by the compactness of L, that

$$\bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)} \neq \varnothing.$$

Suppose $x, y \in \bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}$. Since $g\bar{M} \cap K \neq \varnothing$, there exists $h \in K$ such that

$$g\bar{M} \cap K = h\bar{M} \cap K = h(\bar{M} \cap K) = hM.$$

Thus $x, y \in \overline{\alpha(hM)} = \alpha(h)\overline{\alpha(M)}$, since α is a homomorphism. Hence $xy^{-1} \in \overline{\alpha(M)}$ for all $M \in \mathfrak{Q}$. However $\bigcap_{M \in \mathfrak{Q}} \overline{\alpha(M)} = 1$ by 2.17(ii) and 2.19(i). Hence $x = y$; so $\bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}$ has exactly one point, as required. Thus we define $\bar{\alpha} : (\bar{K}, \mathscr{P}) \to (L, \mathscr{L})$ by $\bar{\alpha}(g) = \bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}$, for each $g \in \bar{K}$.

If $g \in K$ then $\overline{\alpha(g\bar{M} \cap K)} = \overline{\alpha(gM)} = \alpha(g)\overline{\alpha(M)}$. Thus $\alpha(g) \in \bigcap_{M \in \mathfrak{Q}} \overline{\alpha(g\bar{M} \cap K)}$; so $\alpha(g) = \bar{\alpha}(g)$ and $\bar{\alpha}$ extends α.

We now show $\bar{\alpha}$ is a homomorphism. Suppose $g, h \in \bar{K}$ and $M \in \mathfrak{Q}$. Since α is a homomorphism, it is clear that

$$\alpha(g\bar{M} \cap K) \cdot \alpha(h\bar{M} \cap K) \subseteq \alpha(gh\bar{M} \cap K).$$

Hence, by 2.18,

$$\overline{\alpha(g\bar{M} \cap K)} \cdot \overline{\alpha(h\bar{M} \cap K)} \subseteq \overline{\alpha(gh\bar{M} \cap K)}.$$

Intersecting over all $M \in \mathfrak{Q}$ gives $\{\bar{\alpha}(g)\} \cdot \{\bar{\alpha}(h)\} \subseteq \{\bar{\alpha}(gh)\}$ and hence $\bar{\alpha}(gh) = \bar{\alpha}(g)\bar{\alpha}(h)$ as required.

Now we show $\bar{\alpha}$ is continuous. Since $\bar{\alpha}$ is a homomorphism and because of 2.17(ii), 2.19(i) and the definition of the co-Černikov topology on L, it is sufficient to show

$$\bar{\alpha}(\bar{M}) \leq \overline{\alpha(M)} \quad \text{for all} \quad M \in \mathfrak{Q}. \tag{4}$$

Suppose $g \in \bar{M}$, $N \in \mathcal{Q}$ and $N \leq M$. Then

$$\overline{\alpha(g\bar{N} \cap K)} \subseteq \overline{\alpha(\bar{M} \cap K)} = \overline{\alpha(M)}$$

by 2.19(ii). Intersecting over all such N, we obtain $\bar{\alpha}(g) \in \overline{\alpha(M)}$. Hence (4) follows.

If $\bar{\alpha}$ is an injection, it is a closed continuous bijection between the compact spaces $\bar{K}$ and $\overline{\alpha(K)}$ by (ii), and hence is a topological and algebraic isomorphism. Hence α is an algebraic and topological embedding.

Finally we suppose α is an algebraic and topological embedding. Let $g \in \ker \bar{\alpha}$. Then by definition of $\bar{\alpha}$,

$$1 \in \overline{\alpha(g\bar{M} \cap K)} \quad \text{for all} \quad M \in \mathcal{Q}.$$

Since α determines a topological isomorphism between K and $\alpha(K)$, $\alpha(g\bar{M} \cap K) \subseteq_c \alpha(K)$. Thus

$$\overline{\alpha(g\bar{M} \cap K)} \cap \alpha(K) = \alpha(g\bar{M} \cap K) \quad \text{for each} \quad M \in \mathcal{Q}.$$

Hence $1 \in \alpha(g\bar{M} \cap K)$ and since α is injective, $1 \in g\bar{M} \cap K$. Thus $g \in \bar{M}$ for each $M \in \mathcal{Q}$; so, intersecting over all such M and using 2.19(i), we find that $g = 1$ and $\bar{\alpha}$ is injective. This completes the proof.

3. Sylow theory in pro-Černikov groups. In this section we shall show that the classical theorems of Sylow and Hall in finite group theory can be extended to the class of pro-Černikov groups. Our approach uses many of the methods of J. Parker [**14**] and B. Hartley [**7**]. The main difference is that instead of using the theorem of Kuroš [**11**] on inverse limits of finite sets, we have been forced to use 1.1. This involves some technicalities in ensuring that the correct topologies are induced, but these are easily overcome.

To begin, we generalise the idea of a π-group. Let π be a set of primes. A co-Černikov group $(G, \mathcal{N})$ will be called a *generalised π-group* if G/N is a π-group (in the usual sense) for all $N \lhd_c G$ with G/N Černikov. This is analogous to the concept used in [**7**], although there the term "generalised" is omitted.

LEMMA 3.1. *Let $(G, \mathcal{N})$ be a co-Černikov group. Then $(G, \mathcal{N})$ is a generalised π-group if and only if G/N is a π-group for all $N \in \mathcal{N}$.*

Proof. This follows from the definitions and 2.13.

The idea of a generalised π-group certainly depends on the filter base involved. For example, if $\langle g \rangle$ is the infinite cyclic group it has the filter bases $\mathcal{M} = \{\langle g^{2^i} \rangle : i \geq 1\}$ and $\mathcal{N} = \{\langle g^{3^i} \rangle : i \geq 1\}$. However, $(\langle g \rangle, \mathcal{M})$ is a generalised 2-group and $(\langle g \rangle, \mathcal{N})$ is a generalised 3-group. We give several elementary properties of generalised π-groups. A generalised π-group $(H, \mathcal{M})$ that is a subgroup (with subspace topology) of a co-Černikov group $(G, \mathcal{N})$ will be called a *generalised π-subgroup* of $(G, \mathcal{N})$.

LEMMA 3.2. *Let* $(G, \mathcal{N})$ *be a co-Černikov group and* π *a set of primes.*

(i) *If* $(G, \mathcal{N})$ *is a generalised* π*-group and* $H \le G$ *then* $(H, H \cap \mathcal{N})$ *is a generalised* π*-group.*

(ii) *If* $(G, \mathcal{N})$ *is a generalised* π*-group and* $K \lhd_c H \le G$ *then* $(H/K, (H \cap \mathcal{N})K/K)$ *is a generalised* π*-group.*

(iii) *If* $\{(H_i, H_i \cap \mathcal{N}) : i \in I\}$ *is a set of generalised* π*-subgroups of* $(G, \mathcal{N})$ *totally ordered by inclusion, then* $\left(\bigcup_{i \in I} H_i, \left(\bigcup_{i \in I} H_i\right) \cap \mathcal{N}\right)$ *is a generalised* π*-subgroup of* $(G, \mathcal{N})$.

(iv) *If* $(H, H \cap \mathcal{N})$ *is a generalised* π*-subgroup of* $(G, \mathcal{N})$ *then so is* $(\bar{H}, \bar{H} \cap \mathcal{N})$.

(v) *The product of every set of normal generalised* π*-subgroups of* $(G, \mathcal{N})$ *is a normal generalised* π*-subgroup of* $(G, \mathcal{N})$.

Proof. (i) If $N \in \mathcal{N}$ then $H/H \cap N \cong HN/N \le G/N$, a π-group by hypothesis. Hence, since $(H, H \cap \mathcal{N})$ is a co-Černikov group, it is a generalised π-group by 3.1.

(ii) The proof is similar to (i) using 2.12 and 3.1.

(iii) Suppose $L \lhd_c \bigcup_{i \in I} H_i = M$ say, with M/L a Černikov group. For each $i \in I$, $L \cap H_i \lhd_c H_i$ by 2.11. Moreover, $H_i/H_i \cap L$ is a Černikov π-group by hypothesis. Hence M/L is the union of the ascending chain of π-subgroups H_iL/L and so is a π-group. Thus $\left(\bigcup_{i \in I} H_i, \left(\bigcup_{i \in I} H_i\right) \cap \mathcal{N}\right)$ is a generalised π-subgroup of $(G, \mathcal{N})$.

(iv) By 2.3, $\bar{H} \le G$. Suppose $N \lhd_c \bar{H}$ and $\bar{H}/N$ is a Černikov group. Then $H \cap N \lhd_c H$ and $H/H \cap N$ is a Černikov π-group by hypothesis. Since the closure of H in $\bar{H}$ is precisely $\bar{H}$, 2.4 and 2.13 imply $\bar{H} = HN$; so $\bar{H}/N$ is a π-group. Hence $(\bar{H}, \bar{H} \cap \mathcal{N})$ is a generalised π-subgroup of $(G, \mathcal{N})$.

(v) It suffices to show that if $(L, L \cap \mathcal{N})$ and $(M, M \cap \mathcal{N})$ are normal generalised π-subgroups of $(G, \mathcal{N})$ then so is $(LM, LM \cap \mathcal{N})$. If $N \in \mathcal{N}$ then $L/L \cap N$ and $M/M \cap N$ are π-groups. Hence $(LM \cap LN)/LM \cap N$ and $(LM \cap MN)/LM \cap N$ are π-groups. Thus their product, $LM/LM \cap N$, is also a π-group and the result follows by 3.1.

A subgroup P of a co-Černikov group $(G, \mathcal{N})$ will be called a *generalised Sylow* π*-subgroup* of $(G, \mathcal{N})$ if

(i) $P \le_c G$,

(ii) $PN/N \in \mathrm{Syl}_\pi G/N$ for all $N \lhd_c G$ with G/N Černikov.

We shall denote the set of generalised Sylow π-subgroups of $(G, \mathcal{N})$ by $\mathrm{Syl}_\pi(G, \mathcal{N})$. It is not immediately clear that a co-Černikov group possesses even generalised Sylow p-subgroups. However we shall show that a pro-Černikov group does possess them.

By 3.2(iii) and Zorn's lemma, the co-Černikov group $(G, \mathcal{N})$ contains maximal generalised π-subgroups and these subgroups are closed by 3.2(iv). Let $\mathrm{Max}_\pi(G, \mathcal{N})$ denote the set of maximal generalised π-subgroups of $(G, \mathcal{N})$. In prosoluble groups the concept of a generalised Sylow π-subgroup and a maximal generalised π-subgroup are the same [**7**, Lemma 6.1]. We at least have the following lemma.

LEMMA 3.3. *Let* $(G, \mathcal{N})$ *be a co-Černikov group. Then* $\mathrm{Syl}_\pi(G, \mathcal{N}) \subseteq \mathrm{Max}_\pi(G, \mathcal{N})$.

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Proof. Suppose $P \in \mathrm{Syl}_\pi(G, \mathcal{N})$ and $P \leq Q \in \mathrm{Max}_\pi(G, \mathcal{N})$. Then $P, Q \leq_c G$ and $PN \leq QN$ for all $N \in \mathcal{N}$. Since $PN/N \in \mathrm{Syl}_\pi G/N$ and QN/N is a π-group, we must have $PN = QN$ for all $N \in \mathcal{N}$. Hence, by 2.3.

$$P = \cap\{PN : N \in \mathcal{N}\} = \cap\{QN : N \in \mathcal{N}\} = Q,$$

and the result holds.

Equality does not generally hold in 3.3. For an example of this, the reader should consult [**10**, Theorem 2.1]. The example constructed there is a countable, metabelian, residually finite group G with a normal Sylow 3-subgroup T and a Sylow 2-subgroup S with the property that $|G : ST| = 2$. Thus S is not a generalised Sylow 2-subgroup of G. A similar example, due to B. Hartley, has occurred in [**3**, (4.13)]. As mentioned above, it has been shown that $\mathrm{Max}_\pi(G, \mathcal{N}) = \mathrm{Syl}_\pi(G, \mathcal{N})$ for prosoluble groups. Unfortunately the proof of that result does not readily extend to our situation, even when all the Černikov factor groups are soluble.

A locally finite group G is *Sylow π-integrated* (for some set of primes π) if the Sylow π-subgroups of every subgroup of G are conjugate. For co-Černikov groups whose Černikov factor groups are Sylow π-integrated it is easier to check that a given subgroup is a generalised Sylow π-subgroup, as the following result shows.

Lemma 3.4. *Suppose $(G, \mathcal{N})$ is a co-Černikov group and π is a set of primes. Suppose that for each $N \in \mathcal{N}$, G/N is Sylow π-integrated. Then $P \in \mathrm{Syl}_\pi(G, \mathcal{N})$ if and only if*
(i) *$P \leq_c G$,*
(ii) *$PN/N \in \mathrm{Syl}_\pi G/N$ for each $N \in \mathcal{N}$.*

Proof. We suppose (i) and (ii) hold and that $M \lhd_c G$ with G/M a Černikov group. By 2.13 there exists $N \in \mathcal{N}$ such that $N \leq M$; so, by (ii), $P_N = PN/N \in \mathrm{Syl}_\pi G/N$. Now $G_N = G/N$ is Sylow π-integrated and $G_M = G/M \cong G_N/H$, where $H = M/N$. Thus if $Q/H \in \mathrm{Syl}_\pi G_N/H$ there exists $g \in G_N$ such that $Q = HP_N^g$ (by [**6**, Lemma 2.1], for example). Hence $P_N H/H \in \mathrm{Syl}_\pi G_N/H$ and $PM/M \in \mathrm{Syl}_\pi G_M$ (since $P_N H/H \cong PM/M$) as required. The reverse implication is clear, so the result follows.

Of course, no restrictions are needed if $\pi = \{p\}$, a single prime, since a Černikov group is always Sylow p-integrated.

To generalise the results of Sylow and Hall we require some preliminary results.

Lemma 3.5. *Suppose G is a Černikov group and $H \leq G$. Let G have the coset topology and let G/H denote the space of cosets of H in G with the quotient topology. Then the natural map $\alpha : G \to G/H$ is closed and continuous.*

Proof. The map α is certainly continuous from the definition of the quotient topology on G/H. By 2.1 every closed set in G is a finite union of cosets of G; so we need only show that if $K \leq G$ then $\alpha(xK)$ is closed in G/H for each $x \in G$.

Now $\alpha(xK) = \{xkH : k \in K\}$ and hence $\alpha^{-1}(\alpha(xK)) = xKH$. Of course, KH need not be a subgroup but K and H possess radicable parts which we denote by K^0 and H^0 respectively. Also K^0H^0 is a subgroup since $K^0, H^0 \leq G^0$, the radicable part of G. If

$\{x_i\}_{i=1}^m$ is a left transversal to K^0 in K and $\{y_j\}_{j=1}^n$ is a right transversal to H^0 in H then $KH = \bigcup_{i,j} x_i K^0 H^0 y_j$; so $xKH = \bigcup_{i,j} xx_i K^0 H^0 y_j$. This set is closed in G by 2.2(iv) and the definition of the coset topology. It follows from the definition of the quotient topology that α is a closed map.

Corollary 3.6. *Suppose G is a Černikov group and $H \le K \le G$. If G/K and G/H denote, respectively, the quotient spaces of cosets of K and H in G, induced by the coset topology on G, then the natural map $\beta: G/H \to G/K$ is closed and continuous.*

Proof. Let $\alpha_H : G \to G/H$ and $\alpha_K : G \to G/K$ be the natural maps. By 3.5 these maps are closed and continuous and since $\beta \circ \alpha_H = \alpha_K$ it follows that β is closed and continuous.

The following lemma is an extension of [**7**, Lemma 6.2] and is very useful in what follows.

Lemma 3.7. *Let $(G, \mathcal{N})$ be a compact co-Černikov group and suppose that for each $N \in \mathcal{N}$, $X(N)$ is a closed set with the property*:

$$\text{if } M, N \in \mathcal{N} \text{ and } M \le N \text{ then } X(M)N = X(N). \qquad (*)$$

Let $X = \cap\{X(N) : N \in \mathcal{N}\}$. Then, for all $N \in \mathcal{N}$, $XN = X(N)$.

Proof. It is clear that $XN \subseteq X(N)$, for each $N \in \mathcal{N}$. Let $N \in \mathcal{N}$ be fixed. If $x \in X(N)$ and $M \in \mathcal{N}$ is such that $M \le N$ then $x \in X(M)N$ by $(*)$. Hence $xN \cap X(M) \ne \varnothing$. If $M_1, \ldots, M_r \le N$ with $M_i \in \mathcal{N}$ then there exists $M_{r+1} \in \mathcal{N}$ such that $M_{r+1} \le M_1 \cap \ldots \cap M_r$. By $(*)$, $X(M_{r+1})M_i = X(M_i)$ for $i = 1, \ldots, r$. Therefore,

$$\varnothing \ne xN \cap X(M_{r+1}) \subseteq \bigcap_{i=1}^{r} (xN \cap X(M_i)).$$

If $\mathcal{M} = \{M \in \mathcal{N} : M \le N\}$ then $\{xN \cap X(M) : M \in \mathcal{M}\}$ is a set of closed subsets of $(G, \mathcal{N})$ with the finite intersection property. Since G is compact, there exists $y \in G$ such that

$$y \in xN \cap (\cap\{X(M) : M \in \mathcal{M}\}) = xN \cap (\cap\{X(M) : M \in \mathcal{N}\}) = xN \cap X.$$

Hence $x \in XN$ and this proves the result.

We now give our extension of Sylow's theorem.

Theorem 3.8. *Let $(G, \mathcal{N})$ be a compact co-Černikov group. Then G possesses generalised Sylow p-subgroups for each prime p.*

Proof. Let p be a fixed prime and for each $N \in \mathcal{N}$ let $A(N) = \{$Sylow p-subgroups of $G/N\}$. Then $A(N) \ne \varnothing$ and the elements of $A(N)$ are conjugate since the Sylow p-subgroups of a Černikov group are conjugate.

For $N \in \mathcal{N}$, let $G_N = G/N$, give G_N its coset topology and let $P_N \in A(N)$. By the previous remark, we may put the elements of $A(N)$ in 1–1 correspondence with the cosets of $N_{G_N}(P_N)$ in G_N. Suppose $N \le M$ and $N, M \in \mathcal{N}$. Since the Sylow p-subgroups of a Černikov group are homomorphism invariant, the natural map $\alpha_{MN} : G_N \to G_M$ induces a map $\beta_{MN}: A(N) \to A(M)$. Suppose $\beta_{MN}(P_N) = P_M$ and put $G_N^* = G_N/N_{G_N}(P_N)$ as a topological space with the quotient topology. We define a map $\beta_{MN}^*: G_N^* \to G_M^*$ by:

$$\text{if } g \in G \text{ then } \beta_{MN}^*((gN)N_{G_N}(P_N)) = (gM)N_{G_M}(P_M).$$

This map is well defined since $\beta_{MN}(P_N) = P_M$. If $\gamma_N: G_N \to G_N^*$ is the natural map then clearly $\gamma_M \circ \alpha_{MN} = \beta_{MN}^* \circ \gamma_N$. Thus, as in 3.5, β_{MN}^* is closed and continuous. Hence, if $A(M)$ and $A(N)$ are given the topologies induced from G_M^* and G_N^* respectively, the map β_{MN} is closed and continuous. Since G_N is compact and T_1, 3.5 implies that G_N^* is compact and T_1 and hence $A(N)$, with its induced topology, is compact and T_1 for each $N \in \mathcal{N}$. Our aim is to eventually use 1.1 applied to the sets $A(N)$. However, it is first necessary to check that we can choose the representatives $P_N \in A(N)$ consistently and to do this it suffices to show that if $P, Q \in A(N)$ then the topologies induced on $A(N)$ by $G_N/N_{G_N}(P)$ and $G_N/N_{G_N}(Q)$ are the same. So let τ and σ be the topologies induced by $G_N/N_{G_N}(P)$ and $G_N/N_{G_N}(Q)$ on $A(N)$, respectively. We identify an element $P^h \in (A(N), \tau)$ with the right coset $N_{G_N}(P)h$ (where $h \in G_N$). Since P and Q are conjugate in G_N, there exists $g \in G_N$ such that $P^g = Q$.

Let $\{P^{h_i} : i \in J\}$ be a closed subset of $(A(N), \tau)$ for some index set J. Then, by definition, $\bigcup_{i \in J} N_{G_N}(P)h_i \subseteq_c G_N$. Hence, by 2.2(iv),

$$\bigcup_{i \in J} N_{G_N}(Q)g^{-1}h_i = g^{-1}\left(\bigcup_{i \in J} N_{G_N}(P)h_i\right) \subseteq_c G_N.$$

Therefore $\{Q^{g^{-1}h_i} : i \in J\} \subseteq_c (A(N), \sigma)$; whence $\{P^{h_i} : i \in J\} \subseteq_c (A(N), \sigma)$ and $\tau \subseteq \sigma$. It follows by symmetry that $\tau = \sigma$ and consequently the topologies induced on $A(N)$ are the same.

The hypotheses of 1.1 are now satisfied for the inverse system $\{A(N), \beta_{MN} : M, N \in \mathcal{N}\}$; so, by 1.1(a), $\varprojlim A(N) \neq \emptyset$.

Let $(P_N) = (Q_N/N) \in \varprojlim A(N)$. If $N \leq M$, $\beta_{MN}(P_N) = P_M$ and $Q_N M = Q_M$. Put $P = \bigcap\{Q_N : N \in \mathcal{N}\}$. Then $P \leq_c G$ since $Q_N \leq_c G$ by definition. By 3.7, $PN = Q_N$ for all $N \in \mathcal{N}$ and hence $PN/N \in \mathrm{Syl}_p G/N$. By 3.4 and the remark following it, $P \in \mathrm{Syl}_p(G, \mathcal{N})$ and this completes the proof.

Let $\mathfrak{W}$ denote the class of co-Černikov groups $(G, \mathcal{N})$ with the property that if $N \in \mathcal{N}$ then G/N is soluble. The above proof then yields the next theorem.

THEOREM 3.9. *Let $(G, \mathcal{N})$ be a compact $\mathfrak{W}$-group. Then G possesses generalised Sylow π-subgroups for all sets of primes π.*

We now obtain the conjugacy of the various generalised Sylow subgroups.

THEOREM 3.10. *Let $(G, \mathcal{N})$ be a compact co-Černikov group. Then:*

(i) *the generalised Sylow p-subgroups of $(G, \mathcal{N})$ are conjugate for all primes p;*

(ii) *if $(G, \mathcal{N}) \in \mathfrak{W}$ then the generalised Sylow π-subgroups of $(G, \mathcal{N})$ are conjugate, for all sets of primes π.*

Proof. Since the proofs of (i) and (ii) are essentially the same, we merely give the proof of (i).

Let $P, Q \in \mathrm{Syl}_p(G, \mathcal{N})$ so that, for each $N \in \mathcal{N}$, PN/N, $QN/N \in \mathrm{Syl}_p G/N$. Put $X(N) = \{g \in G : P^g N = QN\}$. Then $X(N) \neq \emptyset$ since the Sylow p-subgroups of a Černikov group are conjugate. Now

$$g, h \in X(N) \quad \Rightarrow \quad P^g N = P^h N \quad \Rightarrow \quad gh^{-1} \in N_G(PN).$$

Hence $X(N) = N_G(PN)g$ and so $X(N)$ is closed in $(G, \mathcal{N})$ by 2.5(iii) and the definition of the co-Černikov topology induced by $\mathcal{N}$. Moreover, if $M, N \in \mathcal{N}$ then $X(M \cap N) \subseteq X(M) \cap X(N)$ and so the set $\{X(N) : N \in \mathcal{N}\}$ has the finite intersection property. Since $(G, \mathcal{N})$ is compact, $\cap\{X(N) : N \in \mathcal{N}\}$ contains an element g. Hence if $N \in \mathcal{N}$, $P^g N = QN$ and since P^g, $Q \leq_c G$, 2.3 implies

$$P^g = \cap\{P^g N : N \in \mathcal{N}\} = \cap\{QN : N \in \mathcal{N}\} = Q;$$

so P and Q are conjugate. This completes the proof.

The above method of proof is of course well known in the prosoluble group case.

By a *generalised Sylow basis* of a co-Černikov group $(G, \mathcal{N})$ we shall mean a complete set of generalised Sylow p-subgroups, one for each prime p, with the property that if π is a set of primes then $\langle S_p : p \in \pi \rangle$ is a generalised π-group. This is a somewhat more general definition than that given by Parker [**14**], although in the prosoluble case our definition and that of Parker coincide.

It is possible to prove the next theorem in a similar manner to 3.8.

THEOREM 3.11. *Let $(G, \mathcal{N})$ be a compact $\mathfrak{W}$-group. Then $(G, \mathcal{N})$ possesses generalised Sylow bases.*

The next theorem completes our survey of Hall's results.

THEOREM 3.12. *Let $(G, \mathcal{N})$ be a compact $\mathfrak{W}$-group. Then the generalised Sylow bases of $(G, \mathcal{N})$ are conjugate.*

Proof. Let $\mathbf{S} = \{S_p\}$ and $\mathbf{T} = \{T_p\}$ be generalised Sylow bases and for $N \in \mathcal{N}$ set

$$X(N) = \{g \in G : S_p^g N = T_p N \text{ for all primes } p\}.$$

Because $(G, \mathcal{N}) \in \mathfrak{W}$, a result of Gol'berg [**5**] shows that the Sylow bases of G/N are conjugate; so $X(N) \neq \emptyset$. Moreover if $g \in X(N)$ then

$$X(N) = \bigcap_{p \in \mathbb{P}} N_G(S_p N)g;$$

so $X(N)$ is closed in $(G, \mathcal{N})$ and the sets $X(N)$ are easily seen to have the finite intersection property, as in the proof of 3.10. The result then follows easily.

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REFERENCES

1. R. Baer, Lokal endlich-auflösbare Gruppen mit endlichen Sylowuntergruppen, *J. Reine Angew. Math.* **239/240** (1969), 109–144.

2. R. M. Bryant, The verbal topology of a group, *J. Algebra* **48** (1977), 340–346.

3. M. R. Dixon, *Formation theory in a class of locally finite groups*, Ph.D. thesis, University of Warwick (1979).

4. M. R. Dixon and M. J. Tomkinson, The local conjugacy of some Sylow bases in a class of locally finite groups, *J. London Math. Soc.* (2) **21** (1980), 225–228.

5. P. A. Gol'berg, Sylow bases of infinite groups, *Mat. Sb.* **32** (1953), 465–476.

6. B. Hartley, Sylow subgroups of locally finite groups, *Proc. London Math. Soc.* (3) **23** (1971), 159–192.

7. B. Hartley, Profinite and residually finite groups, *Rocky Mountain J. Math.* **7** (1977), 193–217.

8. P. Higgins, *An introduction to topological groups*, London Math. Soc. Lecture Notes Series **15** (Cambridge University Press, 1974).

9. O. H. Kegel and B. A. F. Wehrfritz, *Locally finite groups* (North-Holland, 1973).

10. L. G. Kovács, B. H. Neumann and H. De Vries, Some Sylow subgroups, *Proc. Roy. Soc. Ser. A* **260** (1961), 304–316.

11. A. G. Kuroš, *Theory of groups*, Vol. 2 (Chelsea, 1956).

12. N. Massey, Locally finite groups with min-p for all p. I, *J. London Math. Soc.* (2) **12** (1975), 7–14.

13. B. H. Neumann, Groups covered by permutable subsets, *J. London Math. Soc.* **29** (1954), 236–248.

14. J. R. Parker, *A topological approach to a class of residually finite groups*, Ph.D. thesis, University of Warwick (1973).

15. G. F. Simmons, *Introduction to topology and modern analysis* (McGraw-Hill, 1963).

16. I. Stewart, Conjugacy theorems for a class of locally finite Lie algebras, *Compositio Math.* **30** (1975), 181–210.

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