Canad. Math. Bull. Vol. **30** (2), 1987

A SYSTEM OF OPERATOR EQUATIONS

BY
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ABSTRACT. Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ and $A_i, B_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \ldots, r$. It is shown that if no nontrivial linear combination of the operators A_i is compact, then there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that $XA_iY = B_i$ for all i. A related (but much milder) result is discussed in other algebras with the unique maximal ideal and an application to elementary operators is given.

1. **Introduction and the main results**. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The main result of this note is

THEOREM 1. *Let A_i, B_i be elements of $\mathcal{B}(\mathcal{H})$, $i = 1, \ldots, r$ (where r is a positive integer). If no nontrivial linear combination of the operators A_i is compact, then there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that*

$$XA_iY = B_i, \; i = 1, \ldots, r. \tag{1.1}$$

This theorem is a little surprising, since the system (1.1) can consist of many equations with only two unknowns X and Y. In a much milder form this theorem holds for more general algebras then $\mathcal{B}(\mathcal{H})$, as will be shown by the following algebraic considerations.

Let $\mathcal{A}$ be a unital algebra over some field $\mathcal{F}$. For each $A \in \mathcal{A}$ the left and the right multiplication by A are linear operators on $\mathcal{A}$ defined by $L_A(X) = AX$ and $R_A(X) = XA$ respectively, for all $X \in \mathcal{A}$. For each $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{A}^r$ and $\boldsymbol{B} = (B_1, \ldots, B_r) \in \mathcal{A}^r$ the elementary operator $E_{\boldsymbol{AB}}$ is defined by

$$E_{\boldsymbol{AB}} = \sum_{i=1}^{r} L_{A_i} R_{B_i} \tag{1.2}$$

(In the past such operators have been vigorously studied; see e.g. the bibliography in [4].) The set of all elementary operators on $\mathcal{A}$, $\mathcal{E}(\mathcal{A})$, is obviously an $\mathcal{F}$-algebra (often called the multiplication algebra of $\mathcal{A}$ [8]). The algebra $\mathcal{A}$ itself can be regarded as an $\mathcal{E}(\mathcal{A})$-module in an obvious way, the submodules of which are precisely the two-sided ideals of $\mathcal{A}$. Thus, if $\mathcal{A}$ contains only one maximal ideal $\mathcal{K}$ (as is the case if $\mathcal{A} = \mathcal{B}(\mathcal{H})$), then $\mathcal{K}$ is the only maximal submodule of the $\mathcal{E}(\mathcal{A})$-module $\mathcal{A}$. For such modules the following variant of the classical Jacobson density theorem can be proved.

Received by the editors August 12, 1985.
Math. Subj. Class. (1980): Primary 47A62, Secondary 16A53.

THEOREM 2. *Let $\mathcal{R}$ be a ring and $\mathcal{M}$ an $\mathcal{R}$-module. Suppose that $\mathcal{M}$ contains a proper submodule $\mathcal{K}$ such that every proper submodule of $\mathcal{M}$ is contained in $\mathcal{K}$ and that $\mathcal{R}(\mathcal{M}/\mathcal{K}) \neq 0$. Let $x_i, y_i \in \mathcal{M}$, $i = 1, \ldots, r$, and assume that the cosets $x_i + \mathcal{K} \in \mathcal{M}/\mathcal{K}$ are linearly independent over the division ring* $\mathrm{End}_{\mathcal{R}}(\mathcal{M}/\mathcal{K})$. *Then there exists an $a \in \mathcal{R}$ such that $ax_i = y_i$ for all i.*

Note that by the lemma of Schur [9] the ring $\mathrm{End}_{\mathcal{R}}(\mathcal{M}/\mathcal{K})$ is indeed a division ring, since $\mathcal{M}/\mathcal{K}$ is a simple module (does not contain any proper non-zero submodule). The Jacobson density theorem for simple modules is a special case of theorem 2, when $\mathcal{K} = 0$. The proof of theorem 2 is essentially the same as the proof of the classical Jacobson density theorem (see [9, p. 221, Exercise 1]) and will be given in the appendix only for the sake of completeness. Here we state a consequence of theorem 2, which is closely related to theorem 1.

COROLLARY 1. *Let $\mathcal{A}$ be an algebra with unit over some field $\mathcal{F}$. Suppose that $\mathcal{K}$ is the only maximal ideal of $\mathcal{A}$ and let $\mathcal{Z}$ be the centre of the algebra $\mathcal{A}/\mathcal{K}$. Let A_i, $B_i \in \mathcal{A}$, $i = 1, \ldots, r$. If the cosets $A_i + \mathcal{K} \in \mathcal{A}/\mathcal{K}$ are linearly independent over $\mathcal{Z}$, then there exists a positive integer m and X_j, $Y_j \in \mathcal{A}$* for *$j = 1, \ldots, m$, such that*

$$\sum_{j=1}^{m} X_j A_i Y_j = B_i,\ i = 1, \ldots, r$$

To prove the corollary, just apply theorem 2 to the $\mathcal{E}(\mathcal{A})$-module $\mathcal{A}$ and note that the division ring $\mathrm{End}(\mathcal{A}/\mathcal{K})$ can be naturally identified with the commutative ring $\mathcal{Z}$. (Indeed, it is well known and easy to see that the map $\mathrm{End}(\mathcal{A}/\mathcal{K}) \rightarrow \mathcal{A}/\mathcal{K}$, $\lambda \rightarrow \lambda(1)$, induces an isomorphism of $\mathrm{End}(\mathcal{A}/\mathcal{K})$ onto the centre $\mathcal{Z}$ of $\mathcal{A}/\mathcal{K}$.)

Note that if $\mathcal{A}$ is a complex normed algebra satisfying the hypothesis of corollary 1, then $\mathcal{Z} = \mathbb{C} \cdot 1$, since $\mathbb{C}$ is (up to an isomorphism) the only complex normed division algebra [3, p. 23].

Several important operator algebras satisfy the hypothesis of corollary 1; for example, the algebras of all bounded operators on the Banach spaces c_0 and l^p $(1 \leq p < \infty)$ [3, p. 95], the algebra $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is not necessarily separable Hilbert space [7], and the von Neumann factors [10, p. 350].

Corollary 1 can be used to generalize some results of [6] and [1] about elementary operators.

COROLLARY 2. *Suppose that $\mathcal{A}$, $\mathcal{K}$, $\mathcal{Z}$ are as in corollary 1. Let $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{A}^r$, $\boldsymbol{B} = (B_1, \ldots, B_r) \in \mathcal{A}^r$ and let $E_{\boldsymbol{AB}}$ be the elementary operator on $\mathcal{A}$ (defined by (1.2)). Assume that the elements $A_i + \mathcal{K}$ of $\mathcal{A}/\mathcal{K}$ are linearly independent over the centre $\mathcal{Z}$ of $\mathcal{A}/\mathcal{K}$. Then: (i) For arbitrary two sided ideal $\mathcal{J}$ of $\mathcal{A}$ the range of the elementary operator $E_{\boldsymbol{AB}}$ is a subset of $\mathcal{J}$ if and only if $B_i \in \mathcal{J}$ for all $i = 1, \ldots, r$.*

(ii) If $E_{\boldsymbol{AB}}$ is an element of some proper ideal $\mathcal{J}$ of the algebra $\mathcal{E}(\mathcal{A})$, then $B_i \in \mathcal{K}$ for all $i = 1, \ldots, r$.

PROOF. (i) If $B_i \in \mathcal{J}$ for all i, then clearly the range of $E_{\boldsymbol{AB}}$ is a subset of $\mathcal{J}$, since $\mathcal{J}$ is an ideal in $\mathcal{A}$. To prove the converse, note first that by corollary 1 there exist X_j,

$Y_j \in \mathcal{A}$ such that

$$\sum_{j=1}^{m} X_j A_i Y_j = \delta_{1i} \cdot 1,\ i = 1, \ldots, r$$

Now we have

$$B_1 = \sum_{i=1}^{r} \left(\sum_{j=1}^{m} X_j A_i Y_j \right) B_i = \sum_{j=1}^{m} X_j \left(\sum_{i=1}^{r} A_i Y_j B_i \right) = \sum_{j=1}^{r} X_j E_{\boldsymbol{AB}}(Y_j)$$

From this we see that $B_1 \in \mathcal{J}$ if the range of E_{AB} is a subset of $\mathcal{J}$. The proof that $B_i \in \mathcal{J}$ for $i = 2, \ldots, r$ is the same.

(ii) Suppose that $E_{\boldsymbol{AB}} \in \mathcal{J}$ for some proper ideal $\mathcal{J}$ of $\mathcal{E}(\mathcal{A})$ and let X_j, $Y_j \in \mathcal{A}$, $j = 1, \ldots, m$, be chosen as in the proof of (i). Then an easy computation shows that

$$\sum_{j=1}^{m} L_{X_j} E_{\boldsymbol{AB}} L_{Y_j} = R_{B_1}$$

This implies that $R_{B_1} \in \mathcal{J}$. If B_1 were not an element of $\mathcal{K}$, then the same argument (but with the right multiplications instead of the left ones) would show that the identity operator I is an element of $\mathcal{J}$. Since $\mathcal{J}$ is a proper ideal, $I \notin \mathcal{J}$, hence $B_1 \in \mathcal{K}$. In the same way it can be shown that $B_i \in \mathcal{K}$ for $i = 2, \ldots, r$. //

REMARKS. In the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ (= the ideal of compact operators on $\mathcal{H}$) corollary 2(i) was proved by Fong and Sourour in [6]. Apostol and Fialkow proved in [1] corollary 2(i) for the general ideal $\mathcal{J}$ in $\mathcal{B}(\mathcal{H})$. The general question, when is the range of an elementary operator contained in a fixed ideal of $\mathcal{B}(\mathcal{H})$ (if A_i are not linearly independent modulo $\mathcal{K}(\mathcal{H})$), seems to be still open, except in some special cases considered in [4].

Corollary 2(ii) shows in particular that for a simple algebra $\mathcal{A}$ the algebra $\mathcal{E}(\mathcal{A})$ is also simple (since $\mathcal{K} = 0$ in this case). This observation applies for example to the Calkin algebra [2]. In particular there are no non-zero compact elementary operators on the Calkin algebra. This last statement was conjectured in [6] and proved in [1] using the well known theorem of Voiculescu.

Theorem 1 will be proved in section 3, while section 2 contains the necessary preliminary result.

2. **Linear independence modulo compact operators**. From now on let $\mathcal{H}$, $\mathcal{L}$ be separable Hilbert spaces, $\mathcal{B}(\mathcal{L}, \mathcal{H})$ the vector space of all bounded linear operators from $\mathcal{L}$ to $\mathcal{H}$, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ the Calkin algebra [2], [3].

For every $A \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ the minimum modulus $m(A)$ is defined by

$$m(A) = \inf\{\|Ax\|;\ x \in L,\ \|x\| = 1\}$$

For each $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ and each $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{B}(\mathcal{L}, \mathcal{H})^r$ denote

$$\lambda \cdot \boldsymbol{A} = \sum_{i=1}^{r} \lambda_i A_i$$

https://doi.org/10.4153/CMB-1987-029-2 Published online by Cambridge University Press

The unit sphere in $\mathbb{C}^r$ is denoted by S_r. For each compact subset S of $\mathbb{C}^r$ and each $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{B}(\mathcal{L}, \mathcal{H})^r$ let

$$d(\boldsymbol{A}; S) = \min\{m(\lambda \cdot \boldsymbol{A}); \lambda \in S\}$$

In the case $S = S_r$ we shall write simply $d(A)$ instead of $d(A; S_r)$. Finally, if $\mathcal{L}$ is a subspace of $\mathcal{H}$ and $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{B}(\mathcal{H})^r$, we denote

$$\boldsymbol{A} \,|\, \mathcal{L} = (A_1 \,|\, \mathcal{L}, \ldots, A_r \,|\, \mathcal{L})$$

(Thus $A \,|\, \mathcal{L} \in \mathcal{B}(\mathcal{L}, \mathcal{H})^r$.) The next proposition will be needed in the proof of theorem 1.

PROPOSITION 1. *If $\boldsymbol{A} = (A_1, \ldots, A_r) \in \mathcal{B}(\mathcal{H})^r$ is such that $\lambda \cdot \boldsymbol{A} \notin \mathcal{K}(\mathcal{H})$ holds for every $\lambda \in \mathbb{C}^r - \{0\}$, then there exists a closed infinite dimensional subspace $\mathcal{L}$ of $\mathcal{H}$ such that $d(\boldsymbol{A} \,|\, \mathcal{L}) > 0$.*

REMARK. If $\lambda \cdot \boldsymbol{A} \notin \mathcal{K}(\mathcal{H})$, then it is well known that there exists an infinite dimensional subspace $\mathcal{L}_\lambda$ of $\mathcal{H}$ such that the operator $\lambda \cdot \boldsymbol{A} \,|\, \mathcal{L}_\lambda$ is bounded below. But the proposition claims more: there is a subspace $\mathcal{L}$ of $\mathcal{H}$ such that all the operators $\lambda \cdot \boldsymbol{A} \,|\, \mathcal{L}$ are bounded below for $\lambda \in \mathbb{C}^r - \{0\}$.

In the proof of proposition 1 a few facts that are either well known or easy to see will be used several times. For the convenience of the reader we now state this facts as lemmas.

Recall that an operator $A \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ is left Fredholm iff there exists $B \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $I - BA$ is a compact operator (where I is the identity operator on $\mathcal{L}$).

LEMMA 1. *An operator $A \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ is not left Fredholm if and only if there exists an infinite dimensional closed subspace $\mathcal{M}$ of $\mathcal{L}$ such that the restriction $A \,|\, \mathcal{M}$ is a compact operator.*

LEMMA 2. *Let $\boldsymbol{A} \in \mathcal{B}(\mathcal{L}, \mathcal{H})^r$. Suppose that S is a compact subset of $\mathbb{C}^r$ such that $0 \notin S$ and such that S intersects every line through $0 \in \mathbb{C}^r$. Then $d(\boldsymbol{A}; S) > 0$ if and only if $m(\lambda \cdot \boldsymbol{A}) > 0$ for every $\lambda \in \mathbb{C}^r - \{0\}$.*

Lemma 1 is well known [3, p. 70]; lemma 2 follows by an obvious compactness argument from the continuity of the function $\lambda \to m(\lambda \cdot \boldsymbol{A})$.

LEMMA 3. *Let $\boldsymbol{A} \in \mathcal{B}(\mathcal{L}, \mathcal{H})^r$. If the operator $\lambda \cdot \boldsymbol{A}$ is left Fredholm for every $\lambda \in S_r$, then there exists a subspace $\mathcal{M}$ of finite codimension in $\mathcal{L}$ such that $d(\boldsymbol{A} \,|\, \mathcal{M}) > 0$.*

PROOF. Since $\lambda \cdot \boldsymbol{A}$ is a left Fredholm operator, there exists a subspace $\mathcal{M}_\lambda$ of finite codimension in $\mathcal{L}$ such that the operator $\lambda \cdot \boldsymbol{A} \,|\, \mathcal{M}_\lambda$ is bounded below [5]. Then each $\lambda \in S_r$ has an open neighborhood U_λ such that the operator $\mu \cdot \boldsymbol{A} \,|\, \mathcal{M}_\lambda$ is bounded below for every $\mu \in U_\lambda$. If $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$ is a finite covering of S_r by such neighborhoods, then the subspace

$$\mathcal{M} = \bigcap_{j=1}^{n} \mathcal{M}_{\lambda_j}$$

is of finite codimension in $\mathcal{L}$ and $m(\lambda \cdot \boldsymbol{A} \mid \mathcal{M}) > 0$ for all $\lambda \in S_r$. The lemma now follows from lemma 2. //

For a subset S of some Banach space let $\bigvee S$ denote the closed linear span of S. As usual, the symbol $\hat{}$ will indicate the non-present term (for example, $(\lambda_1, \ldots, \hat{\lambda}_k, \ldots, \lambda_r) = (\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_r)$).

PROOF OF PROPOSITION 1. *In this proof by a subspace of a Hilbert space we always mean an infinite dimensional closed subspace*. The proof is by an induction on r. In the case $r = 1$ the proposition reduces to the well known fact that a noncompact operator is bounded below on some subspace. Suppose inductively that the proposition holds for arbitrary $\boldsymbol{B} \in \mathcal{B}(\mathcal{H})^{r-1}$. Then we will show that the assumption

$$\text{(a)} \qquad d(\boldsymbol{A} \mid \mathcal{L}) = 0 \text{ for all subspaces } \mathcal{L} \text{ of } \mathcal{H}$$

leads to a contradiction. The proof is divided in three steps.

Step 1. We shall prove that there exist operators $A_1', \ldots, A_r'$ on $\mathcal{H}$ and subspaces $\mathcal{L}_1, \ldots, \mathcal{L}_r$ of $\mathcal{H}$ such that:

(1) The sets $\{A_1', \ldots, A_r'\}$ and $\{A_1, \ldots, A_r\}$ have the same linear span in $\mathcal{B}(\mathcal{H})$;
(2) $A_i' \mid \mathcal{L}_i$ is a compact operator for each $i = 1, \ldots, r$;
(3) $d_i := d(A_1' \mid \mathcal{L}_i, \ldots, \widehat{A_i' \mid \mathcal{L}_i}, \ldots, A_r' \mid \mathcal{L}_i) > 0$ for each $i = 1, \ldots, r$.

Assume inductively that for some i $\in \{1, \ldots, r\}$ the operators A_k' and the subspaces $\mathcal{L}_{i-1,k}$ have been found for all $k = 1, \ldots, i-1$, such that:

(1_{i-1}) The sets $\{A_1', \ldots, A_{i-1}', A_i, \ldots, A_r\}$ and $\{A_1, \ldots, A_r\}$ have the same linear span in $\mathcal{B}(\mathcal{H})$;
(2_{i-1}) $A_k' \mid \mathcal{L}_{i-1,k}$ is a compact operator for all $k = 1, \ldots, i-1$;
(3_{i-1}) $d(A_1' \mid \mathcal{L}_{i-1,k}, \ldots, \widehat{A_k' \mid \mathcal{L}_{i-1,k}}, \ldots, A_{i-1}' \mid \mathcal{L}_{i-1,k}, A_i \mid \mathcal{L}_{i-1,k}, \ldots) > 0$ for all $k = 1, \ldots, i-1$.

We shall then find an operator $A_i' \in \bigvee\{A_1, \ldots, A_r\}$ and subspaces $\mathcal{L}_{ik}$ of $\mathcal{H}$ for $k = 1, \ldots, i$, such that the corresponding conditions $(1_i) - (3_i)$ will be satisfied. Then, putting $\mathcal{L}_k = \mathcal{L}_{rk}$, $k = 1, \ldots, r$, we see, that the conditions (1)−(3) will be satisfied.

Since the proposition holds for any $(r-1)$-tuple of operators, there exists a subspace $\mathcal{M}$ of $\mathcal{H}$ such that

$$\text{(2.1)} \qquad d(A_1' \mid \mathcal{M}, \ldots, A_{i-1}' \mid \mathcal{M}, A_{i+1} \mid \mathcal{M}, \ldots, A_r \mid \mathcal{M}) > 0$$

Note that for at least one $\alpha \in S_r$ the operator $\alpha \cdot \boldsymbol{A} \mid \mathcal{M}$ is not left Fredholm. (If $\lambda \cdot \boldsymbol{A} \mid \mathcal{M}$ were a left Fredholm operator for all $\lambda \in S_r$, then by lemm 3 there would exist a subspace $\mathcal{N}$ of $\mathcal{M}$ such that $d(\boldsymbol{A} \mid \mathcal{N}) > 0$, but this would contradict the assumption (a).) Thus (by lemma 1) there exists a subspace $\mathcal{L}_{ii}$ of $\mathcal{M}$ such that $\alpha \cdot \boldsymbol{A} \mid \mathcal{L}_{ii}$ is a compact operator. Put $A_i' = \alpha \cdot \boldsymbol{A}$. By ($1_{i-1}$) we can write A_i' as

$$\text{(2.2)} \qquad A_i' = \beta_1 A_1' + \cdots + \beta_{i-1} A_{i-1}' + \beta_i A_i + \cdots + \beta_r A_r, \ \beta_j \in \mathbb{C}$$

Observe that $\beta_i \neq 0$. (If β_i were 0, then the operator $A_i' | \mathcal{M}$ would be left Fredholm, since (2.1) would imply that $m(A_i' | \mathcal{M}) > 0$.) Therefore the sets $\{A_1', \ldots, A_i', A_{i+1}, \ldots, A_r\}$ and $\{A_1', \ldots, A_{i-1}', A_i, \ldots, A_r\}$ have the same linear span in $\mathcal{B}(\mathcal{H})$, hence the condition (1_i) is satisfied. The fact that $A_i' | \mathcal{L}_{ii}$ is a compact operator and (2_{i-1}) imply that the conditions (2_i) are satisfied if $\mathcal{L}_{ik}$ is any subspace of $\mathcal{L}_{i-1,k}$ for $k = 1, \ldots, i - 1$. It remains to show that the subspaces $\mathcal{L}_{ik} \subseteq \mathcal{L}_{i-1,k}$ can be chosen in such a way that (3_i) is satisfied, that is, $\mathcal{L}_{ik}$ must be such that

$$(2.3) \qquad d(A_1' | \mathcal{L}_{ik}, \ldots, \widehat{A_k' | \mathcal{L}_{ik}}, \ldots, A_i' | \mathcal{L}_{ik}, A_{i+1} | \mathcal{L}_{ik}, \ldots, A_r | \mathcal{L}_{ik}) > 0$$

for all $k = 1, \ldots, i$.

Observe that for $k = i$ (2.3) holds by (2.1), since $\mathcal{L}_{ii} \subseteq \mathcal{M}$. Let $k = 1, \ldots, i - 1$ be fixed and for every $\tilde{\lambda} = (\lambda_1, \ldots, \hat{\lambda}_k, \ldots, \lambda_r) \in S_{r-1}$ put

$$B(\tilde{\lambda}) = \lambda_1 A_1' + \cdots + \widehat{\lambda_k A_k'} + \cdots + \lambda_i A_i' + \lambda_{i+1} A_{i+1} + \cdots + \lambda_r A_r$$

By lemma 2 the condition (2.3) is equivalent to the requirement

$$(2.4) \qquad m(B(\tilde{\lambda}) | \mathcal{L}_{ik}) > 0 \text{ for all } \tilde{\lambda} \in \mathbb{C}^{r-1} - \{0\}$$

Thus, it suffices to prove the existence of a subspace $\mathcal{L}_{ik}$ in $\mathcal{L}_{i-1,k}$ such that (2.4) is satisfied. Now inserting the expression (2.2) for A_i' into the expression for $B(\tilde{\lambda})$ we obtain

$$(2.5) \qquad B(\tilde{\lambda}) = \mu_1 A_1' + \cdots + \mu_{i-1} A_{i-1}' + \mu_i A_i + \cdots + \mu_r A_r$$

where $\mu_i = \lambda_i \beta_i$, $\mu_k = \lambda_i \beta_k$ and $\mu_j = \lambda_j + \lambda_i \beta_j$ for $j \neq i, k$. Let S be the image of the sphere S_{r-1} under the non-degenerate linear map $(\lambda_1, \ldots, \hat{\lambda}_k, \ldots, \lambda_r) \to \mu_1, \ldots, \hat{\mu}_k, \ldots, \mu_r)$. The condition (3_{i-1}) and lemma 2 imply that

$$\delta_k = d(A_1' | \mathcal{L}_{i-1,k}, \ldots, \widehat{A_k' | \mathcal{L}_{i-1,k}}, \ldots, A_{i-1}' | \mathcal{L}_{i-1,k}, A_i | \mathcal{L}_{i-1,k}, \ldots; S)$$

satisfy

$$\delta_k > 0$$

Since $A_k' | \mathcal{L}_{i-1,k}$ is a compact operator by (2_{i-1}), there exists a subspace $\mathcal{L}_{ik}$ in $\mathcal{L}_{i-1,k}$ such that $\|A_k' | \mathcal{L}_{ik}\| |\beta_k| < \delta_k$. With such a subspace $\mathcal{L}_{ik}$ we have by (2.5)

$$\begin{aligned} m(B(\tilde{\lambda}) | \mathcal{L}_{ik}) &\geq m[(\mu_1 A_1' + \cdots + \widehat{\mu_k A_k'} + \cdots + \mu_{i-1} A_{i-1}' \\ &\qquad + \mu_i A_i + \cdots + \mu_r A_r) | \mathcal{L}_{i-1,k}] - \|\mu_k A_k' | \mathcal{L}_{ik}\| \\ &\geq \delta_k - |\lambda_i \beta_k| \|A_k' | \mathcal{L}_{ik}\| \\ &\geq \delta_k - |\beta_k| \|A_k' | \mathcal{L}_{ik}\| \\ &> 0 \end{aligned}$$

for all $\tilde{\lambda} \in S_{r-1}$, hence for all $\tilde{\lambda} \in \mathbb{C}^{r-1} - \{0\}$. Thus (2.4) is established and this concludes the proof of step 1.

Step 2. Let A_i', $\mathcal{L}_i$ and d_i, $i = 1, \ldots, r$, be as in step 1, so that conditions (1)–(3)

are satisfied. Then, there exist subspaces $\mathcal{M}_i$ of $\mathcal{L}_i$ for $i = 1, \ldots, r$, such that

(2.6) if $i \neq j$, then $\mathcal{M}_i \perp \mathcal{M}_j$ and $\bigvee\{A_k'\mathcal{M}_i;\ k \neq i,\ k = 1, \ldots, r\} \perp \bigvee\{A_k'\mathcal{M}_j;\ k \neq j,\ k = 1, \ldots, r\}$

and

$$\|A_i' \mid \mathcal{M}_i\| < \epsilon,\ i = 1, \ldots, r \tag{2.7}$$

where

$$\epsilon = \frac{1}{2\sqrt{r}} \min\{d_1, \ldots, d_r\}$$

To see this, construct first a sequence of unit vectors $x_{11}, \ldots, x_{r1}, x_{12}, \ldots, x_{r2}, x_{13}, \ldots, x_{r3}, \ldots$ as follows. Choose any $x_{11} \in \mathcal{L}_1$ such that $\|x_{11}\| = 1$. Assume inductively that $x_{jm} \in \mathcal{L}_j$ have been already chosen for all $(j, m) \in N_{i,n}$, where $i \in \{1, \ldots, r\}$ and $n \in N$ are fixed integers and where

$$N_{i,n} = (\{1, \ldots, r\} \times \{1, \ldots, n-1\}) \cup (\{1, \ldots, i-1\} \times \{n\})$$

(if $i = 1$, the set $\{1, \ldots, i-1\}$ is to be interpreted as empty.) Then choose $x_{in} \in \mathcal{L}_i$ so that x_{in} is orthogonal to the finite set

$$\{x_{jm}, A_p^* A_q x_{jm};\ p, q = 1, \ldots, r,\ (j, m) \in N_{i,n}\}$$

This choice of x_{in} is possible, since $\mathcal{L}_i$ is an infinite dimensional space. Now the subspaces

$$\mathcal{M}_i' = \bigvee\{x_{in};\ n = 1, 2, \ldots\},\ i = 1, \ldots, r$$

clearly satisfy (2.6). Since $\mathcal{M}_i' \subseteq \mathcal{L}_i$, the operators $A_i' \mid \mathcal{M}_i'$ are compact by (2), hence there exists for each i a subspace $\mathcal{M}_i$ of $\mathcal{M}_i'$ such that (2.7) holds.

Step 3. Let $U_i: \mathcal{H} \to \mathcal{M}_i$ be arbitrary unitary operators for $i = 1, \ldots, r$ and let

$$\mathcal{M} = \{U_1 x + \cdots + U_r x;\ x \in \mathcal{H}\}$$

We shall show that $m(\lambda \cdot \boldsymbol{A}' \mid \mathcal{M}) > 0$ for every $\lambda \in S_r$, where $\boldsymbol{A}' = (A_1', \ldots, A_r')$. Since the sets $\{A_1', \ldots, A_r'\}$ and $\{A_1, \ldots, A_r)$ have the same linear span, this will imply that $m(\lambda \cdot \boldsymbol{A} \mid \mathcal{M}) > 0$ for all $\lambda \in S_r$, hence $d(\boldsymbol{A} \mid \mathcal{M}) > 0$ by lemma 2. But this will contradict the assumption (a), so the proof of the proposition will be completed.

Now for any $\lambda = (\lambda_1, \ldots, \lambda_r) \in S_r$ and any $z = U_1 x + \cdots + U_r x \in \mathcal{M}$ with $\|z\| = 1$ we have

$$\begin{aligned}
\|(\lambda \cdot \boldsymbol{A}')z\| &\geq \left\| \sum_{j=1}^{r} \Big(\sum_{i \neq j} \lambda_i A_i' \Big) U_j x \right\| - \sum_{j=1}^{r} |\lambda_j| \|A_j' U_j x\| \\
&= \left[\sum_{j=1}^{r} \left\| \Big(\sum_{i \neq j} \lambda_i A_i' \Big) U_j x \right\|^2 \right]^{1/2} - \sum_{l=1}^{r} |\lambda_j| \|A_j' U_j x\| \qquad \text{(by (2.6))} \\
&\geq \left[\sum_{j=1}^{r} d_j^2 \|U_j x\|^2 \right]^{1/2} - \sum_{j=1}^{r} \|A_j' \mid \mathcal{M}_j\| \|U_j x\| \\
&\qquad \text{(by (3), since } \mathcal{M}_i \subseteq \mathcal{L}_i)
\end{aligned}$$

$$\geq \left(\min_{1\leq j\leq r} d_j\right)\left[\sum_{j=1}^{r}\|U_jx\|^2\right]^{1/2} - r\epsilon\|x\| \qquad \text{(by (2.7))}$$

$$= \left(\min_{1\leq j\leq r} d_j\right)\|z\| - r\epsilon\frac{\|z\|}{\sqrt{r}} \qquad \text{(since } \mathcal{M}_j\text{'s are orthogonal by (2.6))}$$

$$= \epsilon\sqrt{r} \qquad \text{(by the definition of } \epsilon\text{, since } \|z\| = 1)$$

It follows that $m(\lambda\cdot A) \geq \epsilon \vee r > 0$, as required. //

3. Proof of Theorem 1. By proposition 1 there exists an infinite dimensional closed subspace $\mathcal{L}$ of $\mathcal{H}$ such that

$$d := d(A \mid \mathcal{L}) > 0$$

Let $(f_n)_{n=1}^{\infty}$ be an orthogonal sequence of vectors in $\mathcal{L}$ such that the subspaces

$$\mathcal{H}_n = \vee\{A_1f_n, \ldots, A_rf_n\},\ n = 1, 2, \ldots$$

of $\mathcal{H}$ are orthogonal. (Such a sequence $(f_n)_{n=1}^{\infty}$ can be constructed inductively as in the proof of step 2 of proposition 1.) Let $\{g_{1n}, \ldots, g_{rn}\}$ be an orthonormal basis of $\mathcal{H}_n$ for each n and define $T_n \in \mathcal{B}(\mathcal{H}_n)$ by

$$T_n(A_if_n) = g_{in},\ i = 1, \ldots, r$$

Since $d = d(A \mid \mathcal{L}) > 0$, the operators T_n are well defined. Moreover, the sequence of their norms is bounded, $\|T_n\| < 1/d$ for all $n = 1, 2, \ldots$. (To see this, it suffices to verify that $\|T_n^{-1}x\| \geq d$ for every unit vector $x \in \mathcal{H}_n$. If $x = \lambda_1 g_{1n} + \cdots + \lambda_r g_{rn}$, where $(\lambda, \ldots, \lambda_r) \in S_r$, then $\|T_n^{-1}x\|^2 = \|\lambda_1A_1f_n + \cdots + \lambda_rA_rf_n\|^2 \geq d^2$ by the definition of d.) It follows that the orthogonal sum $T = T_1 \oplus T_2 \oplus \cdots$ is a bounded operator on the subspace

$$\mathcal{H}' = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$$

of $\mathcal{H}$. Note that $\{g_{in};\ i = 1, \ldots, r,\ n = 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{H}'$. Let $(e_n)_{n=1}^{\infty}$ be any orthonormal basis of $\mathcal{H}$. Define $U:\mathcal{H}' \to \mathcal{H}^r = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ by

$$Ug_{in} = (0, \ldots, e_n, 0, \ldots, 0)$$

where e_n is on the i-th position. Let $B:\mathcal{H}^r \to \mathcal{H}$ be defined by $B(x_1, \ldots, x_r) = B_1x_1 + \cdots + B_rx_r$ (where the operators B_i are as in the statement of the theorem). Then the composite BUT is a bounded operator from $\mathcal{H}'$ to $\mathcal{H}$, hence it can be extended to an operator $X \in \mathcal{B}(\mathcal{H})$, With so defined X we have

$$XA_if_n = BUTA_if_n = BUg_{in} = B(0, \ldots, e_n, 0, \ldots, 0) = B_ie_n$$

for all $i = 1, \ldots, r$ and all $n = 1, 2, \ldots$. Finally, let $Y:\mathcal{H} \to \mathcal{H}$ be an isometry defined by

$$Ye_n = f_n,\ n = 1, 2, \ldots$$

Then $XA_iYe_n = B_ie_n$ for all n, hence $XA_iY = B_i$ for all $i = 1, \ldots, r$ (since (e_n) is an orthonormal basis of $\mathcal{H}$). //

REMARK. Theorem 1 holds also for a non-separable Hilbert space $\mathcal{H}$, but the hypothesis "If no nontrivial linear combination of the operators A_i is compact" must be replaced by "If no nontrivial linear combination of the operators A_i is contained in the maximal ideal of $\mathcal{B}(\mathcal{H})$". The proof is essentially the same as for a separable space, except that transfinite induction has to be used.

4. **Appendix**. Since the author couldn't find any reference for the proof of theorem 2, a sketch of the proof will be given here, although it is essentially the same as the proof of the Jacobson density theorem [8].

PROOF OF THEOREM 2. Denote by $\mathcal{M}^r$ the direct sum of r copies of $\mathcal{M}$ and by $\pi : \mathcal{M} \to \mathcal{M}/\mathcal{K}$ the natural map. It has to be shown that $\mathcal{R}(x_1, \ldots, x_r) = \mathcal{M}^r$ (where $\mathcal{R}(x_1, \ldots, x_r) = \{(cx_1, \ldots, cx_r); c \in \mathcal{R}\}$), and this can be done by an induction on r. For $r = 1$ the theorem clearly holds (by definition of $\mathcal{K}$), so assume inductively that it holds for $r - 1$, where r is a fixed positive integer. It suffices to prove that for each $i = 1, \ldots, r$ there exists an element $z_i \in \mathcal{M} - \mathcal{K}$ such that $(0, \ldots, z_i, \ldots, 0) \in \mathcal{R}(x_1, \ldots, x_r)$, since $z_i \notin \mathcal{K}$ implies that $\mathcal{R}z_i = \mathcal{M}$. To this end assume without loss of generality that $i = r$. Consider the homomorphisms $\phi : \mathcal{R} \to \mathcal{M}^{r-1}$ and $\psi : \mathcal{R} \to \mathcal{M}/\mathcal{K}$ defined by $\phi(c) = (cx_1, \ldots, cx_{r-1})$ and $\psi(c) = \pi(cx_r)$, repectively. The existence of the element z_r is obviously equivalent to the condition Ker $\phi \nsubseteq$ Ker ψ. It will be shown that the assumption Ker $\phi \subseteq$ Ker ψ leads to a contradiction.

Note that the maps ϕ and ψ are onto by the inductive hypothesis and the simplicity of the module $\mathcal{M}/\mathcal{K}$ respectively, hence they induce the isomorphisms $\phi' : \mathcal{R}/\text{Ker}\ \phi \to \mathcal{M}^{r-1}$ and $\psi' : \mathcal{R}/\text{Ker}\ \psi \to \mathcal{M}/\mathcal{K}$. If Ker $\phi \subseteq$ Ker ψ, then we have the natural epimorphism $\theta : \mathcal{R}/\text{Ker}\ \phi \to \mathcal{R}/\text{Ker}\ \psi$, which induces an epimorphism $\lambda : \mathcal{M}^{r-1} \to \mathcal{M}/\mathcal{K}$, $\lambda = \psi'\theta\phi'^{-1}$. Let $\lambda_j : \mathcal{M} \to \mathcal{M}/\mathcal{K}$ be the components of λ, that is, $\lambda(u_1, \ldots, u_{r-1}) = \lambda_1(u_1) + \cdots + \lambda_{r-1}(u_{r-1})$ for all $(u_1, \ldots, u_{r-1}) \in \mathcal{M}^{r-1}$. Since the module $\mathcal{M}/\mathcal{K}$ is simple, Ker λ_j is either $\mathcal{M}$ or the only maximal submodule $\mathcal{K}$ of $\mathcal{M}$. In any case λ_j induces an endomorphism $\lambda_j' \in \text{End}_{\mathcal{R}}(\mathcal{M}/\mathcal{K})$. By the definition of λ we now have

$$\lambda_1'\pi(cx_1) + \cdots + \lambda_{r-1}'\pi(cx_{r-1}) = \pi(cx_r)$$

for all $c \in \mathcal{R}$. The last equality implies in particular that $\lambda_1'\pi(x_1) + \cdots + \lambda_{r-1}'\pi(x_{r-1}) = \pi(x_r)$, but this is a contradiction, since the elements $\pi(x_i)$ are linearly independent over the division ring $\text{End}_{\mathcal{R}}(\mathcal{M}/\mathcal{K})$. //

ACKNOWLEDGEMENT. The results here are closely related to the author's Ph.D. thesis written under the direction of Professor I. Vidav. The author would like to express his gratitude to Professor Vidav for his guidance during the preparation of the thesis.

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